

CALCULUS (Global Governance) A.Y. 2024/25

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FINAL EXAM: 21st JANUARY 2025

Last Name:	First Name:	Student's ID:
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Instructions:

- You have 3 hours to answer the following questions. Students who successfully passed the midterms and are exempt from taking Part I have 1.5 hours (If they do not turn in the exam after 1.5 hours, it will be considered a forfeiture of their midterm score and they will be required to complete the entire final exam).
- Report your answers in the table (for Part I) and in the empty space (for Part II) provided. Any work done on draft sheets will not be considered.
- No notes, books, or any other reference materials is allowed during the test. Electronic devices, including computers, tablets, and cell phones, are strictly prohibited. You are only permitted to use a non-scientific and non-graphing calculator.

Part I (Multiple choice questions)

Answers

1	2	3	4	5	6	7	8	9	10
d	a	c	b	c	b	d	c	d	c

1. Compute the following limit:

$$\lim_{x \rightarrow +\infty} e^{-(x^2+5)} \left(\frac{5x^3 + 2x + 5}{2x^3 + 1} \right) =$$

- (a) $\frac{5}{2}$
 (b) $+\infty$
 (c) $\frac{5}{2} e^{-5}$
 (d) 0
 (e) None of the above

2. Compute the following definite integral:

$$\int_1^2 6x \ln(4x) dx =$$

- (a) $-\frac{9}{2} + 6 \ln 4 + 9 \ln 4$
 (b) $6 \ln 16 - \frac{9}{2} \ln 4$
 (c) $\frac{9}{2} - 6 \ln 4 + \frac{9}{2} \ln 4$
 (d) $-9 + 9 \ln 4$
 (e) $6 \ln 4 + 9 \ln 4$

3. Find the second order partial derivative f_{xy} of the function

$$f(x, y) = 5x^4 y^3 + \frac{2xy}{x-16} e^x$$

- (a) $-\frac{2e^x x}{(32-x)^2} + \frac{5e^x(x+1)}{x-16} + 55x^3 y^2$
 (b) $-\frac{2e^x x}{(15-x)^2} + \frac{3e^x(x+1)}{x-32} + 65x^3 y^2$
 (c) $-\frac{2e^x x}{(16-x)^2} + \frac{2e^x(x+1)}{x-16} + 60x^3 y^2$
 (d) $-\frac{2e^x x}{(48-x)^2} + \frac{7e^x(x+1)}{x-16} + 30x^3 y^2$
 (e) $-\frac{2e^x x}{(14-x)^2} + \frac{4e^x(x+1)}{x-48} + 45x^3 y^2$

4. A pollutant is introduced into a lake, and after t seconds, the concentration of the pollutant in the water is given by $P(t)$ milligrams per cubic centimeter (g/cm^3), where $P(t) = 0.2(1 + 3e^{-0.06t})$. How long (in seconds) will it take for the pollutant concentration to reach $0.5 \text{ g}/\text{cm}^3$?

- (a) $\frac{25}{3} \ln(3)$
 (b) $\frac{50}{3} \ln(2)$
 (c) $\frac{5}{3} \ln(6)$
 (d) $\frac{5}{3} \ln(50) - \frac{1}{3}$
 (e) It will never reach this concentration.

5. Determine the solution to the following separable differential equation:

$$y'(x) = \frac{y-36}{x+6}, \quad y(0) = 6$$

- (a) $y(x) = -5x + 12$
- (b) $y(x) = -34x - 12$
- (c) $y(x) = -5x + 6$
- (d) $y(x) = -11x - 6$
- (e) $y(x) = -11x + 18$

6. Determine the solution to the following linear differential equation

$$y'(x) + \frac{2}{x}y(x) = (3-x)^2, \quad y(1) = 0$$

- (a) $y(x) = \frac{3}{4} \frac{(x^3-1)}{x^2} - \frac{(x^4+1)}{x^2} + \frac{x^5-1}{2x^2}$
- (b) $y(x) = 3 \frac{(x^3-1)}{x^2} - \frac{3}{2} \frac{(x^4-1)}{x^2} + \frac{x^5-1}{5x^2}$
- (c) $y(x) = \frac{18}{5} \frac{(x^3+1)}{x^2} - \frac{5}{2} \frac{(x^4+1)}{x^2} + \frac{x^5-2}{5x^2}$
- (d) $y(x) = \frac{3}{2} \frac{(x^3-1)}{x^2} - \frac{9}{2} \frac{(x^4+1)}{x^2} + \frac{x^5-3}{4x^2}$
- (e) $y(x) = \frac{8}{3} \frac{(x^3+1)}{x^2} - \frac{3}{5} \frac{(x^4-1)}{x^2} + \frac{x^5-1}{3x^2}$

7. Find the critical point of the function

$$f(x, y) = 2x^2 + y^2 + 2xy + 4x + 14y - 21$$

specifying whether it is a relative minimum, maximum or saddle point:

- (a) (11, 16), local maximum
- (b) (5, -5), saddle point
- (c) (8, -5), local maximum
- (d) (5, -12), local minimum
- (e) (6, 9), saddle point

8. Compute the following limit

$$\lim_{x \rightarrow -3} \left(\frac{2x+6}{-3 \ln(-3x^2 - 23x - 41)} \right) =$$

- (a) $+\infty$
- (b) It does not exist
- (c) $\frac{2}{15}$
- (d) 0
- (e) $\frac{1}{5}$

9. Determine the domain of the function

$$f(x) = \ln \left(\frac{2x-13}{5x^2 - 65x + 200} \right)$$

- (a) $(-\infty, 5) \cup (\frac{13}{2}, 8)$
- (b) $(-\infty, 5] \cup [\frac{13}{2}, 8]$
- (c) $[5, \frac{13}{2}) \cup [8, +\infty)$
- (d) $(5, \frac{13}{2}) \cup (8, +\infty)$
- (e) $\mathbb{R} \setminus \{\frac{13}{2}\}$

10. Find all possible equilibrium points y_0 and their stability, (s)table or (u)nstable, for the differential equation:

$$y'(x) = y(x)(1-y(x)) - \frac{1}{4} \frac{y(x)}{(1+y(x))}$$

- (a) $y_0 : 0(u), -\sqrt{\frac{7}{12}}(s), \sqrt{\frac{7}{12}}(s)$
- (b) $y_0 : 0(s), -\sqrt{\frac{3}{4}}(u), \sqrt{\frac{3}{4}}(s)$
- (c) $y_0 : 0(u), -\sqrt{\frac{3}{4}}(s), \sqrt{\frac{3}{4}}(s)$
- (d) $y_0 : 0(u), -\sqrt{\frac{7}{4}}(s), \sqrt{\frac{7}{4}}(s)$
- (e) $y_0 : 0(s), -\sqrt{\frac{7}{12}}(s), \sqrt{\frac{7}{12}}(u)$

Part II (Problems)

Problem 1. Study the following function and sketch its graph:

$$h(x) = \frac{8}{3} \left(\frac{16 - x^2}{4x^2 - 16} \right)$$

In particular, address the following aspects: domain, intersection with axes, continuity, positivity and negativity, asymptotes (vertical, horizontal or oblique), intervals of monotonicity (increase or decrease), local maxima and minima, second derivative and concavity.

Solution:

• **Domain and continuity**

Domain: $4x^2 - 16 \neq 0 \Leftrightarrow 4x^2 \neq 16 \Leftrightarrow x^2 \neq 4 \Leftrightarrow x \neq \pm 2$
 $\mathbb{D} = \mathbb{R} \setminus \{\pm 2\}$

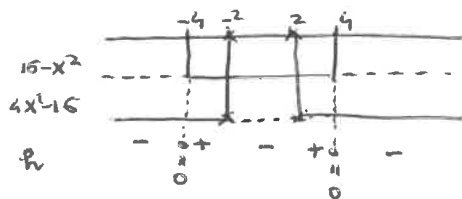
Continuity: h is continuous at all $x \in \mathbb{D}$

• **Intersection with axis and positivity/negativity**

X-AXIS: $16 - x^2 = 0 \Leftrightarrow x^2 = 16 \Leftrightarrow x = \pm 4$ $A = (-4, 0)$ $B = (4, 0)$

Y-AXIS: $h(0) = \frac{8}{3} \cdot \left(\frac{16}{-16} \right) = -\frac{8}{3}$ $C = (0, -\frac{8}{3})$

SIGN:



$h > 0$ in $(-4, -2) \cup (2, 4)$
 $h < 0$ in $(-\infty, -4) \cup (-2, 2) \cup (4, +\infty)$

• **Asymptotes**

VERTICAL: $x = \pm 2$ $\lim_{x \rightarrow -2^+} h(x) = +\infty$ $\lim_{x \rightarrow 2^-} h(x) = +\infty$
 $\lim_{x \rightarrow -2^-} h(x) = -\infty$ $\lim_{x \rightarrow 2^+} h(x) = -\infty$

HORIZONTAL: $\lim_{x \rightarrow \pm\infty} h(x) = \frac{8}{3} \cdot \left(-\frac{1}{4} \right) = -\frac{2}{3}$ HORIZONTAL ASYMPT.: $y = -\frac{2}{3}$ BOTH AT $+\infty$ AND $-\infty$

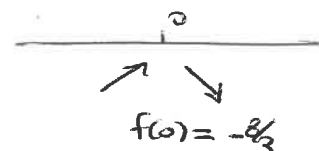
• **First derivative, interval of monotonicity, local maxima/minima**

$$h'(x) = \frac{8}{3} \cdot \frac{-2x(4x^2 - 16) - (16 - x^2)8x}{(4x^2 - 16)^2} = \frac{8}{3} \cdot \frac{-16x}{(4x^2 - 16)^2}$$

$h'(x) > 0$ in $x \in (-\infty, 0) \setminus \{-2\}$ ← INCREASING

$h'(x) < 0$ in $x \in (0, +\infty) \setminus \{2\}$ ← DECREASING

LOCAL MAX AT $x = 0$



• Second derivative and concavity

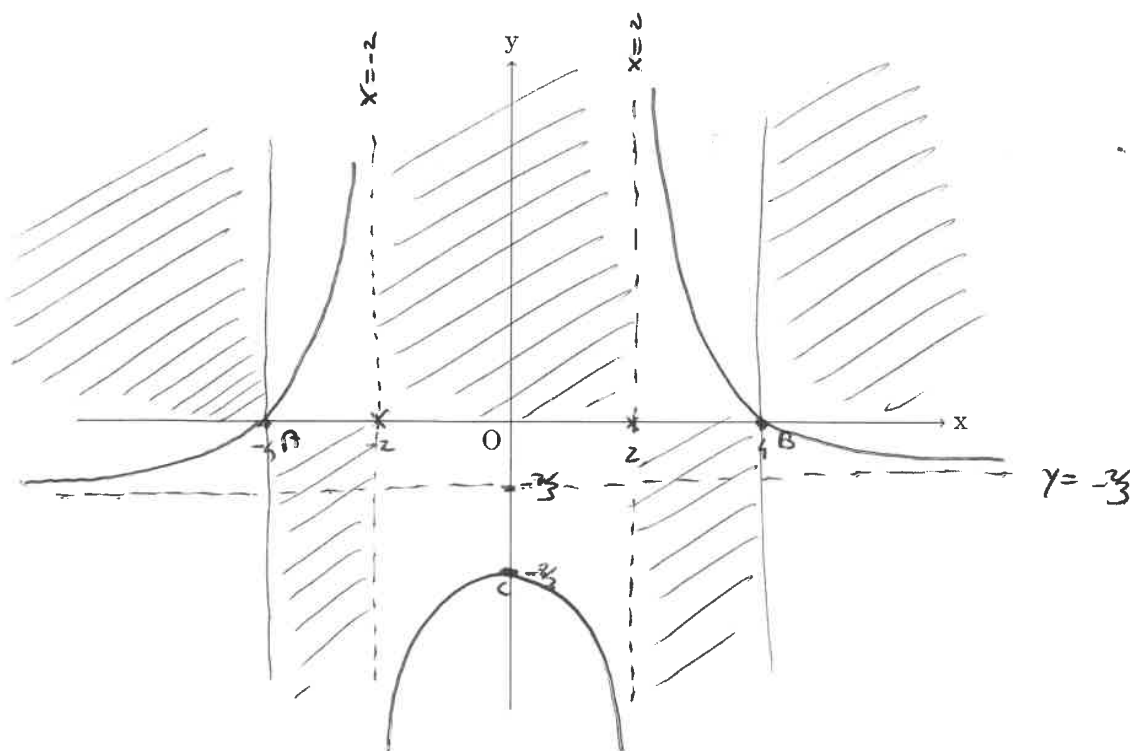
$$R''(x) = -\frac{8}{3} \cdot 95 \frac{1 \cdot (4x^2-16)^2 - x \cdot 2(4x^2-16) \cdot 8x}{(4x^2-16)^3} = -255 \cdot \frac{(-12x^2-16)}{(4x^2-16)^3} = 255 \cdot \frac{12x^2+16}{(4x^2-16)^3}$$

$$R'' > 0 \Leftrightarrow 4x^2 - 16 > 0 \Leftrightarrow x \in (-\infty, -2) \cup (2, +\infty) \quad \text{CONCAVITY UP}$$

$$R'' < 0 \Leftrightarrow 4x^2 - 16 < 0 \Leftrightarrow x \in (-2, 2) \quad \text{CONCAVITY DOWN}$$

$$R''(x) \neq 0 \quad \forall x \in \mathbb{R} \rightarrow \text{NO INFLECTION POINTS}$$

• Graph



Problem 2. The unitary cost C (in hundreds of euros) for producing a certain commodity is modeled by

$$C(h) := 3 \left(1 + \frac{48}{h+3} \right)^{3/2},$$

where h is the thousands of items produced per day. Suppose further that the daily production varies with time as

$$h(t) := \sqrt{t^2 - 6t + 17},$$

where t is the number of months since the start of production.

- What will the unitary cost be after 2 months?
- At what rate is the unitary cost changing with respect to time after 2 months? Is the cost increasing or decreasing at this time?
- At what time t is the unitary cost at its maximum?

Solution

Ⓐ $C(R(2)) = ?$

$$R(2) = \sqrt{4 - 12 + 17} = \sqrt{9} = 3$$

$$C(R(2)) = C(3) = 3 \cdot \left(1 + \frac{48}{3+3} \right)^{3/2} = 3 \cdot \left(1 + \frac{48}{6} \right)^{3/2} = 3 \cdot 9^{3/2} = 3 \cdot 27 = 81$$

Ⓑ $\frac{dC(R(t))}{dt} = \frac{dC(R)}{dR} \cdot \frac{dR(t)}{dt}$ CHAIN RULE

$$\frac{dR}{dt} = \frac{2t-6}{2\sqrt{t^2-6t+17}} = \frac{t-3}{\sqrt{t^2-6t+17}} \rightarrow \frac{dR(2)}{dt} = \frac{-1}{3}$$

$$\frac{dC}{dR} = 3 \cdot \frac{3}{2} \left(1 + \frac{48}{R+3} \right)^{1/2} \cdot \left(-\frac{48}{(R+3)^2} \right) \rightarrow \frac{dC}{dR}(R(2)) = \frac{9}{2} \cdot \left(1 + \frac{48}{3+3} \right)^{1/2} \cdot \left(-\frac{48}{36} \right)$$

$$= \frac{9}{2} \cdot 3 \cdot \left(-\frac{48}{36} \right) = -18$$

$$\Rightarrow \frac{dC(R(2))}{dt} = \frac{dC(R(2))}{dR} \cdot \frac{dR(2)}{dt} = -18 \cdot \left(-\frac{1}{3} \right) = 6$$

SINCE $6 > 0$
 \rightarrow IT IS INCREASING.

Ⓒ $\frac{dC(R(t))}{dt} = 0 \Leftrightarrow \frac{dC}{dR} \cdot \frac{dR}{dt} = 0 \Leftrightarrow \frac{dR}{dt} = 0$

SINCE $\frac{dC}{dR} \neq 0 \quad \forall R \neq -3$

$$\Leftrightarrow \frac{t-3}{\sqrt{t^2-6t+17}} = 0 \Leftrightarrow t = 3$$

\uparrow
IT IS A MAXIMUM SINCE $\frac{dC}{dt} < 0$ FOR $t > 3$
 $\frac{dC}{dt} > 0$ FOR $t < 3$

