

CALCULUS (Global Governance) A.Y. 2024/25
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FINAL EXAM: 11th FEBRUARY 2025

Last Name:	First Name:	Student's ID:
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Instructions:

- You have 3 hours to answer the following questions. Students who successfully passed the midterms and are exempt from taking Part I have 1.5 hours (If they do not turn in the exam after 1.5 hours, it will be considered a forfeiture of their midterm score and they will be required to complete the entire final exam).
- Report your answers in the table (for Part I) and in the empty space (for Part II) provided. Any work done on draft sheets will not be considered.
- No notes, books, or any other reference materials is allowed during the test. Electronic devices, including computers, tablets, and cell phones, are strictly prohibited. You are only permitted to use a non-scientific and non-graphing calculator.

Part I (Multiple choice questions)

Answers

1	2	3	4	5	6	7	8	9	10
d	a	a	a	d	c	c	a	b	a

1. Compute the following limit

$$\lim_{x \rightarrow -\infty} \frac{5x^6 + 9x^5 - 4x + 10}{54x^5 - 18x^4 + 6} =$$

- (a) $-\frac{5}{54}$
(b) $+\infty$
(c) $\frac{5}{54}$
(d) $-\infty$
(e) 0

3. Compute the following limit

$$\lim_{x \rightarrow -3} \left(\frac{4x + 12}{4e^{(-x^2 - 8x - 15)} - 4} \right) =$$

- (a) $-\frac{1}{2}$
(b) $+\infty$
(c) 0
(d) $-\frac{3}{4}$
(e) It does not exist

2. The level of radiation in a contaminated area decreases over time. After t hours, the radiation level $R(t)$ in millisieverts (mSv) is given by $R(t) = 0.1(1 + 6e^{-0.04t})$. How long (in hours) will it take for the radiation level to reach 0.4 mSv?

- (a) $25 \ln(2)$
(b) $\frac{5}{2} \ln(4)$
(c) $\frac{25}{2} \ln(3)$
(d) $\frac{5}{3} \ln(40) - \frac{1}{6}$
(e) It will never reach this concentration.

4. Find the critical point of the function

$$f(x, y) = x^2 + y^2 + 12(x - y)$$

specifying whether it is a relative minimum, maximum or saddle point:

- (a) $(-6, 6)$, local minimum
(b) $(-\frac{13}{2}, \frac{11}{2})$, local minimum
(c) $(-\frac{13}{3}, 6)$, local maximum
(d) $(-6, 4)$, local maximum
(e) $(-\frac{11}{2}, \frac{11}{2})$, local minimum

5. Determine the solution to the following linear differential equation

$$y'(x) + \frac{1}{x}y(x) = 4, \quad y(1) = 0$$

- (a) $y(x) = \frac{7}{4} \frac{x^2-3}{x}$
 (b) $y(x) = 2 \frac{x^2-2}{x}$
 (c) $y(x) = \frac{5}{2} \frac{x^2-1}{x}$
 ► (d) $y(x) = 2 \frac{x^2-1}{x}$
 (e) $y(x) = \frac{4}{3} \frac{x^3-1}{2x}$

6. Find the second order partial derivative f_{xy} of the function

$$f(x, y) = 5x^4y^3 + \frac{2x}{x-4}e^y$$

- (a) $-\frac{3xe^y}{(4-y)^2} + \frac{3e^y}{x-4} + 60x^2y^2$
 (b) $-\frac{3xye^y}{(6-2x)^2} + \frac{2e^y}{y-5} + 30x^3y^3$
 ► (c) $-\frac{2xe^y}{(4-x)^2} + \frac{2e^y}{x-4} + 60x^3y^2$
 (d) $-\frac{2ye^y}{(4+x)^2} + \frac{2e^x}{y-6} + 30x^3y^3$
 (e) $-\frac{2xe^x}{(5-x)^2} + \frac{2e^x}{x-4} + 60x^2y^2$

7. Determine the solution to the following separable differential equation:

$$y'(x) = \frac{y+2}{x-2}, \quad y(0) = 4$$

- (a) $y(x) = 2 - 3x^3$
 (b) $y(x) = 6 - 2x^2$
 ► (c) $y(x) = 4 - 3x$
 (d) $y(x) = 2 - 4x$
 (e) $y(x) = 10 - 2x$

8. Find all possible equilibrium points y_0 and their stability, (s)table or (u)nstable, for the differential equation:

$$y'(x) = y(x)(1 - y(x)) - \frac{1}{3} \frac{y(x)}{(1 + y(x))}$$

- (a) $y_0 : 0(u), -\sqrt{\frac{2}{3}}(s), \sqrt{\frac{2}{3}}(s)$
 (b) $y_0 : 0(s), -\sqrt{\frac{2}{3}}(u), \sqrt{\frac{2}{3}}(s)$
 (c) $y_0 : 0(u), -\sqrt{\frac{5}{3}}(s), \sqrt{\frac{5}{3}}(s)$
 (d) $y_0 : 0(u), -\sqrt{\frac{5}{12}}(s), \sqrt{\frac{5}{12}}(s)$
 (e) $y_0 : 0(s), -\sqrt{\frac{5}{12}}(s), \sqrt{\frac{5}{12}}(u)$

9. Compute the following definite integral:

$$\int_{-1}^0 (x+4)e^{-\frac{x}{6}} dx =$$

- (a) $60 - 54e^{-\frac{1}{6}}$
 ► (b) $-60 + 54e^{\frac{1}{6}}$
 (c) $-60 - 54e^{-\frac{1}{6}}$
 (d) $10 + 60e^{\frac{1}{36}}$
 (e) $-6 + 30e^{-\frac{1}{36}}$

10. Determine the domain of the function

$$f(x) = \sqrt{\frac{x^2 - 8x + 12}{8x - 32}}$$

- (a) $[2, 4) \cup [6, +\infty)$
 (b) $[2, 4] \cup [6, +\infty)$
 (c) $\mathbb{R} \setminus \{4\}$
 (d) $(-\infty, 2] \cup (4, 6]$
 (e) $(-\infty, 2] \cup [4, 6]$

Part II (Problems)

Problem 1. Study the following function and sketch its graph:

$$f(x) = \frac{4(x+5)^2}{x-10}$$

In particular, address the following aspects: domain, intersection with axes, continuity, positivity and negativity, asymptotes (vertical, horizontal or oblique), intervals of monotonicity (increase or decrease), local maxima and minima, second derivative and concavity.

Solution:

• **Domain and continuity**

Domain: $x-10 \neq 0 \Leftrightarrow \mathbb{R} \setminus \{10\} \Leftrightarrow \boxed{D = (-\infty, 10) \cup (10, +\infty)}$

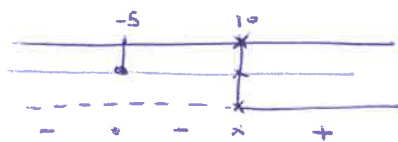
Continuity: f is continuous at every $x \in D$, i.e. $\forall x \in (-\infty, 10) \cup (10, +\infty)$

• **Intersection with axis and positivity/negativity**

INTERSECTION W/ X-AXIS: $f(x)=0 \Leftrightarrow x+5=0 \Leftrightarrow x=-5$ $A = (-5, 0)$

INTERSECTION W/ Y-AXIS: $y=f(0) = \frac{4 \cdot 5^2}{-10} = -10$ $B = (0, -10)$

POSITIVITY/NEGATIVITY: $N: (x+5)^2 \geq 0 \quad \forall x \in D$



$f(x) \leq 0 \quad \forall x \in (-\infty, 10) \setminus \{-5\}$
 $f(x) = 0 \quad \text{for } x = -5$
 $f(x) > 0 \quad \forall x \in (10, +\infty)$

• **Asymptotes**

VERTICAL: $x=10$ $\lim_{x \rightarrow 10^\pm} f(x) = \pm \infty$

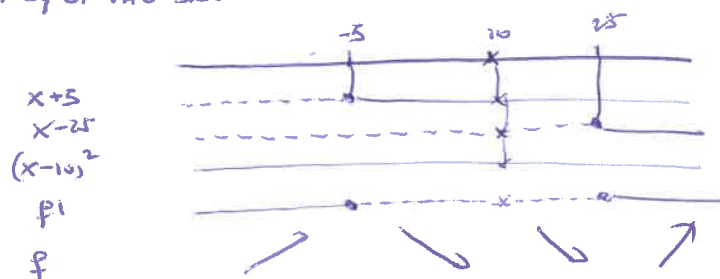
HORIZONTAL: $\lim_{x \rightarrow \pm \infty} f(x) = \pm \infty$ NO HORIZONTAL ASYMPT AT $\pm \infty$

OBlique: $y = mx + q$ $m = \lim_{x \rightarrow \pm \infty} f(x) = 4$
 $q = \lim_{x \rightarrow \pm \infty} (f(x) - 4x) = \lim_{x \rightarrow \pm \infty} \frac{4(x^2 + 25 + 10x) - 4x(x-10)}{(x-10)} = \lim_{x \rightarrow \pm \infty} \frac{80x + 100}{x-10} = 80$
 $\hookrightarrow y = 4x + 80$ BOTH AT $\pm \infty$

• **First derivative, interval of monotonicity, local maxima/minima**

$$\begin{aligned} f'(x) &= 4 \cdot \frac{2(x+5)(x-10) - (x+5)^2}{(x-10)^2} = 4 \cdot \frac{(x+5)(2x-20-x-5)}{(x-10)^2} = 4 \cdot \frac{(x+5)(x-25)}{(x-10)^2} \\ &= 4 \cdot \frac{x^2 - 20x - 125}{(x-10)^2} \end{aligned}$$

STUDY OF THE SIGN



f INCREASING in $(-\infty, -5) \cup (25, +\infty)$

f DECREASING in $(-5, 10) \cup (10, 25)$

$x = -5$ LOCAL MAXIMUM $A = (-5, 0)$

$x = 25$ LOCAL MINIMUM $C = (25, 240) \rightarrow f(25) = 240$

Second derivative and concavity

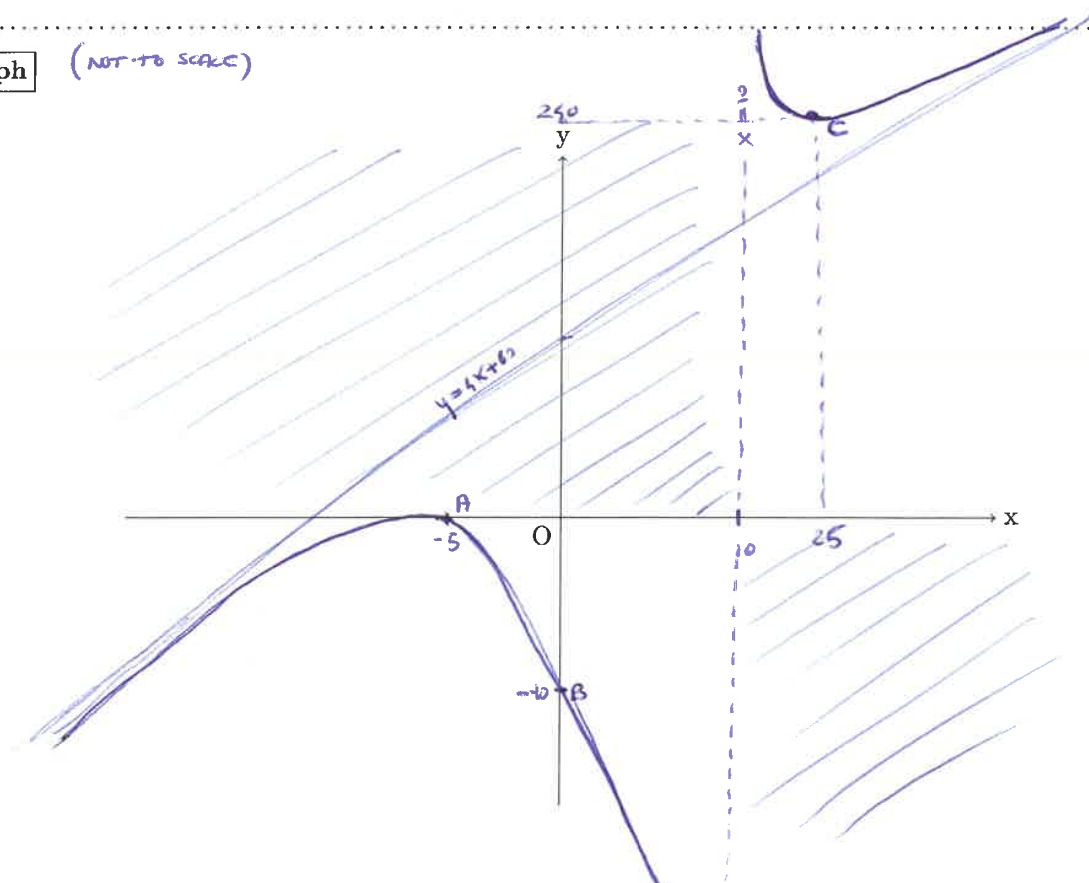
$$f''(x) = \frac{(2x-20)(x-10)^2 - (x^2-20x+125)2(x-10)}{(x-10)^4} = 8 \frac{(x-10) [x^2-20x+125 - 2x^2+40x-250]}{(x-10)^4}$$

$$= 8 \frac{225}{(x-10)^3} = \frac{1800}{(x-10)^3}$$

$f'' > 0 \Leftrightarrow x > 10$ CONCAVITY UP \cup

$f'' < 0 \Leftrightarrow x < 10$ CONCAVITY DOWN \cap

Graph (NOT TO SCALE)



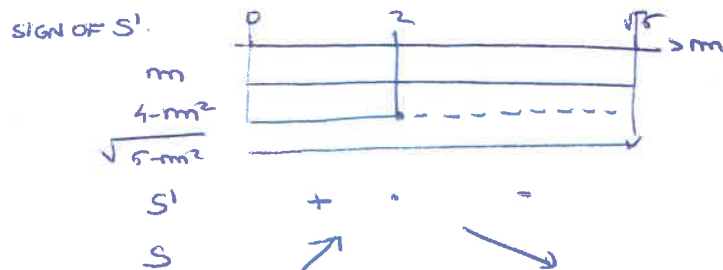
Problem 2. A farmer determines that planting t thousand trees on their land will yield a revenue of $R(t)$ thousands euros, where R is given by the function

$$R(t) := 3t^2 \sqrt{\frac{27}{2} - t^2}.$$

- (a) How many trees should the farmer plant to maximize the revenue? What would the maximum revenue be?
 (b) For what number of planted trees is the revenue increasing most rapidly?

OBSERVE $m \geq 0$ AND $6 - m^2 \geq 0 \Rightarrow 0 \leq m \leq \sqrt{6}$ ← RANGE FOR m

$$S'(m) = 4m \sqrt{6 - m^2} + 2m^2 \frac{(-2m)}{2\sqrt{6 - m^2}} = \frac{4m(6 - m^2) - 2m^3}{\sqrt{6 - m^2}} = \frac{24m - 6m^3}{\sqrt{6 - m^2}} = \frac{6m(4 - m^2)}{\sqrt{6 - m^2}}$$



S HAS A GLOBAL MAXIMUM AT $m = 2$ (MAXIMAL SOCIAL WELFARE SCORE FOR 2 MILLION EUROS)

MAXIMAL SCORE $S(2) = 2 \cdot 4 \sqrt{6 - 4} = 8\sqrt{2} \approx 11,3137$

WE NEED TO STUDY THE MAXIMUM OF $S'(m)$ ↪ RATE OF CHANGE

$$S''(m) = \frac{(24 - 18m^2)\sqrt{6 - m^2} - (24m - 6m^3) \frac{(-2m)}{2\sqrt{6 - m^2}}}{(6 - m^2)^{\frac{3}{2}}} = \frac{(24 - 18m^2)(6 - m^2) + 24m^2 - 6m^4}{(6 - m^2)^{\frac{3}{2}}}$$

$$= \frac{144 - 132m^2 + 18m^4 + 24m^2 - 6m^4}{(6 - m^2)^{\frac{3}{2}}} = \frac{12m^4 - 108m^2 + 144}{(6 - m^2)^{\frac{3}{2}}}$$

$\frac{1}{2}$ solve $12m^4 - 108m^2 + 144 = 0 \Leftrightarrow m^2 = \frac{108 \pm \sqrt{108^2 - 48 \cdot 144}}{24} = \frac{108 \pm \sqrt{4752}}{24}$
 BOTH > 0

$m = \pm \sqrt{\frac{108 \pm \sqrt{4752}}{24}} \rightarrow$ I CONSIDER ONLY + SINCE m MUST BE POSITIVE

THE ONLY SOLUTION IN $0 \leq m \leq \sqrt{6}$

IS $m = \sqrt{\frac{108 - \sqrt{4752}}{24}} \approx 1.276$

IT IS A MAXIMUM FOR S'

POINT AT WHICH S IS INCREASING MOST RAPIDLY

(OBSERVE $\sqrt{\frac{108 + \sqrt{4752}}{24}} \approx 2.715 > \sqrt{6} \approx 2.449$)