Some exercises on functions of two variables

NOTE: I have changed problem 4 since there was something peculiar about the version in the lecture which might be confusing.

A regional development agency is analyzing economic output in a region. The annual GDP (in billions of euros) depends on two policy variables: infrastructure investment I, (in billions of euros) and education spending E (in billions of euros).

They predict GDP will be given by

$$G(I, E) = 23I + 55E - I^2 - 2E^2 + IE.$$

1. (a) Calculate the partial derivatives $\frac{\partial G}{\partial I}$ and $\frac{\partial G}{\partial E}$ and explain their significance.

   (b) If $I = 5$ and $E = 4$, where would an increase in spending have the biggest impact on GDP?

   Solution:

   (a)
   $$\frac{\partial G}{\partial I} = 23 - 2I + E, \quad \frac{\partial G}{\partial E} = 55 - 4E + I; \tag{1}$$
   $\frac{\partial G}{\partial I}$ gives a formula for the predicted increase in GDP in euros for each additional euro of infrastructure spending (which depends on the current levels of infrastructure and education spending), $\frac{\partial G}{\partial E}$ gives a similar formula for the increase for each additional euro of education spending.

   (b) Using the previous result $G_I(5, 4) = 23 - 10 + 4 = 17$, $G_E(5, 4) = 55 - 16 + 5 = 44$; so G_E is larger, and therefore increased education spending gives a bigger improvement.

2. The agenct plans to increase both investments over time according to

   - $I(t) = 3 + 0.5t$,
   - $E(t) = 2 + 0.3t$,

   where t is the time in years.

   How fast is GDP growing after 4 years?

   Solution: One way to find the answer is to use the chain rule, which in this case reads
   $$\frac{dG}{dt} = \frac{\partial G}{\partial I}\frac{dI}{dt} + \frac{\partial G}{\partial E}\frac{dE}{dt}. \tag{2}$$

After four years (i.e. $t = 4$) $I(4) = 5$ and $E(t)3.2$, so

$$\frac{\partial G}{\partial I} = 23 - 2\dot{5} + 3.2 = 16.2, \quad \frac{\partial G}{\partial E} = 55 - 4\dot{3}.2 + 5 = 47.2, \tag{3}$$

and $I'(t) = 0.5$, $E'(t) = 0.3$; putting all of this together

$$\frac{dG}{dt} = 16.2 \times 0.5 + 47.2 \times 0.3 = 22.26, \tag{4}$$

so GDP is growing by 22.26 billion euros per year.

3. What levels of spending would give the highest possible GDP?

   Solution: Setting the partial derivatives equal to zero gives the conditions

$$\begin{cases} 23 - 2I + E = 0 \\ 55 - 4E + I = 0, \end{cases} \tag{5}$$

   and solving this gives one solution (critical point) $I^* = 21, E^* = 19$.

   To confirm that this is a maximum, we calculate the second partial derivatives

$$G_{II} = -2, \quad G_{EE} = -4, \quad G_{IE} = 1, \tag{6}$$

   and so $D_G(I, E) = (-2)\dot{(}-4) - 1^2 = 7 > 0$; therefore this is a local extremum, and since also $G_{II}, G_{EE} < 0$ it is indeed a maximum.

4. Suppose the total budget is fixed at 32 billion euros. Taking this into account, what spending levels would give the highest GDP?

   Solution: We now want to maximize $G(I, E)$ subject to the constraint $I + E = 32$. Using the method of Lagrange multipliers, we have

$$F(I, E, \lambda) = G(I, E) - \lambda[I + E - 32], \tag{7}$$

   so the critical points are the solutions of

$$\begin{cases} 23 - 2I + E - \lambda = 0, \\ 55 - 4E + I - \lambda = 0, \\ 32 - I - E = 0. \end{cases} \tag{8}$$

   Eliminating λ from the first two equations gives

$$23 - 2I + E = 55 - 4E + I \Rightarrow E = \frac{3}{5}I + \frac{32}{5}, \tag{9}$$

   and plugging this into the constraint equation gives

$$I + \frac{3}{5}I + 96 = 32 \Rightarrow I = 16, \tag{10}$$

   and plugging this into the previous equation gives $E = 16$.