Applied Quantitative Analysis

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1 Differential equations

1.1 Models of differential equations in applied sciences

The main role of mathematics in applications often consists in building *mathematical models* which may serve as reference for the description of possibly different type of phenomena. If we believe in Galileo's intuition, that *the world is written in the language of mathematics*, we see how important it is to understand the description of phenomena in mathematical terms. If we are able to do so, we will next use all the strength of mathematics to analyze phenomena, to raise questions and give answers concerning their behavior. Differential equations are at the heart of applied mathematics; indeed, the most natural way to observe phenomena is to report *changes* in what we see. In mathematics, a change of some variable, as well as of related quantities, are described through growth ratios $\frac{\Delta y}{\Delta x}$ and, in the continuous limit, through derivatives.

Definition 1.1 *A differential equation is an equation of the type*

$$F(x, y(x), y'(x), y''(x), \ldots, y^{(n)}(x)) = 0 \qquad x \in (a, b) \subseteq \mathbb{R}$$

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to be satisfied for every $x \in (a,b)$ by the (unknown) function $y(x)$, which is required to be a C^n function in the interval (a,b), $y^{(k)}$ denoting its k-derivative.

We say that the equation has order n if this is the maximal order of derivation of y involved in the equation.

In other words, a differential equation is an identity required to be satisfied by a function together with its derivatives, for all values of the variable x in some interval (a,b).

Most common differential equations are of first or second order, i.e. they involve only up to first or second derivatives of y.

In particular, several phenomena from biology and medicine are described through (experimental) models of first order equations, where the variable x represents time: those models are derived by looking at the variation Δy of some quantity measured at different times. In the following examples, the independent variable is called t in place of x.

Example 1.1 *The decay of radioactive isotopes of Carbonium 14 satisfies the first order differential equation*

$$y' = -\lambda\, y$$

where λ is a positive constant and $y(t)$ is the unknown function (note that the independent variable is denoted by t).

Typically, a law of this kind may be obtained by observing, at different times, a proportionality between Δy and the quantity y itself:

$$y(t+h) - y(t) = -\lambda\, h\, y(t)$$

The proportionality constant is λ for units of time. If this law is observed regularly at different time steps of order h, and holds for h arbitrarily small, it will be reasonable to deduce a law independent of the time-step of our measurements: letting $h \to 0$ one deduces

$$y'(t) = \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} = -\lambda y(t)\,.$$

This is a typical way in which many laws are deduced in the form of differential equations.

The above example of differential equation is one of the most important models of *growth* or *decay.* Since y' represents the (instantaneous) growth rate of the function y, this model describes all phenomena in which *the growth rate of a quantity is proportional to the quantity itself.*

One class of applications is when y stands for the amount of some population: in many natural models the growth rate of the population is proportional to the amount of population. The differential model in this case yields

$$y'(t) = k\, y(t) \tag{1.1}$$

where k is the proportionality constant, which is interpreted as the *intrinsic growth rate*, i.e. the *per capita rate of increase* of the population. Note that $k > 0$ means an effective increase but $k < 0$ means actually a decrease, as it was in the case of radioactive decay.

The solution to (1.1) is always the exponential function $y(t) = c\, e^{kt}$; notice that y is a solution for any possible choice of multiplicative constant $c \in \mathbb{R}$. Indeed, we have one degree of freedom here for the class of solutions. This degree of freedom can be fixed, for example, by prescribing an initial condition $y(0) = y_0$: if y must satisfy this condition, then only one value of c is acceptable, and in this case $y = y_0 e^{kt}$ is the unique solution.

We will come back to the role of initial condition later on. Please observe the analogy with the discrete laws of recursions of first order: given a recursive law $a_n = f(n, a_{n-1})$, one needs a starting value a_0 in order that a_n be well defined.

The exponential growth model (1.1) can be modified to take into account that the intrinsic growth rate be dependent on the population itself. If this is the case, one says that *the intrinsic growth rate is density dependent*, which simply means that the ratio $\frac{y'}{y}$ is a function of y, i.e. $y' = k(y)\, y$ (or possibly even time dependent $y' = k(t, y)\, y$. The simplest case happens when k is a linear function, as in the famous *logistic growth model.*

Example 1.2 *(Logistic growth) A population y is said to evolve with logistic growth if $y(t)$ satisfies*

$$y'(t) = r\, y(t) \left(1 - \frac{y(t)}{K}\right)$$

where r, K are positive constants. The interpretation of those parameters is the following: r is the initial intrinsic growth (since for $y \simeq 0$ we have

$y'(t) \simeq ry(t)$, like in the usual exponential growth), while K is the carrying capacity. Indeed, y is increasing provided $y(t) < K$, otherwise the growth would be decreasing: the value $y(t) = K$ corresponds to $y' = 0$, meaning that the population does not grow. We will see later that the value $y(t) = K$ represents a stable equilibrium for this model.

Example 1.3 *(Newton's cooling law) According to Newton's law of cooling, a body exchanges temperature with the neighborhood environment in a way that the rate of change of the body temperature is proportional to the difference between the temperature of the body and the external temperature. This means that there exists a constant $k > 0$ such that*

$$T'(t) = -k(T(t) - T_e)$$

where $T(t)$ denotes the body temperature at time x and T_e is the external temperature. Notice the minus sign in front of k, meaning that the body will decrease its temperature whenever the external one is lower (and viceversa, it will warm up should the external temperature be higher).

Newton's law of motion gives examples of second order equations, since $F = m\,a$ means that we are given the acceleration, i.e. the second order derivative $y''(t)$, according to the forces which are acting. Of course, only in the simplest case the forces acting are independent of the motion, like for instance for the simplified model of a body falling on earth surface (from not too far...): assuming the gravity force to be constantly proportional to the mass of the body, one has $y''(t) = g$. The solution can be found with two successive integrations, and gives that y is a parabola (which was Galileo's solution). But other forces could play a role, which depend themselves on the motion, like for instance a friction force depending on the velocity of the body. In this case, we should modify the law as

$$my''(t) = mg - \lambda y'(t)$$

where $\lambda > 0$ is a friction coefficient (e.g. related to air resistance). This modification makes the law a truly differential equation, whose solution will be less obvious at first glance.

Example 1.4 *(motion of a spring) The motion of a spring is analyzed through Newton's law by considering the elastic force (proportional to the elongation* *of the spring) and, in case, a friction force due to air resistance. In absence of other external forces, the equation becomes*

$$my''(t) + \lambda y'(t) + ky(t) = 0$$

where $y(t)$ represents the elongation (from rest) of the spring, m is the mass of the body, k (usually called the spring stiffness constant) is related to the intrinsic rigidity of the spring, λ is a friction coefficient. In the simplest situation, those coefficients m, λ, k are assumed to be constants (nonnegative, of course). The analysis of this model will be done in the last section.

1.2 Solving differential equations.

The simplest case of differential equation is given by

$$y' = f(x)\,, \qquad x \in (a, b) \tag{1.2}$$

where (a, b) is some interval in $\mathbb{R}$. Here we know that the solution comes from the fundamental theorem of calculus: if $f(x)$ is continuous, the solution is given by the indefinite integral $y = \int f(x)\,dx$. As we know, the indefinite integral denotes the set of all primitives, and in a given interval (a, b) this has precisely one degree of freedom. So, if $x_0 \in (a, b)$, all solutions of (1.2) are given by

$$y(x) = \int_{x_0}^{x} f(s)ds + c$$

where c varies in $\mathbb{R}$. Observe that a unique solution can be found by fixing the value of y at $x = x_0$. As we saw in the examples of previous section, this is naturally interpreted as the initial condition.

Solving more general differential equations as

$$y' = f(x, y)$$

can be not easy as before. Nevertheless, a general theorem by Cauchy ensures that, whenever f is a regular function, the initial value problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \tag{1.3}$$

is well posed at least in some neighborhood of x_0. Problem (1.3) is usually referred to as the Cauchy's problem.

Theorem 1.1 *Assume that f is a C^1 function of the variables (x, y) in some neighborhood of (x_0, y_0). Then, there exists an interval (a, b), containing x_0, such that the problem (1.3) admits a unique solution $y(x)$ for $x \in (a, b)$.*

The above theorem says that the problem (1.3) is well-posed at least near x_0: whether this solution will exist for every x (before or past x_0) is not guaranteed and will depend on properties of the function $f(x, y)$. If x is interpreted as time, we say that a solution exists at least in short time (past and future), and we talk of global existence in time (in the past or in the future) if the solution lives for all times (before or later x_0).

A case when solutions exist globally is whenever the function f is linear with respect to y, i.e. $f(x) = b(x) - a(x)y$. In this case, the equation reads as

$$y' + a(x)y = b(x) \tag{1.4}$$

and is called a linear nonhomogeneous differential equation of first order.

Theorem 1.2 *Assume that $a(x), b(x)$ are continuous functions. Then, all solutions to (1.4) are given by the formula*

$$y(x) = e^{-A(x)} \left\{ \int e^{A(x)} b(x) dx + c \right\} \tag{1.5}$$

where $A(x)$ is one primitive of the function $a(x)$. In particular, the unique solution to

$$\begin{cases} y' + a(x)y = b(x) \\ y(x_0) = y_0 \end{cases}$$

is given by

$$y(x) = e^{-A(x)} \left\{ \int_{x_0}^{x} e^{A(s)} b(s) ds + y_0 \right\},$$

where $A(x) = \int_{x_0}^{x} a(s) ds$.

Please notice that in formula (1.5) you may take $A(x)$ as any primitive of $a(x)$; in fact, if you add a constant to $A(x)$, the formula won't change. With (1.5), we find all solutions to the differential equation, and again this is a set with one degree of freedom, namely the constant c. By fixing the value at x_0, only one solution is selected, as you see in the last statement.

Proof. Let $A(x)$ be a primitive of $a(x)$. Since $e^{A(x)} > 0$, multiplying by $e^{A(x)}$ both sides the equation is unchanged. Then we get

$$e^{A(x)}(y' + a(x)y) = b(x)e^{A(x)}$$

Now we notice that the left-hand side equals $\frac{d}{dx}(e^{A(x)}y(x))$. Hence

$$\frac{d}{dx}(e^{A(x)}y(x)) = e^{A(x)}(y' + a(x)y) = b(x)e^{A(x)}$$

which means that $e^{A(x)}y(x)$ is a primitive of $b(x)e^{A(x)}$. In other words, we have

$$e^{A(x)}y(x) = \int b(x)e^{A(x)}\,dx\,.$$

Recall that the indefinite integral is the set of all primitives (invariant by addition of a constant), so it is the same if we write

$$e^{A(x)}y(x) = \int b(x)e^{A(x)}\,dx + c\,, \qquad c \in \mathbb{R}\,.$$

From here, multiplying both sides by $e^{-A(x)}$, we find $y(x)$ and formula (1.5) is proved. ∎

To see an example of application of formula (1.5), let us solve the equation

$$y' = \frac{y}{x} + (x+2)^2\,,$$

for $x > 0$.

First notice that here $a(x) = -\frac{1}{x}$ and $b(x) = (x+2)^2$. Then we have $A(x) = \int a(x)dx = -\log|x|$ and, since we are studying $x > 0$, we have $A(x) = -\log x$. Now the formula reads

$$y(x) = x\left\{\int \frac{(x+2)^2}{x}dx + c\right\}$$

We need to compute

$$\int \frac{(x+2)^2}{x}dx = \int \frac{x^2+4x+4}{x}dx = \int (x+4+\frac{4}{x})dx = \frac{x^2}{2} + 4x + 4\log|x|$$

and finally we have (recall that $x > 0$ so here $\log|x| = \log x$)

$$y(x) = \frac{x^3}{2} + 4x^2 + 4x\log x + cx\,. \tag{1.6}$$

If we are required to find the unique solution to the Cauchy's problem

$$\begin{cases} y' = \frac{y}{x} + (x+2)^2 \\ y(1) = 2 \end{cases}$$

it is enough to impose the condition $y(1) = 2$ in formula (1.6):

$$2 = \frac{1}{2} + 4 + c$$

and we find $c = -\frac{5}{2}$. So the unique solution to this Cauchy's problem is given by

$$y(x) = \frac{x^3}{2} + 4x^2 + 4x \log x - \frac{5}{2}x\,.$$

Example 1.5 *A cup of hot chocolate is brought in a room during a cold winter afternoon. Assume that the room temperature T_r is varying as $T_r(t) = 24(1 - \frac{t}{12})$ (time is measured in hours) being 24 (Celsius degrees) its initial temperature, and that the chocolate is, initially, at 70 degrees. Use Newton's law of cooling for the temperature $T(t)$ of the chocolate cup*

$$T'(t) = -k(T(t) - T_r(t))$$

and assume that $k = 1$. What will be the temperature of the chocolate after 15 minutes?

Let us first solve the differential equation:

$$T' + kT = k(24 - 2t)\,.$$

This gives, according to (1.5),

$$T = e^{-kt} \int e^{kt}(24 - 2t)dt + ce^{-kt}$$

We have

$$\int e^{kt}(24 - 2t)dt = \frac{24}{k}e^{kt} - 2\int e^{kt}t\,dt$$

and last integral is solved with integration by parts:

$$\int e^{kt}t\,dt = \frac{e^{kt}}{k}t - \frac{1}{k}\int e^{kt}\,dt = \frac{e^{kt}}{k}t - \frac{1}{k^2}e^{kt}$$

Therefore, altogether we have

$$\int e^{kt}(24 - 2t)dt = \frac{24}{k}e^{kt} - 2\frac{e^{kt}}{k}t + \frac{2}{k^2}e^{kt}\,.$$

The solution is finally

$$T = \frac{24}{k} - \frac{2}{k}t + \frac{2}{k^2} + ce^{-kt}\,.$$

At initial time we have $T(0) = \frac{24}{k} + \frac{2}{k^2} + c$, hence $c = T_0 - \frac{24}{k} - \frac{2}{k^2}$ if T_0 is the initial time. We conclude that

$$T = (\frac{24}{k} + \frac{2}{k^2})(1 - e^{-kt}) + T_0 e^{-kt} - \frac{2}{k}t\,.$$

Now we insert the informations: $k = 1$, $T_0 = 70$ and we find $T = 26(1 - e^{-t}) + 70e^{-t} - 2t$. After 15 minutes, we have $t = \frac{1}{4}$ and we approximately compute $T(\frac{1}{4}) \simeq 59.7$. I hope now you can enjoy your cup of chocolate !!

1.3 Equations with separation of variables

A special class of differential equations which can be solved through direct integration consists of equations of the form

$$y' = f(x)g(y) \tag{1.7}$$

where f, g are continuous functions. Those equations are in a form said as *separation of variables*, since the variables x and y can be separated just dividing by g:

$$\frac{y'}{g(y)} = f(x)\,.$$

Integrating with respect to x both terms we get

$$\int \frac{y'(x)}{g(y(x))}dx = \int f(x)dx\,.$$

But if we change the variable in the first integral by setting $y = y(x)$, since $dy = y'(x)dx$ we can rewrite as

$$\int \frac{1}{g(y)}dy = \int f(x)dx\,.$$

We deduce the following: *whenever $g(y) \neq 0$, all solutions to (1.7) are given by*

$$H(y) = F(x) + c\,, \qquad \text{where } H(y) = \int \tfrac{1}{g(y)}dy \text{ and } F(x) = \int f(x)dx. \tag{1.8}$$

The above formula gives, in principle, the solution y, always up to one degree of freedom, which is the constant c. Notice that, since we are assuming that $g(y) \neq 0$, the function $H(y)$ will be strictly monotone and therefore injective. This means that we can recover y through the inverse function $y = H^{-1}(F(x) + c)$.

The procedure described above allows one to solve first order equations with separation of variables. However, be careful to the condition $g(y) \neq 0$ which was assumed: one should often check this condition at the initial time. In particular, we have

Proposition 1.1 *If g is a C^1 function in a neighborhood of y_0, then the Cauchy problem*

$$\begin{cases} y' = f(x)g(y) \\ y(x_0) = y_0 \end{cases} \tag{1.9}$$

has a unique solution y defined at least in some neighborhood of x_0.

(i) If $g(y_0) = 0$, the unique solution is the constant function $y(x) = y_0$.

(ii) If $g(y_0) \neq 0$, then the unique solution is given by the implicit relation

$$\int_{y_0}^{y} \frac{1}{g(y)} dy = \int_{x_0}^{x} f(s) ds\,. \tag{1.10}$$

The above theorem tells us that, if we start with $y = y_0$ and $g(y_0) = 0$, then the solution will remain constantly equal to y_0. It is natural to interpret those values as equilibrium points for the evolution, and they are often very important in applications. We will discuss this issue in the next paragraph.

Example 1.6 The growth of some species of fishes seems to obey the so-called Von Bertalanffy law. In particular, the lenght y of the fish depends on time t according to the equation

$$y'(t) = k(L - y(t))$$

where k, L are positive constant. The fish will increase its length since the birth, so we understand that $y_0 < L$, where y_0 is the initial value of y. By solving the equation, according to (1.10),

$$\int_{y_0}^{y} \frac{1}{L-y} dy = \int_0^t k\, ds$$

we get

$$-\log|L-y|+\log|L-y_0|=kt \quad \Rightarrow \quad |L-y|=(L-y_0)e^{-kt}$$

Since $y_0 < L$, it is never possible that $y(t) = L$, so $y(t) < L$ for all times. We write then $|L-y| = L-y$ and we finally find

$$y(t) = L-(L-y_0)e^{-kt}$$

which is the unique solution starting from y_0.

From this solution we can interpret L as the maximal length of the fish, an upper bound to be possibly reached only in infinite times. In math terms, L is the horizontal asymptote of $y(t)$ as $t \to \infty$. The constant k is related to the growth rate: here the maximal growth rate (i.e. $\max y'(t)$) is at the initial time and equals to $k(L-y_0)$.

Example 1.7 The height of a tree evolves according to the law

$$y'(t) = \frac{k}{t^2}\, y(t)$$

meaning that the intrinsic growth rate $\frac{y'}{y}$ is decreasing with time. By separating the variables and integrating, as in (1.8), we get

$$\int \frac{1}{y} dy = \int \frac{k}{t^2} dt \quad \Rightarrow \log|y(t)| = -\frac{k}{t} + c$$

Since $y > 0$ we have $|y| = y$, so we obtain

$$y(t) = e^c\, e^{-\frac{k}{t}}\,.$$

Notice that $\lim_{t\to 0^+} y(t) = 0$, regardless of the value of c. This means that, in this model, the initial condition is necessarily $y(0) = 0$; the degree of freedom is instead related to the limiting growth of the height. In fact, we have $\lim_{t\to\infty} y(t) = e^c$; by calling $e^c = L$, this horizontal asymptote for $y(t)$ is, similar to the example before, the upper bound of the growth.

The solution can therefore be written as

$$y(t) = Le^{-\frac{k}{t}}$$

It is a good exercise to compare this growth to the growth in length of the fish presented above. How would you state the differences and similarities of those two growth models ?

Example 1.8 *In a simplified model of spreading of a contagious disease, $y(t)$ denotes the percentage of infected individuals, and $1-y(t)$ the complementary set of non-infected (we used a normalization condition that the totality be equal 1). If r is a coefficient of transmission rate of the disease, the evolution of y is given by*

$$y' = ry(1-y)$$

meaning that the increase of infected individuals is proportional to how many infected meet non-infected individuals.

We can solve this differential equation which is in separation of variables form. From (1.8) we get

$$\int \frac{1}{y(1-y)} dy = \int r\, dt = rt + c$$

To solve the first integral in the right, observe that

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$$

so

$$\int \frac{1}{y(1-y)} dy = \int \frac{1}{y}\, dy + \int \frac{1}{1-y} dy = \log|y| - \log|1-y| = \log\left|\frac{y}{1-y}\right|$$

Therefore we get

$$\log\left|\frac{y}{1-y}\right| = rt + c \quad \Rightarrow \quad \left|\frac{y}{1-y}\right| = e^{rt}\, e^{c}$$

By setting $y(0) = y_0$, we identify the constant e^c since when $t = 0$ we have $e^c = \left|\frac{y_0}{1-y_0}\right|$. Moreover, we are here considering values $y \in (0,1)$, so we can drop the absolute value since $\left|\frac{y}{1-y}\right| = \frac{y}{1-y}$. We deduce

$$\frac{y}{1-y} = \frac{y_0}{1-y_0} e^{rt}$$

and finally we find the solution

$$y(t) = \frac{y_0}{y_0 + (1-y_0)e^{-rt}}$$

We notice that, in this model, the disease will eventually spread over the totality of individuals if $t \to \infty$. Namely, we have $\lim_{t\to\infty} y(t) = 1$.

Of course, the velocity of spreading depends on the transmission rate r. Can you find how much time is needed, since the initial moment, to double the infected population?

The method of integration used in the previous example is quite important. This method is part of a general rule to compute the integral of a rational function $\frac{P(y)}{Q(y)}$, where P and Q are polynomials. Since the general rule is too complicated for the purposes of this course, let us analyze at least one more example of this method similar to what we did before. Assume we wish to compute

$$\int \frac{3y-1}{y^2-y-2}dy$$

The strategy consists in

(i) a factorization of the denominator as product of simple monomials. In this example, by finding the roots of the parabola, we know that

$$y^2 - y - 2 = (y-2)(y+1)$$

(ii) rewriting the ratio as sum of simpler fractions, each with a monomial as denominator.

In this example, since $\frac{3y-1}{y^2-y-2} = \frac{3y-1}{(y-2)(y+1)}$, we wish now to read this ratio as a sum of simpler fractions. Namely, we hope that

$$\frac{3y-1}{y^2-y-2} = \frac{3y-1}{(y-2)(y+1)} = \frac{A}{y-2} + \frac{B}{y+1}$$

for some A and B.

Of course, it may not be immediate to find which A, B are suitable. So we first compute the sum in abstract terms, and then simply *impose* that the right and left sides be equal. So first we compute

$$\frac{3y-1}{y^2-y-2} = \frac{Ay + A + By - 2B}{(y-2)(y+1)}$$

and then we give the needed conditions for the numerators to be equal. By the principle of identities of polynomials, this means that

$$3y = (A+B)y \qquad \text{and} \qquad -1 = A - 2B$$

The result is a system of conditions

$$\begin{cases} A + B = 3 \\ A - 2B = -1 \end{cases}$$

which we can solve to find A and B. Here we get $A = \frac{5}{3}$ and $B = \frac{4}{3}$. Therefore, we found that

$$\frac{3y-1}{y^2-y-2} = \frac{\frac{5}{3}}{y-2} + \frac{\frac{4}{3}}{y+1}$$

(iii) Now we can integrate easily

$$\int \frac{3y-1}{y^2-y-2}\,dy = \int \frac{\frac{5}{3}}{y-2}dy + \int \frac{\frac{4}{3}}{y+1}dy = \frac{5}{3}\log|y-2| + \frac{4}{3}\log|y+1| + c$$

Ex: The logistic growth equation

$$y' = ry\left(1 - \frac{y}{K}\right)$$

can be reduced to the equation studied in Example 1.8 by rescaling the unknown y. In fact, the function $\tilde{y} = \frac{y}{K}$ satisfies

$$\tilde{y}' = r\tilde{y}(1-\tilde{y})$$

Using the solution of Example 1.8, compute $\tilde{y}$ and find the solution y of the logistic growth equation.

Example 1.9 *The logistic growth equation is just one model of density dependent intrinsic growth rate $\frac{y'}{y}$. In the logistic model, the ratio $\frac{y'}{y}$ depends on y, and more precisely it is decreasing with respect to y (the simpler case is therefore a decreasing linear function).*

In recent times, over-population phenomena led to models with a possibly increasing density dependent intrinsic growth. An example is when the intrinsic growth is proportional to a positive power of y. This means that $\frac{y'}{y} \propto y^\alpha$, i.e. there exists $k > 0$:

$$y' = ky^{1+\alpha}$$

for some $\alpha > 0$.

By solving this equation, show that, whatever $\alpha > 0$ you take, this growth is not sustainable for all times in the future.

Let us give the solution of this exercise. Integrating the equation leads to

$$\int y^{-1-\alpha} dy = k \int dt = kt + c$$

hence

$$-\frac{y^{-\alpha}}{\alpha} = kt + c\,.$$

When $t = 0$ we find $c = -\frac{1}{\alpha} y_0^{-\alpha}$, where y_0 is the initial value for y. Then we obtain

$$\frac{y^{-\alpha}}{\alpha} = -kt + \frac{1}{\alpha} y_0^{-\alpha}$$

and so we find y as

$$y^{\alpha} = \frac{1}{y_0^{-\alpha} - \alpha kt}\,.$$

We realize from this formula that the solution exists only up $t_0 = \frac{y_0^{-\alpha}}{\alpha k}$ since $\lim_{t \to t_0^-} y(t) = +\infty$. We interpret this fact by saying that the model is sustainable only in a finite maximal time in the future; the solution y exists only for $t < t_0$ and given by $y = \left(\frac{1}{y_0^{-\alpha} - \alpha kt}\right)^{\frac{1}{\alpha}}$.

1.4 Stability of equilibrium points.

As we saw in the last paragraph, when studying the equation

$$y' = g(y) \tag{1.11}$$

a special role is played by the zeroes of the function g. Any y^* such that $g(y^*) = 0$ is an equilibrium point for this equation, since a possible solution in this case is the constant $y(t) = y^*$. By Theorem 1.1, we know that when $g \in C^1$ in a neighborhood of y^*, the unique solution starting from y^* would be the constant one. In this case, even starting from a different initial value, it is not possible that the solution y crosses the value $y = y^*$ in finite time: this is a consequence of formula (1.10). Indeed, since

$$\int_{y_0}^{y} \frac{1}{g(y)} dy = \int_{x_0}^{x} ds = (x - x_0)$$

if $\lim_{x \to b^-} y(x) = y^*$ for some finite b, we would have

$$\lim_{y \to y^*} \int_{y_0}^{y} \frac{1}{g(y)} dy = (b - x_0) .$$

But the right-hand side would be finite while the left-hand side would be infinite, leading to a contradiction. In fact, since g is C^1 we have $|g(y)| \leq k|y^* - y|$ by Lagrange theorem; suppose for example that $y_0 < y < y^*$, then

$$\begin{aligned} \int_{y_0}^{y} \frac{1}{g(y)} dy &\geq \int_{y_0}^{y} \frac{1}{k(y^* - y)} dy \\ &= \frac{1}{k} \left[\log(y^* - y_0) - \log(y^* - y)\right] \overset{y \to y^*}{\to} +\infty \end{aligned}$$

Therefore, a fundamental remark is the following: *if $g \in C^1$ and y^* is such that $g(y^*) = 0$, then either $y(t) = y^*$ for all times or $y \neq y^*$ for all times.* And this latter case implies, from the intermediate values theorem, that y can never reach in finite time the value y^*.

On account of this remark, we know that when $g \in C^1$ the points of equilibria for the dynamics can only be reached at infinity.

Example 1.10 In Example 1.8, we have $g(y) = ry(1-y)$ and the equilibria are $y = 0$ and $y = 1$. We may notice, from the explicit solution, that $\lim_{t \to +\infty} y(t) = 1$, regardless of the initial value y_0. A significant difference is therefore observed between the two equilibria: here $y = 0$ is *unstable* and $y = 1$ is *stable*, since the dynamics will go far from $y = 0$ and will tend to $y = 1$ whatever choice of initial condition is taken.

Let us now generalize this situation. We say that an equilibrium point y^* is unstable for the dynamics if the solution, even starting close to y^*, tends to move away from this value; we say that y^* is stable if the solution tends to return to y^* as time evolves.

Assume now that the function g has a finite number of zeroes $y_0, y_1, \ldots y_N$. Do we have a method to understand whether one zero is stable or not without computing the solution of the differential equation? The answer is yes, and we only need to look at the function g to understand the stability of an equilibrium. We have indeed the following criterion:

Proposition 1.2 *Assume that $g \in C^1$ and let y^* such that $g(y^*) = 0$. Then we have, with reference to the dynamics (1.11):*

(i) If $g'(y^*) < 0$ then the equilibrium is stable.

(ii) If $g'(y^*) > 0$ then the equilibrium is unstable.

Notice that, in the case of $g'(y^*) < 0$, if the dynamics enters a suitable neighborhood of y^*, then it will converge to this equilibrium with exponential rate. This can be observed by replacing g with its linear approximation. Since

$$g(y) \simeq g(y^*) + g'(y^*)(y - y^*)$$

and since $g(y^*) = 0$, one has

$$y' = g(y) \simeq g'(y^*)(y - y^*)$$

and therefore

$$(y - y^*)' \simeq k(y - y^*) \qquad \text{with } k = g'(y^*)$$

which means that $y - y^*$ behaves like e^{kx}. Since $k < 0$, it will converge to zero exponentially fast.

Example 1.11 *The Solow growth model in economics assumes that the output Q is given in terms of capital and labour as*

$$Q = Q(K, L)$$

and a fraction of the production is invested in capital, so that

$$K'(t) = \kappa Q(K, L)$$

for some constant $\kappa > 0$, while labour grows according to

$$L' = \lambda L$$

for some $\lambda > 0$. If we assume that Q obeys the Cobb Douglas function with constant return to scale: $Q = cK^\alpha L^{1-\alpha}$, with $\alpha \in (0, 1)$, find the differential equation satisfied by the function $y = \frac{K}{L}$ (ratio capital to labor) and the equilibria for its time evolution.

Let us first give a general way to proceed in order to find the equation solved by y. From the chain rule we have

$$y' = \frac{K'}{L} - \frac{K}{L^2}L'$$

and so, using the conditions on K' and L' given by the Solow growth model, we have

$$y' = \kappa\frac{Q}{L} - \lambda\frac{K}{L} = \kappa\frac{Q}{L} - \lambda y$$

Now we observe that

$$\frac{Q}{L} = c(\frac{K}{L})^{\alpha} = cy^{\alpha}$$

and so we obtain the differential equation for y as

$$y' = c\,\kappa y^{\alpha} - \lambda y$$

You may notice that the assumption of *constant return to scale* (for the Cobb-Douglas model) played a crucial role in the above reduction.

Now we are asked to study the equilibria of this autonomous equation, which is in the form

$$y' = g(y)$$

with $g(y) = c\,\kappa y^{\alpha} - \lambda y$. Equilibrium points are given by the solutions of

$$c\,\kappa y^{\alpha} - \lambda y = 0$$

which gives $y_0 = 0$ and $y_1 = \left(\frac{c\kappa}{\lambda}\right)^{\frac{1}{1-\alpha}}$. By computing the derivative of g we find that $g'(0) = +\infty$ and $g'(y_1) < 0$. Therefore, y_0 is an unstable equilibrium while y_1 is a stable equilibrium.

As a conclusion, we found that in the Solow growth model the ratio capital to labor given by

$$\frac{K}{L} = \left(\frac{c\kappa}{\lambda}\right)^{\frac{1}{1-\alpha}}$$

represents a stable attractive equilibrium point for the evolution of this ratio.

We conclude this paragraph with a remark concerning the assumption $g \in C^1$ which we made all along the previous discussion. It is worth pointing out that, if the function g was not C^1, then the Cauchy problem (1.9) may have more than one solution, and new phenomena can happen: for example, it is possible to move instantaneously from an equilibrium, as the following example shows.

Example 1.12 *The equation*

$$\begin{cases} y' = \sqrt[3]{y} \\ y = 0 \end{cases}$$

admits an infinite number of solutions. Indeed, for any $a > 0$*,* $y(x) = \max(0,(x-a))^3$ *is a solution of the above initial value problem.*

1.5 Second order linear equations with constant coefficients

In this section we briefly analyze what happens for second order equations which are linear and with constant coefficients, namely

$$ay''(x) + by'(x) + cy(x) = f(x)\,, \qquad a, b, c \in \mathbb{R}\,,$$

where a, b, c are constants.

For the sake of simplicity, we only consider the case that $f = 0$, namely

$$ay''(x) + by'(x) + cy(x) = 0\,, \qquad a, b, c \in \mathbb{R}\,. \tag{1.12}$$

We first notice the following fundamental facts:

(i) the equation is linear with respect to y; therefore, any linear combination of solutions will remain a solution. In other words: *if* y_1 *and* y_2 *are two solutions of (1.12), it follows that* $y = \alpha y_1 + \beta y_2$ *is also a solution, for any* $\alpha, \beta \in \mathbb{R}$*.*

(ii) The problem

$$\begin{cases} ay''(x) + by'(x) + cy(x) = 0 \\ y(x_0) = y_0 \\ y'(x_0) = \tilde{y}_0 \end{cases}$$

has a unique solution.

As a consequence of (i) and (ii), it can be proved that the set of all solutions to (1.12) is a linear vector space which has dimension two (since any solution is determined up to 2 degrees of freedom). Therefore, if we find two independent solutions y_1 and y_2, then all solutions will be described by linear combinations of those two.

It remains to answer the main practical question: how do we find those two independent solutions which generate all the others? The main hint here is given by the following remark:

$$y(x) = e^{\lambda x} \qquad \Rightarrow \qquad ay''(x) + by'(x) + cy(x) = e^{\lambda x}(a\lambda^2 + b\lambda + c)$$

Therefore, $y(x) = e^{\lambda x}$ *is a solution if and only if* λ *is a zero of the associated parabola* $a\lambda^2 + b\lambda + c$.

In particular, when the parabola has two real roots λ_1 and λ_2, we immediately find two different solutions $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$. When the parabola has just one single root (i.e. $\lambda_1 = \lambda_2$), the two solutions of course coincide and we still need to find one more. But we can observe that, by rules of derivation, we have

$$y(x) = xe^{\lambda x} \quad \Rightarrow \quad ay''(x) + by'(x) + cy(x) = e^{\lambda x}(a\lambda^2 + b\lambda + c) + e^{\lambda x}(2a\lambda + b)$$

If the parabola has only root λ_1, then we know that $\lambda_1 = -\frac{b}{2a}$, therefore we conclude that the function $y(x) = xe^{\lambda_1 x}$ is also a solution in this case. So we found the second solution whenever the parabola had only one root.

Finally, the case that the parabola has no real roots can be dealt with as the first situation provided we use imaginary numbers. In fact, in this case the parabola has two different imaginary roots and since the exponential of imaginary numbers gives the trigonometric functions, one is able to find two different solutions as well.

We summarize the whole stuff in the following result, which gives a general formula for the set of solutions of (1.12).

Theorem 1.3 *Let* $a, b, c \in \mathbb{R}$. *Then we have*

1. *If* $b^2 - 4ac > 0$, *all solutions to (1.12) are given by the formula*

$$y(x) = \alpha e^{\lambda_1 x} + \beta e^{\lambda_2 x}$$

where $\alpha, \beta \in \mathbb{R}$ *and* λ_1, λ_2 *are the two real solutions to* $a\lambda^2 + b\lambda + c = 0$.

2. *If* $b^2 - 4ac = 0$, *all solutions to (1.12) are given by the formula*

$$y(x) = \alpha e^{\lambda_1 x} + \beta x e^{\lambda_1 x}$$

where $\alpha, \beta \in \mathbb{R}$ *and* λ_1 *is the unique solution to* $a\lambda^2 + b\lambda + c = 0$.

3. If $b^2 - 4ac < 0$, all solutions to (1.12) are given by the formula

$$y(x) = \alpha e^{-\frac{b}{2a}x} \cos(\omega x) + \beta e^{-\frac{b}{2a}x} \sin(\omega x)$$

where $\alpha, \beta \in \mathbb{R}$ and $\omega = \frac{\sqrt{|b^2-4ac|}}{2a}$.

The above theorem gives the general solution of (1.12) up to the constants α, β; those can be fixed by two conditions, for instance by prescribing the initial value for y and y'.

Example 1.13 Find the solution to the Cauchy problem

$$\begin{cases} y'' + 2y' + 2y = 0 \\ y(0) = \frac{1}{2} \\ y'(0) = 1 \end{cases}$$

We start by finding all solutions to the equation. Since $\lambda^2 + 2\lambda + 2 = 0$ has no real roots and $\lambda = -1 \pm \sqrt{-1}$ are the (imaginary) roots, we are in case 3 of the above theorem. Therefore all solutions are described by

$$y = \alpha e^{-x} \cos x + \beta e^{-x} \sin x\,.$$

Imposing the initial conditions given we have

$$y(0) = \alpha = \frac{1}{2}\,, \qquad y'(0) = -\alpha + \beta = 1$$

so $\alpha = \frac{1}{2}$ and $\beta = \frac{3}{2}$. The unique solution is therefore

$$y = \frac{1}{2}e^{-x} \cos x + \frac{3}{2}e^{-x} \sin x\,.$$

Thanks to the above theorem, we know the qualitative behavior of the general solution of a linear equation like (1.12). A typical application is the study of the motion of a spring.

Example 1.14 *With reference to Example 1.4, let us study the equation*

$$my''(t) + \lambda y'(t) + ky(t) = 0$$

We first notice that whenever $\lambda^2 \geq 4km$, solutions behave like exponentials, since we are in case 1 and 2 of Theorem 1.3. This means that no oscillations *are observed, and we can interpret this as a consequence of either too much resistance in the air (i.e. λ too big) or a too small combination of mass and elasticity, which prevents oscillations to happen. Of course, this case is only possible if some friction force is present.*

Therefore, we conclude that oscillatory solutions exist if and only if

$$\lambda^2 - 4km < 0\,.$$

In this case, the solutions are given by the formula

$$y = e^{-\frac{\lambda}{2m}t}\left[\alpha\cos(\omega t) + \beta\sin(\omega t)\right]\,, \qquad \omega = \sqrt{\frac{4km-\lambda^2}{4m^2}}$$

which can be rewritten as

$$y(t) = e^{-\frac{\lambda}{2m}t}\sqrt{\alpha^2+\beta^2}\cos(\omega t + \phi)$$

for some phase angle ϕ.

We recognize that this function describes an oscillatory motion with period $T = \frac{2\pi}{\omega}$ multiplied by a (varying in time) amplitude $e^{-\frac{b}{2a}t}\sqrt{\alpha^2+\beta^2}$. Notice that, as $t \to \infty$, the amplitude of oscillation will decrease to zero (an effect of air friction). The frequency of oscillation is

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{4km-\lambda^2}{4m^2}}\,.$$

We recognize that the frequency decreases as the friction coefficient λ increases, and the maximal frequency is attained in absence of air resistance, i.e. when $\lambda = 0$. In this case, the frequency is $\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ (the intrinsic frequency of the spring) and the motion is globally periodic with period $T = 2\pi\sqrt{\frac{m}{k}}$.

1.6 Exercises

1. Solve the following Cauchy problems, where $y = y(x)$:

$$\begin{cases} y' = \frac{x}{1+x} \\ y(1) = 2 \end{cases} \qquad \begin{cases} y' = \frac{\ln x}{x} \\ y(1) = -1 \end{cases} \qquad \begin{cases} y' = 1 - y \\ y(1) = 0 \end{cases}$$

$$\begin{cases} y' = 2y^3 \\ y(0) = 1 \end{cases} \qquad \begin{cases} y' = \frac{y}{x+1} \\ y(1) = 1 \end{cases} \qquad \begin{cases} y' + 2y = x \\ y(0) = 1 \end{cases}$$

$$\begin{cases} y' = \frac{y\ln y}{2x} \\ y(1) = 2 \end{cases} \qquad \begin{cases} y' = y(2-y) \\ y(0) = 1 \end{cases} \qquad \begin{cases} y' - xy = 2x \\ y(0) = 1 \end{cases}$$

2. Find the general solution of the following differential equations:

$$y'' - y = 0 \qquad\qquad y'' + y' - 2y = 0$$

$$y'' + 4y = 0 \qquad\qquad 2y'' + 6y' + 4y = 0$$

$$y'' - y' = 0 \qquad\qquad y'' - 2y' + 5y = 0$$

$$y'' + 2y' + y = 0 \qquad\qquad 3y'' + \frac{1}{3}y = 0$$

3. A hot engine is cooling according to the law

$$T'(t) = -k(T(t) - T_e)$$

where the external temperature is $T_e = 2$ degrees. If the engine is 120 degrees at initial time and 70 degrees after 3 minutes, find the value of the proportionality constant k in the law. Then find out how much time is needed for the engine to be at the external temperature up to 0.1 degrees.

4. A population is growing according to the logistic growth with initial intrinsic rate $r = 0.4$. If the initial population is 10^6, find the time when the population will be half of the carrying capacity.

5. A disease is spreading according to the law

$$y' = ry(1 - y)$$

where $y \in (0, 1)$ denotes the fraction of population which is infected and time is measured in days. If initially we have $y = 0.1$, find what is the least transmission rate r such that half of population would be infected after 1 month.

6. A population y is growing according to the law

$$y' = ky^2$$

for some constant $k > 0$. Knowing that the population is 2 millions at the initial time and 4 millions after 1 year, find the maximal sustainable time for this population (i.e. the blow-up time for the amount of population)

7. A population is growing with a density dependent per capita growth rate
$$\frac{y'}{y} = k(y)$$
where k is a positive function defined in $(0, \infty)$. Show that the growth is sustainable for all times in the future if and only if the function k satisfies
$$\int_1^\infty \frac{1}{yk(y)} dy = \infty$$
Deduce that there exists models of sustainable growth with an ever increasing per capita growth rate: exhibit examples.

8. A population y grows according to a logistic equation perturbed by an additional term due to effects of predation:
$$y' = y(1 - \frac{y}{5}) - \frac{ky}{1+y}$$
Find all possible equilibrium points and discuss their stability. Try to give interpretation in terms of the model.

9. Consider the Solow growth model (Example 1.11) with $\lambda = 1$, $c = 1$ and $\alpha = \frac{2}{3}$, then
$$y' = \kappa y^{\frac{2}{3}} - y$$
where $y = \frac{K}{L}$ is the ratio capital to labour. Solve the equation through the following recipe:

(i) Set $z = \sqrt[3]{y}$, compute z' with the chain rule and find the equation satisfied by z.

(ii) Solve the equation of z and deduce the expression for $y = z^3$.