CALCULUS
DIFFERENTIAL EQUATIONS, ADDITIONAL EXERCISES NO.8

1. Equilibria and stability

(1) Find the equilibria and their stability for the following differential equations:

$$y'(x) = y(x)(1 + y(x) - 22), \quad y(x_0) = y_0 \tag{1.1}$$

$$y'(x) = y(x)(1 - (y(x) - 17)), \quad y(x_0) = y_0 \tag{1.2}$$

$$y'(x) = y(x)(1 + y(x) - 98), \quad y(x_0) = y_0 \tag{1.3}$$

(2) Find the values of the constants a, b such that $y = 114$ becomes a stable equilibrium of the following differential equation:

$$y'(x) = y(x)(1 + a(y(x) - b)), \quad y(x_0) = y_0 \tag{1.4}$$

(3) i) Find the solution (with parameters a, b) of the previous differential equation for initial condition $y(0) = 114$; ii) calculate the solution for the parameters a, b found in the previous example and iii) check that $y(x) = 114$ for all times x.

2. Exercise questions

(1) A disease is spreading according to the law:

$$y'(t) = ry(t)(1 - y(t)) \tag{2.1}$$

where $y \in (0, 1)$ denotes the fraction of the population which is infected and time t is measured in days. If initially we have $y(0) = 0.1$, find what is the least transmission rate r such that half of the population would be infected after 1 month.

(2) Consider the Solow growth model

$$y' = \kappa y^{\frac{2}{3}} - y \tag{2.2}$$

where $y = K/L$ is the ratio capital to labour. Solve the equation through the following recipe: i) set $z = y^{\frac{1}{3}}$ and compute z' using the chain rule and find the equation that satisfies z. ii) Solve the equation for z and deduce the expression for $y = z^3$; iii) provide the solution in terms of $y = y(t)$ for $y(0) = 1/2$.