8 - PUBLIC PRODUCTION GOODS AND EXCESS TAX BURDEN: THE MCGUIRE-OLSON MIXED ECONOMY

APPENDIX 8.A - PROOF THAT BOTH LINDAHL COST SHARES AND GROSS INCOME SHARES ADD TO UNITY

Proof that both Lindahl cost shares and income shares add to unity

Concerning eq (8.30) we show, using the simple geometry of **F8.3**, that eqs (8.32+8.33) have a unique solution $\sum_i s_i^* = \sum_i \sigma_i^* = 1$.

Rewrite eq (8.32) putting $\sum_i s_i = x, \sum_i \sigma_i = y$

8.35 $$h'(G^*)x - \widehat{Q}'(G^*)y = h'(G^*) - \widehat{Q}'(G^*)$$

Its solution set in the space (x, y) is the *blue straight line*

8.36 $$y = \frac{\widehat{Q}'(G^*) - h'(G^*)}{\widehat{Q}'(G^*)} + \frac{h'(G^*)}{\widehat{Q}'(G^*)}x$$

with *negative intercept* $\frac{\widehat{Q}'(G^*) - h'(G^*)}{\widehat{Q}'(G^*)} < 0$ on the vertical axis. The negative value of the intercept can be derived from **F8.1**, where we can see that $h'(\bullet) = \widehat{Q}'(\bullet)$ at point $\underset{\max C}{G}$, while $h'(\bullet) > \widehat{Q}'(\bullet)\ \forall\ G > \underset{\max C}{G}$, including G^*.

Next rewrite in the same way eq (8.33)

8.37 $$h(G^*)x - \widehat{Q}(G^*)y = h(G^*) - \widehat{Q}(G^*)$$

Its solution set in the (x, y) space is the *red straight line*

8.38 $$y = \frac{\widehat{Q}(G^*) - h(G^*)}{\widehat{Q}(G^*)} + \frac{h(G^*)}{\widehat{Q}(G^*)}x$$

with *positive intercept* $\frac{\widehat{Q}(G^*) - h(G^*)}{\widehat{Q}(G^*)} > 0$ on the vertical axis. The positive value < 1 of the intercept $\forall\ G > 0$ can again be seen in **F8.1**.

Here in **F8.3** we can then see that there is only one pair $(x = y)$ lying on both straight lines, and this is the pair $(x = y = 1)$. *Formally,* $\forall\ b \neq 0\ [x = y \Leftrightarrow x = y = 1]$ because

8.39 $$\begin{aligned} & ax - by = a - b \\ & \rightarrow y = \frac{b-a}{b} + \frac{a}{b}x \\ & x = y \rightarrow x = \frac{b-a}{b} + \frac{a}{b}x \\ & \rightarrow x = 1 \\ & x = 1 \rightarrow y = 1 \end{aligned}$$

Thus, as shown in the Figure, the *2*-eqs, *2*-variables system (8.35+8.37) has a *unique solution*

8.40 $$x^* = y^* = 1\ \blacksquare.$$