8 - Public production goods and excess tax burden: the McGuire-Olson mixed economy

1 - Introduction

In Chapter 7 (*Public consumption goods: the Samuelson-Lindahl mixed economy*) we discuss the *Samulelson-Lindahl mixed economy* of public goods, SLE for brevity, in which the public good G has the role of a **pure consumption good**, though non-rival and non-excludable in its physical nature. It enters individual *preferences*, and therefore also the *social welfare function*, but **not the production process** for producing output Q. The SLE is still the standard economy for explaining the conditions for **the efficient resource allocation between private and public uses**. However, while it is sufficient for deriving basic results concerning such conditions, its usefulness is significantly diminished by **two unrealistic limitations**.

1) While in the case of rival (private) goods the separation between their use for consumption or production purposes may be relatively straightforward (though with exceptions, especially in the area of health and education), the same cannot be said for non-rival (public) goods, consisting primarily of internal and external security, jurisdiction and law enforcement, protection of the environment, infrastructural facilities and services, transportation and information networks, public health and public education, etc. At our present level of aggregation/approximation the simplest assumption to be made about G is that it acts *simultaneously* as a **consumption good** and a **productive input**. Therefore we shall assume here that output Q is an *increasing function of* G. The analysis is based on the stylized economy introduced by **McGuire** & **Olson (1996)**, with some refinements and extensions. We shall denote our slightly modified version the **McGuire-Olson mixed economy**, MOE for brevity. Our revision concerns both its interpretation and its analytics. For the purpose of *their* investigation (the *positive economics of rent-seeking ruling interests*) McGuire-Olson assume G to have *only a production role.* For our present *normative* purpose we assume G to have *simultaneously a production and a consumption role.*

2) Both the MOE and the SLE are *static one-period economies.* If used for *normative* purposes they must *incorporate by construction* the balanced budget constraint, BB for brevity, which raises the problem of the excess burden of taxation, EB for brevity. For maximum simplicity the standard version of the SLE tacitly assumes **no** EB, but this (hidden) assumption makes it *more unrealistic than it is apparent at first sight.* Formally the assumption of a **no** EB tax system requires, either

1) only *lump-sum taxes* fully disconnected from all types of economic activities, or

2) taxes defined not through *tax rates and tax bases*, but through *individual shares* in the cost of G, *charged directly* as such on taxpayers.

But in real economies almost no taxes are like that. Most of them are defined through tax rates and tax bases, and both theory and evidence indicate that these are distortionary.

A note on excess burden, short&medium run interaction between aggregate demand and supply, burden of high tax pressure in general

12/20: aggiungere nota su

1) teoria convenzionale dell'EB,

2) no ADxAS under full employment, ma spiegare se e in che senso un'alta pressione tributaria possa comunque abbassare la produttività dell'economia, anche in assenza di EB - 12/20

2 - The structure of the MOE economy

The structure of production is represented here graphically in F8.2, which repproduces - with minor adaptations - the corresponding F3.5 of Chapter 3 (on *Public rent*). We repeat here the two eqs (**3.9-10**) of that Chapter, dropping the subscript B appearing there because in this *normative* Chapter the BB condition is contained by construction in the one-period MOE.

The *first eq* defines the BB condition and is shown in F8.2bottom

$$\begin{aligned} T(G,\tau) &= \tau\eta(\tau)X(G) = h(G) \\ &\rightarrow G(\tau),\ \tau \in [0,1] \\ &\rightarrow \tau(G),\ G \in [0,\bar{G}] \end{aligned} \tag{8.1}$$

The *second eq* defines the MOE *consumption frontier*, CFR-MOE for brevity, drawn as the *thick blue curve* in F8.2top

$$\begin{aligned} Q\left(G,\tau(G)\right) &= \eta\left(\tau(G)\right)X(G) = \hat{Q}\left(G\right) \\ &= C + h(G) \\ &\rightarrow C = \eta(\tau(G))X(G) - h(G) \\ &= Q(G,\tau(G)) - h(G) = \hat{Q}(G) - h(G) \\ \hat{C}(G) &= \hat{Q}(G) - h(G) \end{aligned} \tag{8.2}$$

Here the list of notations:

$T(G,\tau)$ is total tax revenues,

τ is the everage tax burden

$\eta(\tau)$ is the *macro coefficient of excess tax burden*, CEB for brevity. It is a decreasing function of the everage tax burden τ, ranging over $[1,0]$ as τ ranges over $[0,1]$, with shapes like those drawn in F8.1. The % $(1-\eta(\tau))$ represents the % reduction of potential output caused by the excess burden of the tax system, EB for brevity.

$G(\tau)$ is the implicit function $\tau \rightarrow G$ generated by the BB condition (**8.1**). The condition implies that when τ increases beyond a critical level $\bar{\tau}$, tax revenues begin to decrease instead of increasing, because the negative impact of τ

on the tax base $Q(\cdot)$ outweighs its positive impact on tax revenues. It follows that under the BB requirement there is a maximum level $\bar{G} < G\max$ that the government can produce. This maximal $\bar{G}$ is shown in the Figure. Since the upper (decreasing) part of $G(\tau)$ is irrelevant we restrict it to its lower (rising) part. Inverting this lower part yields $\tau(G)$, the implicit function $G \to \tau$ used in MOE's eqs.

$X(G)$, drawn in *black*, is total (full employment=potential) output as an increasing function of G. This function is a purely 'technological' relationship, free of any distortions that might decrease the economy's productivity,

$h(G)$, drawn in *red*, is the opportunity cost of G,

$Q(G, \tau(G))$, drawn in *thick black*, is the *output function*, OF for brevity, divided into the 'commercial' part C and the 'non-commercial' part $h(G)$ measured as $G's$ opportunity cost. It is positive in G and negative in τ,

$C(G, \tau(G))$, drawn in *thick blue*, is the *consumption frontier*, CFR for brevity,

Notice that we use of the notation $\hat{f}(x)$ to indicate a *composite* function of x.

The slope of the CFR

The slope of this CFR is obtained by differentiating the *composite* $\hat{C}(G) = \hat{Q}(G) - h(G)$ over G

$$\begin{aligned} \hat{C}'(G) &= \hat{Q}'(G) - h'(G) = (Q'_G(\cdot) + Q'_\tau(\cdot)\tau'(G)) - h'(G) \qquad (8.3) \\ \to \underset{C(G)}{MRT}(G) &= MSC(G) = -\hat{C}'_G = h'(G) - (Q'_G(\cdot) + Q'_\tau(\cdot)\tau'(G)) \end{aligned}$$

where the braketed term $(Q'_G(\cdot) + Q'_\tau(\cdot)\tau'(G))$ is the derivative over G of the composite function $\hat{Q}(G)$. This slope defines the MRT=MSC of producing G, measured in units of the numeraire C, when the marginal increase in G is accompanied by the marginal increase in τ required to keep the budget balanced. It tells how many units of C are lost when the economy produces one additional unit of G while *keeping the budget balanced*, taking into account the fact that while $G's$ impact on productivity is positive $T's$ impact is negative.

Notice in particular that by construction the CFR-MOE reaches $\bar{G}$, the maximum level of G that can be provided under the BB condition, at some point **4** where total output and consumption are not zero: *total output* is the distance **2-$\bar{G}$**, *total taxes* are the distance **5-$\bar{G}$**, and *total consumption* is the distance **2-$\bar{G}$** - **5-$\bar{G}$** = the distance **2-5** .

A note on the spending review

12/20: aggiungere nota su spending review nel MOE

The consumption frontiers of the Samuelson-Lindahl and Atkinson-Stiglitz economies: PFR-SLE and PFR-ASE

F8.2 also shows the consumption frontiers corresponding to the different scenarios. The CFR-SLE, where output Q is a fixed level, unaffected by either G or taxes. For completeness we've also drawn a CFR-ASE, which is the CFR where G remains a pure consumption good of the SLE type, so that output is a simpler function $Q(\tau) = \eta(\tau)X$, for a fixed f.e. output X independent of G but negatively affected by τ, and τ is determined by the BB. We label this CFR-ASE because the case is summarily drawn in **Atkinson & Stiglitz 1980**, Figure 16-3, p 493, with only few words of explanation and a questionable - and unnecessary - use of a 'transformation curve' between a 'private utility' and G.

Notice that even in this case the implicit relation $G(\tau)$ maintains a shape like that shown in **panel B**.

This type of scenario - pure consumption G and distortionary taxes - has been specifically investigated by **Atkinson & Stern** (**1974**) in an economy which differs from the present one in an interesting and important way: a variable labour supply and a *tax on consumption*. Their economy has been reproduced with minor variations in later texts (such as **Atkinson & Stiglitz 1980**, Lecture 16: "Public Goods and Publicly Provided Private Goods", pp. 487-94, **Myles 1995**, Chapter 9: "Public goods", pp. 290-5). In another chapter of these lectures we explain their model using a generalized framework where a *tax on consumption* is placed side by side with a *tax on labour income*, and the meaning of the analytical results is clarified by means of *full graphical illustrations.*

3- The two-subjects geometry of the efficiency conditions.

The 'total' diagram

In the **'total' diagram** of F8.2 we can visualize the new conditions for a Pareto allocation, PA for brevity. Graphically they are obtained in the same way as in the SLE of Chapter 7, but this time the CFR over which the taxpayers' indifference curves must be drawn to find the PAs is the CFR-MOE, which takes into account both the *positive* effects on productivity of increasing G and the *negative* effect on productivity of increasing τ to cover the cost of increasing G.

Notice that in the Figure we are assuming, for visual clarity, the special case of the *maximum* EB, $\eta(\tau) = 1 - \tau$.

As shown in the Figure, and as already known from the SLE, the PAs are obtained graphically in the usual way:

1) start with an arbitrary indifference curve $\mathbf{I}_A$ of **A**,

2) construct the residual consumption frontier, RCFR for brevity, the *small blue bell-shaped curve*, and then

3) attain the highest indifference curve $\mathbf{I}_B$ of **B**, tangent to such *small blue bell-shaped curve*, with the usual result that the *marginal social benefit* or *marginal social rate of substitution* or *marginal social willingness to pay*, MSB=MSRS=MSWP for brevity, equal to vertical sum of the *individual marginal benefits* or *marginal rates of substitution* or *marginal willingness to pay*, MB=MRS=MWP for brevity, equates the MRT of the CFR:

$$\begin{aligned} \underset{c_A(G)}{MSB(G)} = MRS_A(G, c_A) + \underset{c_B(G)}{MRS_B}(G, c_B) &= MSC(G) = \underset{C(G)}{MRT}(G) \quad (8.5) \\ &= h'(G) - \hat{Q}'(G) \end{aligned}$$

One such PA would be at point **1**, with the public good at G^* and private consumption being distributed between **A** and **B** according to points **3** and **6**. Of course, in general, starting with a different indifference curve of **A** the final PA allocation will be different. Notice that the marginal social benefit MSB is the same thing as the *marginal social willingness to pay*, MSWP for brevity.

The 'per-unit' diagram

The same type of information is obtained using the **'per-unit' diagram** of F8.3, which is the MOE counterpart of the standard SLE Figure F7.6 of Chapter 7. Instead of representing *total* variables we represent their *marginal* counterparts (**Layard&Walters** p.39):

$h'(G)$ is the *Marginal production cost* of G**,** MPC for brevity, the increase in budget cost to the government when it increases G by one unit. To simplify the picture we assume here the MPC to be constant instead of rising, drawing it as the *horizontal black line*.

$Q'_G(\cdot) = \eta(\tau(G))X'(G)$ is the *Marginal Product* of G**,** MP for brevity, equal to the *marginal* increase in output $Q(\cdot)$ caused by a *marginal* increase in G, **keeping τ constant**.

$Q'_\tau(\cdot)\tau'(G) = \eta'(\tau(G))\, X(G)\tau'(G)$ is the *Marginal Output Loss***,** MOL for brevity, caused by the *marginal* increase in τ required to keep the budget balanced. The MOL is *by construction a negative value* because $Q'_\tau(\cdot) < 0$ and $\tau'(G) > 0$.

The MSC is the *rising black line*:

$$\begin{aligned} \underset{C(G)}{MRT}(G) &= MSC(G) \\ &= MPC(G) - (MP(G) + MOL(G)) \quad (8.4) \\ &= h'(G) - [Q'_G(\cdot) + Q'_\tau(\cdot)\tau'(G)] \\ &= h'(G) - \hat{Q}'(G) \end{aligned}$$

It is obtained in two steps. First we subtract $Q'_G(\cdot)$ (the *downward* pointing arrow) from the constant MPC line $h'(G)$, obtaining the *green line*. Then we subtract $Q'_\tau(\cdot)\tau'(G)$ (the *upward* pointing arrow, because $Q'_\tau(\cdot)\tau'(G)$ is negative) obtaining the rising black line MSC. The point G_3 is that level of public investment where its positive impact on output is exactly matched by tha negative impact on output cause by the increase in taxes required to keep thye budget balanced. The MSC of G is equal to the MPC of G minus the MP(positive) of G + the MOL(negative) of G.

4 - The general formalism of the efficiency conditions

The extension of the result to n agents is obtained via the *formalization* of the optimization procedure already used for the SLE, by maximizing the standard **additive social welfare function**, ASWF for brevity, over $G, (c_i)$

$$W(u_i(\cdot)) = \sum_i u_i(G, c_i)$$

subject to the new production constraint **(8.2)**, incorporating the $\tau(G)$ function generated by the BB condition **(8.1)**

$$\max_{G, c_i} \sum_i u_i(G, c_i) \text{ subject to } C + h(G) = \hat{Q}(G)$$

The *Lagrangean*

$$L(\cdot) = \sum_i u_i(G, c_i) - \lambda(C + h(G) - \hat{Q}(G))$$

yields the following *first-order conditions*, FOC for brevity

$$L'_G(\cdot) = 0 : \sum_i MU_{iG}(G, c_i) = \lambda(h'_G - \hat{Q}'_G(G)) \quad (8.6)$$
$$L'_{c_i}(\cdot) = 0 : MU_{ic_i}(G, c_i) = \lambda \quad (8.7)$$
$$L'_\lambda(\cdot) = 0 : C + h(G) = \hat{Q}(G)$$

These are $1 + n + 1$ eqs in the $2 + n$ variables $G, (c_i), \lambda$. Under the usual *implicit function conditions* they yield a *unique* solution $G^*, (c_i^*), \lambda^*$, where $G^*, (c_i^*)$ is the *socially optimal* allocation, while λ^* has the usual meaning of Lagrangean multipliers, with which we are not concerned here.

For an extensive treatment of the **implicit function theorem** see **Chiang 2005** Chapters 4,5,8 - in particular pp. 194-204, or my **Chapter MAT 3** on *Static maximization*.

Notice here the peculiarity of eq (8.6) compared to the corresponding eq in the SLE. In the SLE eq, Q disappears because it is fixed, whereas here it doesn't disappear because it changes with G.

To concentrate on **efficiency** we need to **eliminate altogether from the FOC the individual utilities** - and by implication also **social welfare**. To do so we first divide **(8.6)** by **(8.7)**, and then reduce the number of eqs **(8.7)** from n to $n - 1$ by eliminating λ. This yields the following $1 + n$ eqs in the $1 + n$ variables $G, (c_i)$, with the same unique socially optimal allocation $G^*, (c_i^*)$

as before:

$$\sum_i \frac{MU_{iG}(G,c_i)}{MU_{ic_i}(G,c_i)} = \sum_i \underset{c_i(G)}{MRS_i}(G,c_i) = h'(G) - \hat{Q}'(G) \quad (8.8)$$

$$MU_{ic_i}(G,c_i) = MU_{jc_j}(G,c_j), \forall i \neq j \quad (8.9)$$

$$C + h(G) = \hat{Q}(G) \quad (8.10)$$

Then we eliminate the individual marginal utilities, MU for brevity, altogether by eliminating the $n-1$ eqs **(8.9)**, obtaining

$$\sum_i \frac{MU_{iG}(G,c_i)}{MU_{ic_i}(G,c_i)} = \sum_i \underset{c_i(G)}{MRS_i}(G,c_i) = h'(G) - \hat{Q}'(G) \quad (8.11)$$

$$C + h(G) = \hat{Q}(G) \quad (8.12)$$

where **(8.11)** is simply the same as (**8.5**), extended to n agents. Using (**8.4**) it may be rewritten as

$$\sum_i \underset{c_i(G)}{MRS_i}(G,c_i) = MSC(G) = MPC(G) - MNP(G) \quad (8.13)$$
$$= MPC(G) - (MP(G) + MOL(G))$$

Eqs (8.11-12) have a central status in this Chapter. These are 2 eqs in the $1+n$ variables $G, (c_i)$. Under the usual implicit function conditions they yield a *unique* solution only when $n = 1$. When $n > 1$ they yield an infinite set of PAs $G^*, (c_i^*)$ such that

$$\sum_i \underset{c_i(G)}{MRS_i}(G^*, c_i^*) = h'(G^*) - \hat{Q}'(G^*) = MSC(G^*) \quad (8.14)$$

$$= h'(G^*) - Q'_G(\tau(G^*), G^*) - Q'_\tau(\tau(G^*), G^*)\tau'(G^*) \quad (8.15)$$
$$= h'(G^*) - [Q'_G(\tau(G^*), G^*) + Q'_\tau(\tau(G^*), G^*)\tau'(G^*)]$$

$$C^* + h(G^*) = \hat{Q}(G^*) \quad (8.16)$$

whose components are described in detail under eq **(8.3)** above and summarized in eq **(8.4)**. Efficiency requires that the MSB of G (the *vertical sum of the marginal individual benefits*) be equal, not to the MPC of G, but to its MSC, which is equal to MPC minus MNP=MP(positive) + MOL(negative).

Comparing these MOE conditions represented in F8.2 with the corresponding SLE conditions in Chapter 7 (F7.6) we see that

1) if the MSB schedule were the *solid low one* denoted $\left[\text{MSB}_L\right]$ the government should provide an amount/quality of G *higher* than the amount/quality that should be provided on the basis of the sole MPC, as shown by the change (**1,** G_1^*) $\rightarrow$ (**2,** G_2^*), while

2) if the MSB schedule were the *dotted high one* denoted $\left[\text{MSB}_H\right]$ the government should provide an amount/quality of G *lower* than the amount/quality that should be provided on the basis of the sole MPC, as shown by the change (**4,** G_4^*) $\longleftarrow$ (**5,** G_5^*).

An example: improving jurisdiction

By way of example, we consider a possible interpretation of the movement (**1,** G_1^*) $\rightarrow$ (**2**, G_2^*). Suppose G is the current amount/quality of society's *civil and criminal jurisdiction*, and ask the question: how much G should a benevolent government produce? The standard cost-benefit assessment of a particular public expenditure programme consists in confronting the MSB of that programme with its MSC. G's MPC is the horizontal black solid line, while its MSC is the solid black rising line. Now assume the marginal social benefit to be the the lower [MSB$_L$] schedule. If a benevolent government considered only the MPC of this expenditure programme it should provide G up to point (**1**, G_1^*). But if it considered the full MSC of the programme, then it should increase the provision of G up to point (**2**, G_2^*).

20.12.21: Aggiungere commento: la MWP per G si riferisce ai cittadini come votanti, non alle imprese come produttori. Se si considerassero anche i produttori l'abbassamento dell'MPC dovuto a $Q'_G(\cdot)$ andrebbe sostuito da un innalzamento della curve [MSB].

5 - The Lindahl existence theorems revisited. Extension of LET I

In Chapter 7 (on the SLE) we prove the *two versions* of the Lindahl existence theorem, LET I and LET II for brevity, within the SLE.

LET I (corresponding to the so-called *1st* TWE) says that in a mixed (private-public) economy for every *given initial* gross income distribution (y_{i0}) there exists a vector of personalized cost shares (s_i) ensuring **Lindahl unanimity** at a PA.

LET II (corresponding to the so-called *2nd* TWE) says that in a mixed economy for every PA there exists a vector of gross incomes (y_i) and personalized cost shares (s_i) which converts it into a **Lindahl unanimity equilibrium**.

Both versions can be extended to the present MOE - but with non-trivial adaptations required by the fact that now the GDP level (output Q) is not fixed, but changes with G and τ. We summarize below the procedure for proving them.

We begin with the extension of LET I. Unlike in the SLE case, where we started with an arbitrary *exogenously determined gross income distribution* of the given $GDP = Q$

$$(y_{i0}),\ \sum_i y_{i0} = Q$$

here, where output $Q(\tau(G), G) = \hat{Q}(G)$ changes with G, we must start with predetermined, exogenously given gross income distribution *shares*,

$$y_{i0} = \sigma_{i0}\hat{Q}(G),\ \sum_i \sigma_{i0} = 1 \tag{8.17}$$

under the assumption that these are generated by the market and that the government takes them as a given 'parameter' of the system. The government enters the picture as the agent responsible for its own, 'constitutionally' unique job: the provision of G and the *coercive* covering of its cost through its *exclusive power to tax*. Assuming the government to be a *benevolent* one, we suppose that what it wants to do is to determine a PA $(G^*, (c_i^*))$, and to distribute its cost among taxpayers - under the BB requirement - in such a way as to obtain their *unanimous consent* on that efficient G level (a perfect application of the *benefit principle,* BP for brevity). Assuming further - for convenience - that this benevolent government has *all necessary information* on the taxpayers' incomes and preferences, we ask: can it do that? In other words: does there exist an n-vector (s_i^*) of *personalized cost shares* in the *production cost* $h(G)$ s.t. 1) their sum *adds to unity*, and 2) it ensures a *Lindahl unanimity equilibrium* at a PA?

It is easy to show that such an n-vector (s_i^*) does indeed exist.

Individual demand eqs

. Here we must *redefine individual demand functions* under the new scenario of an individual gross income that changes with τ and G:

$$\max_{G,c} u(G,c) \text{ subject to } c + sh(G) = \sigma\hat{Q}(G) \tag{8.18}$$

Lagrangean

$$L(\cdot) = u(G,c) - \lambda(c + sh(G) - \sigma\hat{Q}(G)) \tag{8.19}$$

FOC

$$L'_G(\cdot) = 0 : MU_G(G,c) = \lambda(sh'(G) - \sigma\hat{Q}'_G(G)) \tag{8.20}$$
$$L'_{c_i}(\cdot) = 0 : MU_c(G,c_i) = \lambda \tag{8.21}$$
$$L'_\lambda(\cdot) = 0 : c + sh(G) = \sigma\hat{Q}(G) \tag{8.22}$$

Eliminating λ yields the *standard individual demand* eqs

$$\underset{c(G)}{MRS}(G,c) = sh'(G) - \sigma\hat{Q}'(G) \tag{8.23}$$
$$c = \sigma\hat{Q}(G) - sh(G) \tag{8.24}$$

This is a system of 2 eqs in 2 variables G, c and 2 parameters s, σ. It therefore yields the individual demands for G and c as functions of s and σ.

$$\begin{array}{l} G^D(s,\sigma) \\ c^D(s,\sigma) \end{array} \tag{8.25}$$

Notice how these eqs **(8.23+8.24)** differ from the corresponding eqs **(7.21+7.22)** of Chapter 7 (on the SLE), where Q is fixed. With Q fixed, individual demands depend on the cost *share* and on individual gross *income*. With $\hat{Q}(G)$ individual demands depend on the cost *share* and on the gross income *share*.

Proof

First we insert the given *gross income distribution shares* **(8.17)** into the individual demand eqs **(8.25)**, which yields **individual demands** as functions of s_i only

$$\begin{array}{l} G_i^D(s_i, \sigma_{i0}) \\ c_i^D(s_i, \sigma_{i0}) \end{array} \tag{8.26}$$

Then we set the **system**

$$\sum_i s_i = 1 \tag{8.27}$$

$$G_i^D(s_i, \sigma_{i0}) = G_j^D(s_j, \sigma_{j0}), \ \forall i \neq j \tag{8.28}$$

(notice that the number of the last eqs is not n but $n-1$) and ask: does there exist a (unique) solution n-vector (s_i^*) s.t. the resulting quantities $(G^*, (c_i^*))$

$$\left\{ \begin{array}{c} G_i^D(s_i^*, \sigma_{i0}) = G^* \\ c_i^D(s_i^*, \sigma_{i0}) = c_i^* \end{array} \right\} \forall i \tag{8.29}$$

satify the PA conditions **(8.11+8.12)**?

1) system **(8.27+8.28)** has n eqs in the n variables (s_i), and under the usual implicit function assumptions it yields a (unique) unanimity-ensuring solution n-vector (s_i^*) of individual cost shares.

2) substituting all these values $((\sigma_{i0}), (s_i^*), G^\circ, (c_i^*))$ into the individual demand eqs **(8.23+8.24)** we see that they must necessarily satisfy the PA conditions **(8.11+8.12)**. Indeed, since $\sum_i s_i^* = \sum_i \sigma_{i0} = 1$, eqs

$$\underset{c_i(G)}{MRS}(G^*, c_i^*) = s_i^* h'(G^*) - \sigma_{i0} \hat{Q}'(G^*) \tag{8.23}$$

imply **(8.14)**

$$\begin{aligned} \sum_i \underset{c_i(G)}{MRS}(G^*, c_i^*) &= \sum_i s_i^* h'(G^*) - \sum_i \sigma_{i0} \hat{Q}'(G^*) \\ &= h'(G^*) - \hat{Q}'(G^*) \end{aligned} \tag{8.14}$$

and eqs

$$c_i^* + s_i^* h(G^\circ) = \sigma_{i0} \hat{Q}(G^*) \tag{8.24}$$

imply **(8.16)**

$$\begin{aligned} \sum_i c_i^* + \sum_i s_i^* h(G^*) &= \sum_i \sigma_{i0} \hat{Q}(G^*) \rightarrow \\ C^* + h(G^*) &= \hat{Q}(G^*) \end{aligned} \tag{8.16}$$

which completes the proof ■

A note on the tax price in the MOE economy

In Chapter 7 (on the SLE) we give a detailed definition of the concept of **tax price**. In the SLE scenario the definition is straightforward:

The tax price p of a taxpayer is the marginal increase in his tax liability following a marginal increase in government spending.

Introducing the BB constraint, a marginal increase in G causes a marginal increase in government spending equal to

$$h'(G)$$

and if the taxpayer's share in government spending is

$$t(G) = sh(G)$$

the taxpayer's tax price is

$$t'(G) = sh'(G) = p$$

But in the MOE scenario the concept becomes slightly more complicated because it splits into *two distinct ones*, that we may label p and P respectively. The tax price p is the same as the former

$$sh'(G) = p$$

while the other tax price follows from the new eqs **(8.18-25)**, in particular **(8.23+8.24)**, that in the MOE define the individual demand functions. In the SLE Chapter the eq defining the tax price is eq **(8.21)**

$$\underset{c(G)}{MRS}(G, c) = sh'(G) = p$$

but here in the MOE this eq becomes

$$\underset{c(G)}{MRS}(G, c) = sh'(G) - \sigma\hat{Q}'(G) = P \tag{8.23}$$

Now, both p and P have a rightful claim to be called tax prices. p may be called the tax price of a marginal increase in G because it is the marginal increase in the taxpayer's *tax liability*. But also P may be called the tax price because it is his actual **net marginal payment** for G, obtained by subtracting from the marginal increase in the taxpayer's tax liability ($sh'(G)$) the (*positive* or *negative*) impact of G on his gross income ($\sigma\hat{Q}'(G)$).

The meaning of the distinction between the two types of tax prices can be visualized in F8.2. Tax price P is the one defining the individual MB schedules, and therefore relevant for *ensuring voting unanimity*. Indeed, the *blue* (taxpayer **A**) and *red* (taxpayer **B**) curves labeled $s_A^* h'(G) - \sigma_{A0}\hat{Q}'(G)$

and $s^*_B h'(G) - \sigma_{B0}\hat{Q}'(G)$ drawn in the Figure, obtained by inserting the pair (s^*_i, σ_{i0}) into **(8.23)**

$$\begin{aligned} \underset{c_i(G)}{MRS}(G, c_i) &= s^*_i h'(G) - \sigma_{i0}\hat{Q}'(G) = P_i \\ i &= A, B \end{aligned} \tag{8.23}$$

are precisely the individual *net shares* in the MSC intersecting the respective MB curves where the desired level of the public good $G = G^*_2$ is the same for both. These are, in the MOE, the optimal *net cost shares* based on the benefit principle BP, and are the precise counterpart of the corresponding eqs of the SLE (Chapter 7, eq **(7.21)** and F7.6)

$$\begin{aligned} \underset{c(G)}{MRS}(G, c_i) &= s^*_i h'(G) = p_i \\ i &= A, B \end{aligned}$$

For completeness we have added the *horizontal straight blue* and *red lines*

$$\begin{aligned} s^*_A h'(G) &= p_A \\ s^*_B h'(G) &= p_B \end{aligned}$$

In the 'economy' of the Figure they have no particular meaning. However there are two things that we can say about them:

1) by the present theorem LET I we know that $s^*_A + s^*_B = 1$, which implies that they must add up to the (horizontal) MPC line,

2) along the vertical in G^*_2 they must lie above points **6** and **9**, respectively, because as long as $G < G_3$ the terms $\sigma_{A0}\hat{Q}'(G)$ and $\sigma_{B0}\hat{Q}'(G)$, respectively, are positive.

6 - Extension of LET II

As before, since output is not fixed, gross income distribution can only be defined in terms of *shares in output*

$$\begin{aligned} y_i &= \sigma_i \hat{Q}(G) \\ \sum\nolimits_i \sigma_i &= 1 \end{aligned}$$

We start from an arbitrary PA $(G^*, (c^*_i))$ satisfying conditions **(8.11+8.12)**, and then ask: does there exist a (unique) $2n$ vector (s^*_i, σ^*_i) of personalized *cost shares* s^*_i in the *production cost* $h(G)$, and individual *gross income shares* σ^*_i in output $\hat{Q}(G)$, s.t. 1) they transform such PA into a *Lindahl unanimity equilibrium*, and 2) they satisfy the *share condition* ?

$$\sum\nolimits_i s^*_i = \sum\nolimits_i \sigma^*_i = 1 \tag{8.30}$$

It is again easy to show that such (unique) $2n$ vector (s^*_i, σ^*_i) does indeed exist.

Proof

The proof consists in simply in *inverting the procedure for deriving the demand functions* **(8.25)**: instead of going from (s_i, σ_i) to (G, c_i) we go from (G, c_i) to (s_i, σ_i). Taking an arbitrary PA $(G^*, (c_i^*))$ from eqs **(8.11+8.12)**

$$\sum_i \underset{c_i(G)}{MRS_i}(G, c_i) = h'(G) - \hat{Q}'(G) \tag{8.11}$$

$$C + h(G) = \hat{Q}(G) \tag{8.12}$$

and inserting it into the $2n$ individual demand eqs **(8.23+8.24)**

$$\underset{c(G)}{MRS}(G, c) = sh'(G) - \sigma\hat{Q}'(G) \tag{8.23}$$

$$c = \sigma\hat{Q}(G) - sh(G) \tag{8.24}$$

we obtain the following system of $2n$ eqs in the $2n$ variables (s_i, σ_i)

$$\begin{aligned} G_i^D(s_i, \sigma_i) &= G^* \\ c_i^D(s_i, \sigma_i) &= c_i^* \end{aligned} \tag{8.31}$$

which, under the usual implicit function assumptions, yields a (unique) solution $2n$-vector (s_i^*, σ_i^*).

To complete the proof we still must show that also **(8.30)** holds. To do so we insert (s_i, σ_i) into **(8.23)** and **(8.11)**, which yields

$$\sum_i s_i h'(G^*) - \sum_i \sigma_i \hat{Q}'(G^*) = \sum_i \underset{c_i(G)}{MRS_i}(G^*, c_i^*) = h'(G^*) - \hat{Q}'(G^*) \tag{8.32}$$

and into **(8.24)** and **(8.12)**, which yields

$$\sum_i s_i h(G^*) - \sum_i \sigma_i \hat{Q}(G^*) = -\sum_i c_i^* = -C^* = h(G^*) - \hat{Q}(G^*) \tag{8.33}$$

Then, in APPENDIX 8.1 we show, using simple geometry, that these *two* eqs in the *two* variables $\sum_i s_i = x, \ \sum_i \sigma_i = y$ have precisely a unique solution $\sum_i s_i^* = x^* = \sum_i \sigma_i^* = y^* = 1$. ■

The difference between the theorems of welfare economics in the commercial economy and the corresponding Lindahl existence theorems in the mixed economy

The reader is reminded of the *fundamental difference* between the TWE in a purely **commercial/private economy**, and the LET in a **mixed private/public economy**, explained in detail in Chapter 7 (*Sections 4 & 5*).

Moreovery, notice that the difference between the two versions of the TWE is **weaker** than the difference between the two versions of the LET because in the

TWE scenario the whole **mechanism is impersonal** while in the public economy scenario it requires an **active government role**:

1) in the LET *version I* we (the government) take the gross income shares as given, and search only for a set of cost shares ensuring unanimity at a PA (G^*, C^*),

2) in the LET *version II* we start from some arbitrary PA and then search for a set of gross income shares in $\hat{Q}(G)$ and cost shares in $h(G)$ ensuring unanimity at that particular PA.

Thus in the LET *version I* the government has - so to speak - an easier and more practical task, because it has a more limited degree of freedom than in the LET *version II.*

7 - A special property of proportional income taxation

In Chapter 7 (Section 3 *Cost shares, tax prices, tax systems*) we discuss how individual cost shares are (in general) not fixed directly, but arise out of the particular tax system embedded in the economy. Following **Stiglitz (2015** pp. 232-4), we have seen that with a simple tax system consisting exclusively of a **general proportional income tax on all individual gross incomes**, the resulting individual cost shares s_i turn out to be the same as the respective *shares of individual gross incomes* into total output/*GDP*. As expected, if we suppose τ to be not only the *average tax pressure* in the economy (resulting from whatever tax system is embedded therein), but exactly the very tax rate of a proportional income tax system, we find the same result also in the present MOE. The *proof* is trivial. From the BB condition

$$\tau(G)\hat{Q}(G) = h(G)$$

we obtain

$$\begin{aligned} s_i h(G) &= \tau(G)\sigma_i\hat{Q}(G) = \sigma_i\tau(G)\hat{Q}(G) = \sigma_i h(G) \\ &\rightarrow s_i = \sigma_i \end{aligned} \tag{8.34}$$

Clearly, with such a tax system it will in general be impossible to obtain the sort of Lindahl unanimity of *Section 5* and F8.2, because the individual cost shares are predetermined by the gross income distribution according to some rule based on the *ability to pay principle* APP principle, and cannot be adjusted to conform to the BP principle. Consider taxpayers **A** and **B**. Assume they have equal gross income shares

$$\sigma_A = \sigma_B = \frac{1}{2}$$

With a proportional income tax system they woul have the same cost share in the MPC

$$s_A = s_B = \frac{1}{2}$$

The resulting individual *net cost share* in the MSC would be the same for both

$$\frac{1}{2}MSC(G) = \frac{1}{2}\left(h'(G) - \hat{Q}'(G)\right)$$

In the Figure this common net cost share is the thin black curve going through points **8** and **7**, corresponding to different levels of G preferred by the two taxpayers. ■

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