Variety Expansion Model

Alessandra Pelloni

Tei

December 2015

Introduction

- Firms do R&D for profit.
- The simplest models of endogenous technological change are those in which R&D expands the variety of inputs or machines used in production (Romer, 1990): *process innovation*.
- Alternative: *product innovation* (Grossman and Helpman (1991a,b)):invention of new goods.

Key Insights

- Innovation as generating new blueprints or *ideas* for production.
- Three important features (Romer):
    1. Ideas and technologies *nonrival*—many firms can benefit from the same idea.
    2. Increasing returns to scale—constant returns to scale to capital, labor, material etc. and then ideas and blueprints are also produced.
    3. Costs of research and development paid as fixed costs upfront.
- We must consider models of *monopolistic competition*, where firms that innovate become monopolists and make profits.

Demographics, Preferences, and Technology

- Infinite-horizon economy, continuous time.
- Representative household with preferences:

$$\int_0^\infty \exp\left(-\rho t\right) \frac{C\left(t\right)^{1-\sigma} - 1}{1-\sigma} dt. \tag{1}$$

- L =total (constant) population of workers. Labor supplied inelastically.
- Representative household owns a balanced portfolio of all the firms in the economy.
  Budget constraint: $\dot{A} = rA + wL - C$

Demographics, Preferences, and Technology I

- Unique consumption good, produced with aggregate production function:

$$Y(t) = A\left[\int_0^{N(t)} x(i,t)^{\alpha} di\right] L^{1-a}, \tag{2}$$

where

  - $N(t)$=number of varieties of inputs (machines) at time t,
  - $x(i,t)$=amount of input (machine) type i used at time t.

- The x's depreciate fully after use.
- For given $N(t)$, which final good producers take as given, (2) exhibits constant returns to scale.
- Final good producers are competitive.

Demographics, Preferences, and Technology II

- The resource constraint of the economy at time t is

$$C(t) + X(t) + Z(t) \leq Y(t), \tag{3}$$

  where $X(t)$ is investment on inputs at time t and $Z(t)$ is expenditure on R&D at time t.

- Once the blueprint of a particular input is invented, the research firm can create one unit of that machine at marginal cost equal to $1 > 0$ units of the final good.

Innovation Possibilities Frontier and Patents I

- *Innovation possibilities frontier*:

$$\dot{N}(t) = Z(t)/\eta, \tag{4}$$

  where $\eta > 0$, and the economy starts with some $N(0) > 0$.
- There is free entry into research: any individual or firm can spend one unit of the final good at time t in order to generate a flow rate $1/\eta$ of the blueprints of new machines.
- The firm that discovers these blueprints receives a *perpetual patent* on this machine.
- A firm that invents a new machine variety i sets a profit-maximizing price of $p(i,t)$ at time t to maximize profits.
- Since machines depreciate after use, $p(i,t)$ can also be interpreted as a "rental price" or the user cost of this machine.

The Final Good Sector

- Maximization by final producers (price of final good numeraire):

$$\max_{[x(i,t),L]} A\left[\int_0^{N(t)} x(i,t)^{\alpha} di\right] L^{1-\alpha} - \int_0^{N(t)} p\,(i\,,t)\,x(i,t)di - w\,(t)\,L. \tag{5}$$

- Demand for machines:

$$x(i,t) = L\left(\frac{A\alpha}{p(i,t)}\right)^{\frac{1}{1-\alpha}}, \tag{6}$$

- Sketch of proof next slide.
- Isoelastic demand for machines.

Apply a variational approach:
Imagine x^* is an optimal "path" (ie for each i $x^*(i)$ is the profit maximizing choice) and consider the arbitrary function η. Then for any real ε we can write:

$$\Pi(\varepsilon) = A\left[\int_0^{N(t)} \left(x^*(i,t) + \varepsilon\eta(i)\right)^{\alpha}\right] L^{1-\alpha} di - \int_0^{N(t)} p\,(i\,,t)\left(x^*(i,t) + \varepsilon\eta(i)\right) di - w\,(t)\,L$$

$$\Pi'(\varepsilon) = A\left[\int_0^{N(t)} \alpha\left(x^*(i,t) + \varepsilon\eta(i)\right)^{\alpha-1}\eta(t)\right] L^{1-\alpha} di - \int_0^{N(t)} p\,(i\,,t)\,\eta(i) di$$

$$\Pi'(0) = A\left[\int_0^{N(t)} \alpha x^*(i,t)^{\alpha-1}\eta(i)\right] L^{1-\alpha} di - \int_0^{N(t)} p\,(i\,,t)\,\eta(i) di = 0.$$

Since x^* is an optimal choice.
For this to be true for any arbitrary η we must have
$\alpha x^*(i,t)^{\alpha-1} L^{1-\alpha} = p\,(i\,,t)$,for all i.

Profit Maximization by Technology Monopolists I

- The monopolist chooses an investment plan starting from time t to maximize the discounted value of profits:

$$V(i,t) = \int_t^\infty \exp\left[-\int_t^s r(v)\,dv\right] \pi(i,s)\,ds \tag{7}$$

where

$$\pi(i,t) \equiv p(i,t)x(i,t) - x(i,t)$$

denotes profits of the monopolist producing intermediate i at time t, $r(t)$ is the market interest rate at time t.

- For future reference, the discounted value of profits can also be written in the alternative Hamilton-Jacobi-Bellman form (Prove this as an exercise):

$$r(t)\,V(i,t) - \dot{V}(i,t) = \pi(i,t). \tag{8}$$

Profit Maximization by Technology Monopolists II

- This equation shows that the discounted value of profits may change because of two reasons:
    1. Profits change over time
    2. The market interest rate changes over time.

Characterization of Equilibrium I

- Since (6) defines isoelastic demands, the solution to the maximization problem of any monopolist i involves setting the same price in every period:

$$p(i,t) = \frac{1}{\alpha} \text{ for all } i \text{ and } t. \tag{9}$$

- Profit-maximization also implies that each monopolist rents out the same quantity of machines in every period, equal to

$$x(i,t) = LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}} \equiv x. \text{ for all } i \text{ and } t. \tag{10}$$

- .Substituting (6) and the machine prices into (2) yields:

$$Y = NLA^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}, \pi = LA^{\frac{1}{1-\alpha}}\alpha^{\frac{2}{1-\alpha}}(\frac{1}{\alpha} - 1). \tag{11}$$

Characterization of Equilibrium II

- Even though the aggregate production function exhibits constant returns to scale from the viewpoint of final good firms (which take $N(t)$ as given), there are *increasing returns to scale* for the entire economy;
- An increase in $N(t)$ raises the productivity of labor and when $N(t)$ increases at a constant rate so will output per capita.
- Equilibrium wages:

$$w = N(1-\alpha)A^{\frac{1}{1-\alpha}}\alpha^{\frac{2\alpha}{1-\alpha}}. \tag{12}$$

- It can be shown that:

$$V(i,t) = \eta \tag{13}$$

where $V(\nu, t)$ is given by (7): this is the consequence of free entry in the research sector.

Characterization of Equilibrium III

- The total expenditure on machines is

$$X(t) = xN(t). \tag{14}$$

Since V is given and profits at each date are given the interest rate is constant (In Acemoglu the proof is assigned as an exercise) :

$$V = \eta = \frac{\pi}{r}. \tag{15}$$

It can then be easily calculated that: r=L$\frac{1}{\eta}A^{\frac{1}{1-\alpha}}\alpha^{\frac{1+\alpha}{1-\alpha}}(1-\alpha)$

- Finally, the representative household's problem is standard and implies the usual Euler equation:

$$\frac{\dot{C}\,(t)}{C\,(t)} = \frac{1}{\sigma}(r(t) - \rho) \tag{16}$$

and the transversality condition

$$\lim_{t\to\infty}\left[\exp\left(-\int_0^t r(s)ds\right)A(t)\right] = 0. \tag{17}$$

- The capital market equilibrium implies: A=VN=ηN

Equilibrium and Balanced Growth Path I

- We can now define an equilibrium as time paths for all the variables such that profit maximizing conditions for firms and optimizing conditions for households are satisfied and all markets clear.
- We define a *balanced growth path (BGP)* as an equilibrium path where variables grow at a constant rate. Such an equilibrium can alternatively be referred to as a "steady state", since it is a steady state in transformed variables.

Balanced Growth Path I

- A balanced growth path (BGP) requires that consumption grows at a constant rate, say g_C. This will always be the case when r is constant though time, from the Euler equation:noticing

$$g_C = \frac{1}{\sigma}(r - \rho). \tag{18}$$

- Note given concavity of period U and dynamic budget constraint this condition, together with the transversality condition, fully characterizes the optimal consumption plans of the consumer.
- Moreover In BGP, consumption grows at the same rate as N: (from $C(t) + X(t) + \eta \dot{N} = Y(t)$ if $\dot{N}/N = g_N$ where g_N is a constant we notice that we can write: $g_N = cons\tan ts - C(t)/\eta N(\mathrm{t})$, since $X(t)/N(\mathrm{t})$ and $Y(t)/N(\mathrm{t})$ have been shown to be constant at all time. But then C and N have to grow at the same rate: $g_C = g_N \equiv g$.)

Finally we have to check these two conditions:

$$r > \rho \text{ and } (1-\sigma)\, r < \rho, \tag{19}$$

the first will ensure that $g > 0$ and the second that the transversality condition is satisfied.

- An important feature of this class models is the presence of the *scale effect*: the larger is L, the greater is the growth rate.
- From a formal point of view it can be shown that there is no transitional dynamics

Pareto Optimal Allocations I

The competitive equilibrium is not necessarily Pareto optimal.

-
  1. There is a markup over the marginal cost of production of inputs.
  2. The number of inputs produced at any point in time may not be optimal.
- The first inefficiency is familiar from models of static monopoly, while the second emerges from the fact that the social surplus from innovations is not fully appropriable.

First of all let us note that the social planner will maximize period net output : $Y_s = AL^{1-\alpha} \int_0^{N_s} X_s(i)^{\alpha} di - \int_0^{N_s} X_s(i) di, X_s(i) =$
$X_s(i) = \left(1/AL^{1-\alpha}\alpha\right)^{1/(\alpha-1)}$
This implies : $X_s(i) = A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} L$ ok and
$Y_s = AL^{1-\alpha} \left(A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} L\right)^{\alpha} N_s - A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} LN_s =$
$\mathrm{L}N_s A^{\frac{1}{1-\alpha}} \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right) LN_s \equiv \mathrm{L}N_s A^{\frac{1}{1-\alpha}} \beta LN_s$
The maximization problem of the social planner can then be written as

$$\max \int_0^{\infty} \frac{C_s^{1-\sigma} e^{-\rho t}}{1-\sigma} dt$$

subject to

$$\dot{N}_s\left(t\right) = \frac{1}{\eta}\left(A^{\frac{1}{1-\alpha}} \beta LN_s - C_s\right)$$

- In this problem, $N_s(t)$ is the state variable, and $C_s(t)$ is the control variable. The current-value Hamiltonian is:

$$\begin{aligned}\hat{H}(N_s, C_s, \mu) &= \frac{C_s(t)^{1-\sigma}}{1-\sigma} \\ &\quad + \mu(t)\frac{1}{\eta}\left(A^{\frac{1}{1-\alpha}}\beta L N - C_s\right).\end{aligned}$$

- The conditions for a candidate Pareto optimal allocation are:

$$\begin{aligned}\eta C_s(t)^{-\sigma} - \mu(t) &= 0 \\ A^{\frac{1}{1-\alpha}}\beta L/\eta &= \rho\mu(t) - \dot{\mu}(t) \\ \lim_{t\to\infty}\left[\exp(-\rho t)\,\mu(t)\,N_s(t)\right] &= 0.\end{aligned}$$

- It can be verified easily that conditions are also sufficient for an optimal solution.

- Combining these conditions:

$$\frac{\dot{C}_s(t)}{C_s(t)} = \frac{\frac{A^{\frac{1}{1-\alpha}}\beta L}{\eta} - \rho}{\sigma}. \tag{20}$$

Comparison of Equilibrium and Pareto Optimum

- The comparison to the growth rate in the decentralized equilibrium, (**??**), boils down to that of

$$\frac{A^{\frac{1}{1-\alpha}}\beta L}{\eta} \text{ to } \frac{(1-\alpha)}{\eta}A^{\frac{1}{1-\alpha}}\alpha^{\frac{1+\alpha}{1-\alpha}}L,$$

- The socially-planned economy will always grow faster than the decentralized economy the former is always greater since $\beta = \alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} = \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha) > (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}$,by virtue of the fact that $\alpha \in (0,1)$.

Comparison

- Why is the equilibrium growing more slowly than the optimum allocation?
- 1)Because the social planner values innovation more
- 2)The social planner is able to use the machines more intensively after innovation.

Policies

- What kind of policies can increase equilibrium growth rate?

1. *Subsidies to Research:* the government can increase the growth rate of the economy, and this can be a Pareto improvement if taxation is not distortionary and there can be appropriate redistribution of resources so that all parties benefit.
2. *Subsidies to Capital Inputs:* inefficiencies also arise from the fact that the decentralized economy is not using as many units of the machines/capital inputs (because of the monopoly markup); so subsidies to capital inputs given to final good producers would also increase the growth rate.

- But note, the same policies can also further distort allocations.