Quantitative Methods III - Practice 5
Introduction to Time Series

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Exercise All of the following questions refer to the time series of closing prices of the SP500 stock index. The data, on a daily basis and referring to the week from 17 October 2022 to 21 October 2022, are shown in the following table:

Date	17/10/22	18/10/22	19/10/22	20/10/22	21/10/22
Quote S&P 500	3.37	3.62	3.62	3.65	3.74

1. Find the first, second and third lag.
2. Calculate the first difference and the second difference.
3. Calculate the daily returns (the growth rate, or percentage change, of y_t represents a return).
4. Calculate the autocorrelation coefficients up to order 4.
5. Draw the correlogram.
6. Estimate the coefficients of an AR(1) model.

Solution

1. Let y_t be the value of the series at the generic instant of time t. The first lag, indicated with y_{t-1} represents the value of the series at the time before t. The second and third lags, denoted by y_{t-2} and y_{t-3}, are defined similarly. The values are shown in the last columns of the following table:

Date	Price S&P500 (y_t)	y_{t-1}	y_{t-2}	y_{t-3}
17/10/22	3.37			
18/10/22	3.62	3.37		
19/10/22	3.62	3.62	3.37	
20/10/22	3.65	3.62	3.62	3.37
21/10/22	3.74	3.65	3.62	3.62

2. The first difference is given by $\Delta y_t = y_t - y_{t-1}$. Therefore, for example, in $t = 18/10/22$, the first difference is obtained by the difference between the value of $y_t = y_{18/10/22} = 3.62$ and the value of the variable the day before, $y_{t-1} = y_{17/10/22} = 3.37$.

   We will have that: $\Delta y_t = \Delta y_{18/10/22} = 3.62 - 3.37 = 0.25$.

   The second difference is given by $\Delta^2 y_t = \Delta y_t - \Delta y_{t-1}$. The first date in which we can calculate it is $t = 19/10/22$, where: $\Delta y_t = \Delta y_{19/10/22} = 0$ and $\Delta y_{t-1} = \Delta y_{18/10/22} = 0.25$. Hence: $\Delta^2 y_t = \Delta y_t - \Delta y_{t-1} = 0 - 0.25 = -0.25$.

   The values of the first and second difference are shown in the following table:

Dates	Price S&P500 (y_t)	Δy_t	$\Delta^2 y_t$	$y_t - y_{t-2}$	$\Delta^2 y_t = y_t - 2y_{t-1} + y_{t-2}$
17/10/22	3.37				
18/10/22	3.62	0.25			
19/10/22	3.62	0	-0.25	0.25	-0.25
20/10/22	3.65	0.03	0.03	0.03	0.03
21/10/22	3.74	0.09	0.06	0.12	0.06

Note that the second difference, $\Delta^2 y_t = \Delta(\Delta y_t) = y_t - 2y_{t-1} + y_{t-2} \neq y_t - y_{t-2}$. The calculations that prove it are shown in the last two columns of the previous table.

3. The daily returns, calculated as a growth rate (or percentage change) can be calculated as:

$$\frac{\Delta y_t}{y_{t-1}}$$

or

$$ln(y_t) - ln(y_{t-1}).$$

As can be seen from the following table, the two definitions lead to similar results.

Date	Price S&P500 y_t	Growth Rate $\frac{\Delta y_t}{y_{t-1}}$	Growth Rate $ln(y_t) - ln(y_{t-1})$
17/10/22	3.37		
18/10/22	3.62	0.074	0.072
19/10/22	3.62	0.000	0.000
20/10/22	3.65	0.008	0.008
21/10/22	3.74	0.025	0.024

4. The autocorrelation coefficient of order j is defined as

$$\rho_j = Corr(y_t; y_{t-j}) = \frac{Cov(y_t; y_{t-j})}{Var(y_t)} \quad j = 1, 2, 3, 4$$

We have to calculate the mean and the variance of y_t.

The mean is:

$$\bar{y} = \frac{1}{T}\sum_{i=1}^{T} y_t = \frac{18}{5} = 3.6.$$

The variance is:

$$Var(y_t) = \frac{1}{T}\sum_{i=1}^{T} y_t^2 - \bar{y}^2 = \frac{64.88}{5} - 3.6^2 = 0.015.$$

The formula to calculate the autocovariance is:

$$Cov(y_t; y_{t-j}) = \frac{1}{T}\sum(y_t - \bar{y})(y_{t-j} - \bar{y}) \quad \text{with} \quad j = 1, 2, 3, 4$$

Date	Price (y_t)	y_t^2	$(y_t-\bar{y})(y_{t-1}-\bar{y})$	$(y_t-\bar{y})(y_{t-2}-\bar{y})$	$(y_t-\bar{y})(y_{t-3}-\bar{y})$	$(y_t-\bar{y})(y_{t-4}-\bar{y})$
17/10	3.37	11.36				
18/10	3.62	13.10	-0.0046			
19/10	3.62	13.10	0.0004	-0.0046		
20/10	3.65	13.32	0.0010	0.0010	-0.0115	
21/10	3.74	13.99	0.0070	0.0028	0.0028	-0.0322
Total	18	64.876	0.0038	-0.0008	-0.0087	-0.0322

As an example, the value in $t = 18/10/2022$ would be:

$$(y_t-\bar{y})(y_{t-1}-\bar{y}) = (3.62-3.6)(3.37-3.6) = 0.02 \times (-0.23) = -0.0046$$

It follows that the first autocovariance will be:

$$Cov(y_t; y_{t-1}) = \frac{1}{T}\sum(y_t-\bar{y})(y_{t-j}-\bar{y}) = \frac{0.0038}{5} = 0.00076$$

.

The other three autocovariances will therefore be:

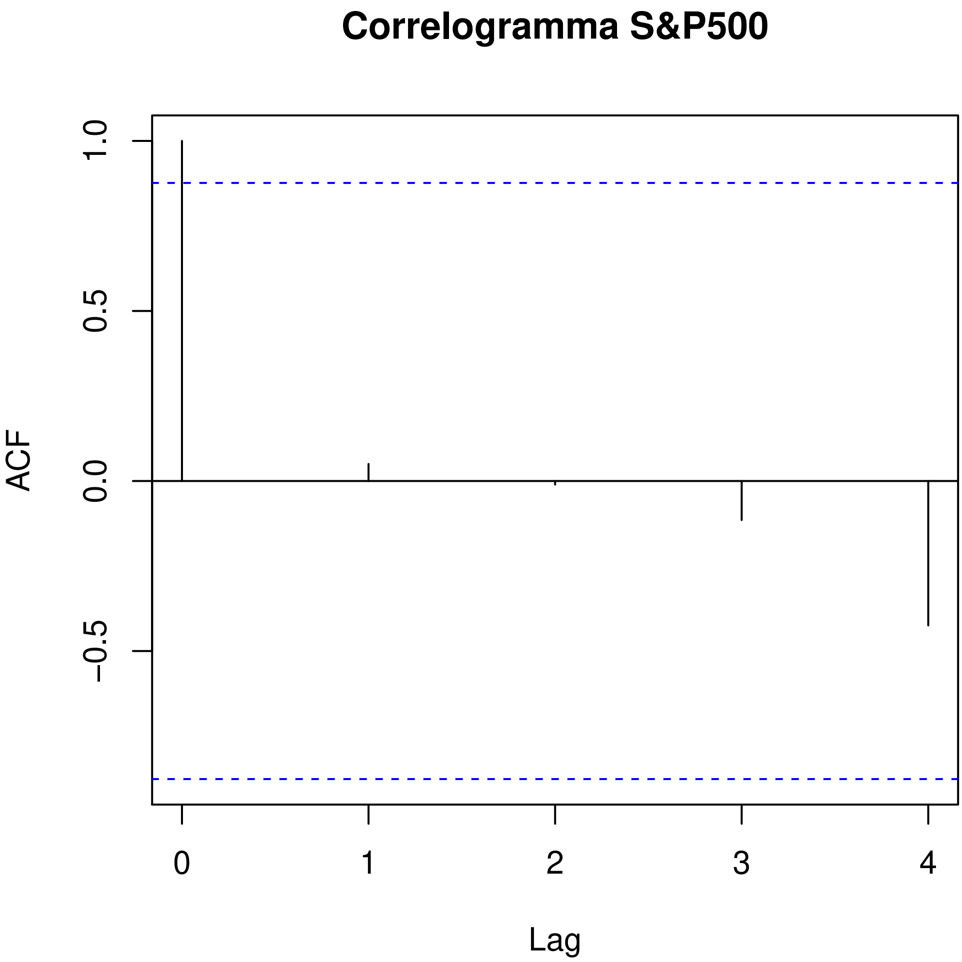
$$Cov(y_t; y_{t-2}) = \frac{-0.0008}{5} = -0.00016$$
$$Cov(y_t; y_{t-3}) = \frac{-0.0087}{5} = -0.00174$$
$$Cov(y_t; y_{t-4}) = \frac{-0.0322}{5} = -0.00644$$

It follows that the autocorrelation coefficients up to order 4 will be:

$$\rho_1 = \frac{Cov(y_t; y_{t-1})}{V(y_t)} = \frac{0.00076}{0.015} = 0.050$$
$$\rho_2 = \frac{Cov(y_t; y_{t-2})}{V(y_t)} = \frac{-0.00016}{0.015} = -0.011$$
$$\rho_3 = \frac{Cov(y_t; y_{t-3})}{V(y_t)} = \frac{-0.00174}{0.015} = -0.115$$
$$\rho_4 = \frac{Cov(y_t; y_{t-4})}{V(y_t)} = \frac{-0.00644}{0.015} = -0.425$$

5. The correlogram is the graphical representation of the autocorrelation structure.

   Using what obtained in the previous point then the correlogram will be:



6. The AR(1) model is:

$$y_t = \beta_0 + \beta_1 y_{t-1} + u_t$$

and the estimates of β_0 and β_1 are obtained using the OLS estimator on the observations relating to the dates in which we have data on both variables involved in the model.

It follows that:

$$\hat{\beta}_1 = \frac{Cov(y_t; y_{t-1})}{V(y_{t-1})} = \frac{\frac{1}{4}52.17 - 3.658 \times 3.565}{\frac{1}{4}50.89 - 3.565^2} = \frac{0.0030}{0.0128} = 0.231$$

and

$$\hat{\beta}_0 = \bar{y}_t - \beta_1 \bar{y_{t-1}} = 3.658 - 0.231 \times 3.565 = 2.834$$

The calculations for the estimates are shown in the table below.

Date	y_t	y_{t-1}	$y_t \times y_{t-1}$	y_{t-1}^2
18/10/22	3.62	3.37	12.20	11.36
19/10/22	3.62	3.62	13.10	13.10
20/10/22	3.65	3.62	13.21	13.10
21/10/22	3.74	3.65	13.65	13.32
Total	14.63	14.26	52.17	50.89
Average	3.658	3.565	13.04	12.72