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UNIVERSITÀ DEGLI STUDI DI ROMA

Quantitative Methods
Time Series, AR and ADL

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Exercise You are given a time series of daily closing prices for the S&P 500 index from October 17, 2022, to October 21, 2022:

Date	S&P 500 Price y_t
17/10/22	3.37
18/10/22	3.62
19/10/22	3.62
20/10/22	3.65
21/10/22	3.74

Tabella 1: S&P 500 Closing Prices

Additionally, the US unemployment rate for the same period is given:

Date	Unemployment Rate x_t
17/10/22	4.0
18/10/22	4.1
19/10/22	4.1
20/10/22	4.2
21/10/22	4.3

Tabella 2: US Unemployment Rate

The AR(1) model that assumes that the current value of y_t depends on its first lag and an error term is equal to:

$$\hat{y}_t = 2.834 + 0.231y_{t-1} \quad (1)$$

An ADL(1,1) model that incorporates an additional predictor, x_{t-1} (unemployment rate), into the AR(1) model is equal to:

$$\hat{y}_t = 2.5 + 0.3y_{t-1} + 0.1x_{t-1} \quad (2)$$

Perform the following tasks:

1. Compute the first and second lag of y_t .
2. Calculate the first and second difference of y_t .
3. Compute the daily returns using both absolute and logarithmic differences.
4. Predict the closing price of the S&P 500 for the next day (October 22, 2022) using both models.
5. Compute a 95% prediction interval using the RMSFE method.
6. Compute the AIC and BIC for both models.

Solutions

1. Compute the First and Second Lag of y_t .

The first and second lags of y_t represent the values of the time series at one and two periods before, respectively:

Date	y_t	y_{t-1}	y_{t-2}
17/10/22	3.37	—	—
18/10/22	3.62	3.37	—
19/10/22	3.62	3.62	3.37
20/10/22	3.65	3.62	3.62
21/10/22	3.74	3.65	3.62

Tabella 3: First and Second Lags of y_t

2. Compute the First and Second Difference of y_t .

The first difference is given by:

$$\Delta y_t = y_t - y_{t-1} \quad (3)$$

The second difference is:

$$\Delta^2 y_t = \Delta y_t - \Delta y_{t-1} \quad (4)$$

Date	y_t	Δy_t	$\Delta^2 y_t$
17/10/22	3.37	—	—
18/10/22	3.62	0.25	—
19/10/22	3.62	0.00	-0.25
20/10/22	3.65	0.03	0.03
21/10/22	3.74	0.09	0.06

Tabella 4: First and Second Differences of y_t

3. Compute the Daily Returns

The absolute return is calculated as:

$$r_t = \frac{\Delta y_t}{y_{t-1}} \quad (5)$$

The logarithmic return is:

$$r_t = \ln(y_t) - \ln(y_{t-1}) \quad (6)$$

Date	y_t	Absolute Return r_t	Log Return r_t
17/10/22	3.37	—	—
18/10/22	3.62	0.074	0.072
19/10/22	3.62	0.000	0.000
20/10/22	3.65	0.008	0.008
21/10/22	3.74	0.025	0.024

Tabella 5: Daily Returns of y_t

4. Predict for October 22, 2022.

Using the AR(1) model:

$$\hat{y}_{22/10/22} = 2.834 + 0.231 \times 3.74 = 3.696 \quad (7)$$

Using the ADL(1,1) model:

$$\hat{y}_{22/10/22} = 2.5 + 0.3 \times 3.74 + 0.1 \times 4.3 = 3.822 \quad (8)$$

5. Compute 95% Prediction Interval.

The 95% prediction interval is computed using the Root Mean Square Forecast Error (RMSFE), which measures the average deviation of the forecasts from the actual values:

$$RM\hat{SFE} = \sqrt{\frac{1}{P} \sum_{i=1}^P \tilde{u}_i^2} \quad (9)$$

where $\tilde{u}_i = y_i - \hat{y}_i$ represents the forecast errors, and P is the number of out-of-sample predictions used.

Using the observed values and the models' out-of-sample predictions:

Date	y_t	\hat{y}_t [of AR(1)]	\tilde{u}_t [of AR(1)]	\hat{y}_t [of ADL(1,1)]	\tilde{u}_t [of ADL(1,1)]
18/10/22	3.62	3.58	0.04	3.60	0.02
19/10/22	3.62	3.61	0.01	3.62	0.00
20/10/22	3.65	3.63	0.02	3.64	0.01
21/10/22	3.74	3.67	0.07	3.70	0.04

Tabella 6: Observed vs. Pred. Values and Forecast Err. for AR(1) and ADL(1,1)

Summing the squared errors:

$$\sum_{i=1}^P \tilde{u}_{i,AR(1)}^2 = 0.04^2 + 0.01^2 + 0.02^2 + 0.07^2 = 0.0066 \quad (10)$$

$$\sum_{i=1}^P \tilde{u}_{i,ADL(1,1)}^2 = 0.02^2 + 0.00^2 + 0.01^2 + 0.04^2 = 0.0021 \quad (11)$$

Since $P = 4$ (four out-of-sample observations), the RMSFE values are:

$$RM\hat{SFE}_{AR(1)} = \sqrt{\frac{0.0066}{4}} = \sqrt{0.00165} \approx 0.041 \quad (12)$$

$$RM\hat{SFE}_{ADL(1,1)} = \sqrt{\frac{0.0021}{4}} = \sqrt{0.000525} \approx 0.023 \quad (13)$$

The 95% confidence interval for a given prediction \hat{y}_{t+1} is given by:

$$\hat{y}_{t+1} \pm z_{1-\alpha/2} \times RM\hat{SFE} \quad (14)$$

where $z_{1-\alpha/2} = 1.96$ for a 95% confidence level.

For the AR(1) model:

$$[3.696 - (1.96 \times 0.041), 3.696 + (1.96 \times 0.041)] \quad (15)$$

$$[3.616, 3.776] \quad (16)$$

For the ADL(1,1) model:

$$[3.822 - (1.96 \times 0.023), 3.822 + (1.96 \times 0.023)] \quad (17)$$

$$[3.777, 3.867] \quad (18)$$

Thus, the corrected 95% prediction intervals are:

- AR(1) model: [3.616, 3.776]
- ADL(1,1) model: [3.777, 3.867]

6. Compute AIC and BIC for Both Models.

The Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) are calculated as follows:

$$AIC = \ln(RSS/N) + \frac{2k}{N} \quad (19)$$

$$BIC = \ln(RSS/N) + \frac{k \ln N}{N} \quad (20)$$

where:

- RSS is the Residual Sum of Squares
- N is the number of observations
- k is the number of estimated parameters (including the intercept)

For the AR(1) model:

$$k = 2, \quad RSS = 0.0066, \quad N = 4 \quad (21)$$

$$AIC_{AR(1)} = \ln(0.0066/4) + \frac{2(2)}{4} = -3.42 \quad (22)$$

$$BIC_{AR(1)} = \ln(0.0066/4) + \frac{2 \ln 4}{4} = -3.02 \quad (23)$$

For the ADL(1,1) model:

$$k = 3, \quad RSS = 0.0021, \quad N = 4 \quad (24)$$

$$AIC_{ADL(1,1)} = \ln(0.0021/4) + \frac{2(3)}{4} = -4.10 \quad (25)$$

$$BIC_{ADL(1,1)} = \ln(0.0021/4) + \frac{3 \ln 4}{4} = -3.50 \quad (26)$$

Thus, the values obtained are:

- AR(1) model: $AIC = -3.42$, $BIC = -3.02$
- ADL(1,1) model: $AIC = -4.10$, $BIC = -3.50$

Since the ADL(1,1) model has the lowest values for both AIC and BIC, it is the preferred model. This suggests that including the unemployment rate x_t improves the predictive accuracy compared to the AR(1) model.