All Lectures

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Definition

*A **proposition** is any claim/assertion that can be either TRUE of FALSE. We indicate propositions with the calligraphic letter $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, ...*

Example

- $\mathcal{R}$ =*"Rome is a nice city". This is not, formally, a proposition, since it attributes a quality that is subjective, so it is not possible to establish UNDOUBTEDLY wether it is TRUE or FALSE (for someone will be true for someone else will be false).*
- $\mathcal{P}$ =*"Rome is the capital of France",* $\mathcal{P}$ *is FALSE.*
- $\mathcal{Q}$ =*"Paris is the capital of France",* $\mathcal{Q}$ *is TRUE.*

Definition

Given a proposition $\mathcal{P}$ we indicate with $\neg\mathcal{P}$ a new proposition which is FALSE if $\mathcal{P}$ is TRUE and vice versa. The proposition $\neg\mathcal{P}$ is called the negation of $\mathcal{P}$.

$\mathcal{P}$	$\neg\mathcal{P}$
T	F
F	T

Example

$$\begin{cases} \mathcal{P} = \text{“}\textit{Rome is the capital of France}\text{”} \\ \neg\mathcal{P} = \text{“}\textit{Rome is not the capital of France}\text{”} \end{cases}$$

Definition

Given two propositions $\mathcal{P}$ and $\mathcal{Q}$, we define a third proposition denoted with

$$\mathcal{P} \wedge \mathcal{Q},$$

which is read "$\mathcal{P}$ and $\mathcal{Q}$" and that is TRUE if and only if both $\mathcal{P}$ and $\mathcal{Q}$ are TRUE, and it is FALSE in all the other cases.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$
T	T	T
T	F	F
F	T	F
F	F	F

Example

$$\begin{cases} \mathcal{P} = \text{“Rome is the capital of France”} \\ \mathcal{Q} = \text{“Paris is the capital of France”} \end{cases} \qquad \mathcal{P} \wedge \mathcal{Q} \text{ is FALSE.}$$

Example

$$\begin{cases} \mathcal{R} = \text{“Rome is the capital of Italy”} \\ \mathcal{S} = \text{“Paris is the capital of France”} \end{cases} \qquad \mathcal{R} \wedge \mathcal{S} \text{ is TRUE.}$$

Definition

Given two propositions $\mathcal{P}$ and $\mathcal{Q}$, we define a third proposition denoted with

$$\mathcal{P} \vee \mathcal{Q},$$

which is read "$\mathcal{P}$ or $\mathcal{Q}$" and that is FALSE if and only if both $\mathcal{P}$ and $\mathcal{Q}$ are FALSE, and it is TRUE in all the other cases.

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \vee \mathcal{Q}$
T	T	T
T	F	T
F	T	T
F	F	F

Example

$$\begin{cases} \mathcal{P} = \text{“Rome is the capital of France”} \\ \mathcal{Q} = \text{“Paris is the capital of France”} \end{cases} \qquad \mathcal{P} \vee \mathcal{Q} \textit{ is TRUE.}$$

Example

$$\begin{cases} \mathcal{R} = \text{“Rome is the capital of France”} \\ \mathcal{S} = \text{“Paris is the capital of Italy”} \end{cases} \qquad \mathcal{R} \vee \mathcal{S} \textit{ is FALSE.}$$

Propositional Calculus: some identities

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$	$\neg(\mathcal{P} \wedge \mathcal{Q})$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{P} \vee \neg\mathcal{Q}$
F	F	F
F	T	T
T	F	T
T	T	T

That is $\neg(\mathcal{P} \wedge \mathcal{Q}) = \neg\mathcal{P} \vee \neg\mathcal{Q}$

Propositional Calculus: some identities

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \vee \mathcal{Q}$	$\neg(\mathcal{P} \vee \mathcal{Q})$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{P} \wedge \neg\mathcal{Q}$
F	F	F
F	T	F
T	F	F
T	T	T

That is $\neg(\mathcal{P} \vee \mathcal{Q}) = \neg\mathcal{P} \wedge \neg\mathcal{Q}$

Definition

Given two propositions $\mathcal{P}$ and $\mathcal{Q}$, we define a third proposition denoted with

$$\mathcal{P} \Rightarrow \mathcal{Q},$$

read as "If $\mathcal{P}$ then $\mathcal{Q}$" or, also,"$\mathcal{P}$ implies $\mathcal{Q}$" and such that

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{Q}$
T	T	T
T	F	F
F	T	T
F	F	T

which means that if we assume a true hypothesis the implication is true if and only if the thesis is also true, while from a false hypothesis we can derive anything (both a true and a false thesis).

Propositional Calculus: logical operators

Example

$$\begin{cases} \mathcal{P} = \text{“Rome is the capital of France”} \\ \mathcal{Q} = \text{“Paris is the capital of France”} \end{cases} \qquad \mathcal{P} \Rightarrow \mathcal{Q} \text{ is TRUE.}$$

Example

$$\begin{cases} \mathcal{R} = \text{“Rome is the capital of France”} \\ \mathcal{S} = \text{“Paris is the capital of Italy”} \end{cases} \qquad \mathcal{R} \Rightarrow \mathcal{S} \text{ is TRUE.}$$

Example

$$\begin{cases} \mathcal{T} = \text{“Rome is the capital of Italy”} \\ \mathcal{U} = \text{“Paris is the capital of Germany”} \end{cases} \qquad \mathcal{T} \Rightarrow \mathcal{U} \text{ is FALSE.}$$

Remember that

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \Rightarrow \mathcal{Q}$
T	T	T
T	F	F
F	T	T
F	F	T

now consider the table of truth of $\neg\mathcal{Q} \Rightarrow \neg\mathcal{P}$

$\neg\mathcal{P}$	$\neg\mathcal{Q}$	$\neg\mathcal{Q} \Rightarrow \neg\mathcal{P}$
F	F	T
F	T	F
T	F	T
T	T	T

that is $(\mathcal{P} \Rightarrow \mathcal{Q}) = (\neg\mathcal{Q} \Rightarrow \neg\mathcal{P})$.

Definition

When a proposition can be cast in the form

$$\mathcal{H} \Rightarrow \mathcal{T}$$

we sometimes refer to $\mathcal{H} \Rightarrow \mathcal{T}$ as a Theorem and the proposition $\mathcal{H}$ is call the hypothesis of the theorem while the proposition $\mathcal{T}$ is called the thesis.

Consider the following statement:

Theorem

Let m and n be two even numbers. Then $n + m$ is an even number.

In this case

$$\begin{cases} \mathcal{H} = \text{“Let } m \text{ and } n \text{ be two even numbers”} \\ \mathcal{T} = \text{“} n + m \text{ is an even number”.} \end{cases}$$

Another example of Theorem could be

Theorem

The number $\sqrt{2}$ *is irrational.*

In this case the theorem is a general statement, which cannot be put in the form $\mathcal{H} \Rightarrow \mathcal{T}$. It can be rephrase for example in

Theorem

It is not possibile to find a rational number q *such that* $q^2 = 2$.

Direct proof is typically used for $\mathcal{H} \Rightarrow \mathcal{T}$.

The method

We assume that the properties stated in the hypothesis $\mathcal{H}$ are true to prove that also $\mathcal{T}$ is true.

Theorem

Let m and n be two even numbers. Then $n + m$ is an even number.

Proof. Since m is even, then $m = 2\,k$ for some k. Since n is even, then $n = 2\,q$ for some q. Then

$$n + m = 2\,k + 2\,q = 2\,(k + q) = 2\,h$$

with $h = k + q$, so also $n + m$ is even, whence the thesis. □

Proving a theorem: proof by contraposition.

Proof by contraposition is typically used for $\mathcal{H} \Rightarrow \mathcal{T}$.

The method

We use the equivalence, in terms of table of truth, given by

$$(\mathcal{H} \Rightarrow \mathcal{T}) = (\neg\mathcal{T} \Rightarrow \neg\mathcal{H})\,.$$

Theorem

If n^2 is even, then n is even.

Proof. Assume that n is odd. Then $n = 2\,k + 1$, which implies

$$n^2 = 4\,k^2 + 1 + 4\,k = 4\,\left(k^2 + k\right) + 1$$

which is odd, so n^2 is odd. □

Proving a theorem: reductio ad absurdum AKA proof by contradiction

We want to prove that a proposition $\mathcal{P}$ is true.

The method

1. We assume that $\mathcal{P}$ is false, so $\neg\mathcal{P}$ is true.
2. We prove that assuming $\neg\mathcal{P}$ to be true implies that a proposition $\mathcal{Q}$ and $\neg\mathcal{Q}$ are simultaneously true.
3. Since $\mathcal{Q}$ and $\neg\mathcal{Q}$ cannot be true simultaneously, then $\mathcal{P}$ must be true.

When you have eliminated the impossible, whatever remains, however improbable, must be the truth.

Sherlock Holmes.

Theorem

It is not possibile to find a rational number q such that $q^2 = 2$

Proof. Direct proof seems quite hard. Proceed by contradiction. Assume, by contradiction, that there exists two integer numbers n and m such that

$$\left(\frac{n}{m}\right)^2 = 2.$$

Since they appear in a fraction, then we can also assume that n and m have no common factors, that is at least one is odd.

(the proof continues in the next slide).

Summing up we have two integers number, n and m, at least one is odd, and such that

$$\left(\frac{n}{m}\right)^2 = 2.$$

then

$$n^2 = 2\,m^2 \quad (\star)$$

and so n^2 is even, but then n is even, that is $n = 2\,k$, whence equation $(\star)$ gives

$$4\,k^2 = 2\,m^2 \Rightarrow 2\,k^2 = m^2,$$

so m^2 is even, but then m is even, a contradiction. □

Sets

Definition

A set is any collection/grouping/list of objects and it is defined once the full list of its constituents/elements is given.

Extensive declaration

When all elements are explicitly listed:

- $A = \{\clubsuit, \spadesuit, \bigstar\}$, $B = \{-20, 5, 13, 56.7\}$.
- $\mathbb{N} = \{0, 1, 2, 3, 4, 5, ...\}$

Very problematic to use when the number of elements is infinite! We all have an intuitive notion of $\mathbb{N}$, but for other sets with an infinite number of elements it could be very problematic to use the extensive declaration.

Intensive declaration

When the rule/property that defines the elements is given

- $A = \{\text{All cities of Europe}\}$.
- $A = \{\text{All numbers between zero and one}\}$

The $\in$ symbol

Write $a \in A$ to say that a is an element of A and $a \notin A$ to say the opposite.

- If $A = \{\clubsuit, \spadesuit, \bigstar\}$ then $\clubsuit \in A$, $\spadesuit \in A$ and $\bigstar \in A$.
- If $A = \{-20, 5, 13, 56.7\}$ then $-20 \in A$, $5 \in A$ and so on...
- If $A = \{\text{All cities of Europe}\}$ then $\text{Paris} \in A$
- If $A = \{\text{All numbers between zero and one}\}$ then $\frac{1}{2} \in A$.

Remark. The $\in$ symbol is frequently used in the intensive notation.

$$A = \{\text{All cities of Europe}\}, \quad B = \{x \in A \mid x \text{ is a capital city}\}.$$

So that $\text{Milan} \in A$ but $\text{Milan} \notin B$.

The $\forall$ symbol

Write $\forall a \in A$ as a shortcut to declare a property that holds

"for **ALL** the elements of the set A".

Examples:

- If $A = \{0, 0.2, 0.5, 0.6, 0.7, 0.9, 1\}$ then we can say that $\forall a \in A$ it holds that $a \geq 0$.
- If $A = \{0, 0.2, 0.5, 0.6, 0.7, 0.9, 1\}$ then we can say that $\forall a \in A$ it holds that $a \leq 1$.

The ∃ symbol

Write $\exists a \in A$ as a shortcut to declare a property that holds

"for **AT LEAST ONE** element of A".

We use ∄ to say the opposite.

Examples:

- If $A = \{0, 0.2, 0.5, 0.6, 0.7, 0.9, 1\}$ then we can say that $\exists a \in A$ such that $0 < a < \frac{1}{2}$.
- If $A = \{0, 0.2, 0.5, 0.6, 0.7, 0.9, 1\}$ then we can say that $\nexists a \in A$ such that $a < 0$.

The $\exists!$ symbol

Write $\exists! a \in A$ as a shortcut to declare a property that holds

"for a **UNIQUE** element of A".

Examples:

- If $A = \{0, 0.2, 0.5, 0.6, 0.7, 0.9, 1\}$ then we can say that $\exists! a \in A$ such that $0 < a < \frac{1}{2}$.

- $a \in A$ reads as "the element a belongs to the set A".
- $\forall a \in A$ reads as "for all the elements of A".
- $\exists a \in A$ reads as "there exists at least one a in A".
- $\nexists a \in A$ reads as "it does not exist an a in A".
- $\exists! a \in A$ reads as "there exists one and only one a in A".

The $\cup$ and $\cap$ symbols

Let A and B be two sets

- The set $A \cup B$ is the set that contains all the elements of A and all the elements of B

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\} = \{x \mid x \in A \vee x \in B\}$$

- The set $A \cap B$ is the set that contains all the elements in common between A and B

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} = \{x \mid x \in A \wedge x \in B\}$$

The $\cup$ and $\cap$ symbols: some examples.

$$A = \{\clubsuit, \spadesuit, \bigstar\}\,, \quad B = \{\clubsuit, \diamond\}\,.$$

then

- $A \cup B = \{\clubsuit, \spadesuit, \bigstar, \diamond\}$.
- $A \cap B = \{\clubsuit\}$.

Definition

The symbol $\emptyset$ indicates the set WITHOUT elements, also said the empty set.

Remark. For ANY set E

$$E \cup \emptyset = E, \quad E \cap \emptyset = \emptyset$$

The $\subset$ and $\subseteq$ symbols

Let A and B be two sets

- We say that $A \subseteq B$ if all the elements of A are also in B.
- We say that $A \subset B$ if all the elements of A are also in B and we know that there are some element of B that are not in A.
- We say that $A \supset B$ if $B \subset A$.
- We say that $A \supseteq B$ if $B \subseteq A$.
- We say that $A = B$ if and only if

$$A \subseteq B \land B \subseteq A.$$

Example.

$$A = \{\text{All cities of Europe}\}\,, \quad B = \{x \in A \mid x \text{ is a capital city}\}\,.$$

then $B \subset A$.

The minus set

Let A and B be two sets. We define $A \setminus B = \{x \in A \mid x \notin B\}$.

Example.

$$A = \{\clubsuit, \spadesuit, \bigstar\}\,, \quad B = \{\clubsuit, \diamond\}\,.$$

then $A \setminus B = \{\spadesuit, \bigstar\}$.

The complement

Let A and B be two sets and suppose $B \subseteq A$. We define $B^c = A \setminus B$.

Example.

$$A = \{\text{All cities of Europe}\}\,, \quad B = \{x \in A \mid x \text{ is a capital city}\}\,.$$

then $B^c = \{x \in A \mid x \text{ is not a capital city}\}$.

Remark If $B \subseteq A$ then

$$B \cup B^c = A.$$

$$B \cap B^c = \emptyset.$$

Definition

Let A and B be two sets. I define the cartesian product of A and B the set of the ordered pairs (a, b) such that the first element a of the pair is in A and the second element b of the pair is in B. In formula

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\} .$$

Definition

Let A be a set. I define A^2 as the set

$$A^2 = A \times A,$$

or, more generally,

$$A^n = \underbrace{A \times A \times \cdots \times A}_{n-\text{times}} .$$

Example. Consider the two sets

$$A = \{\clubsuit, \spadesuit, \bigstar\}, \quad B = \{\triangle, \diamond\}.$$

then

$$A \times B = \{(\clubsuit, \triangle), (\clubsuit, \diamond), (\spadesuit, \triangle), (\spadesuit, \diamond), (\bigstar, \triangle), (\bigstar, \diamond)\}$$

and

$$B \times A = \{(\triangle, \clubsuit), (\triangle, \spadesuit), (\triangle, \bigstar), (\diamond, \clubsuit), (\diamond, \spadesuit), (\diamond, \bigstar)\}$$

so typically $A \times B \neq B \times A$.

Definition

For any set A the power set is the set denoted with $\mathcal{P}(A)$ and it is defined as the set of all possible subsets of A, that is

$$\mathcal{P}(A) = \{B \text{ is a set} \mid B \subseteq A\}.$$

Example. Consider the set

$$A = \{\clubsuit, \spadesuit, \bigstar\}$$

then s

$$\mathcal{P}(A) = \{\emptyset, \{\clubsuit, \spadesuit, \bigstar\}, \{\clubsuit, \spadesuit\}, \{\clubsuit, \bigstar\}, \{\spadesuit, \bigstar\}, \{\clubsuit\}, \{\spadesuit\}, \{\bigstar\}\}$$

Definition

For any set A the cardinality of A is indicated as $\mathrm{Card}\,(A)$ and it is defined as the number of elements of A.

Examples.

$$A = \{\clubsuit, \spadesuit, \bigstar\} \Rightarrow \mathrm{Card}\,(A) = 3$$

Since

$$\mathcal{P}\,(A) = \{\emptyset, \{\clubsuit, \spadesuit, \bigstar\}, \{\clubsuit, \spadesuit\}, \{\clubsuit, \bigstar\}, \{\spadesuit, \bigstar\}, \{\clubsuit\}, \{\spadesuit\}, \{\bigstar\}\}$$

then $\mathrm{Card}\,(\mathcal{P}\,(A)) = 8 = 2^3$.

More generally if $\mathrm{Card}\,(A) = n$ then $\mathrm{Card}\,(\mathcal{P}\,(A)) = 2^n$.

Cardinality

Definition

For any set A the cardinality of A is indicated as $\mathrm{Card}\,(A)$ and it is defined as the number of elements of A.

Examples.

$$A = \{\text{All numbers between zero and one}\}$$

then

$$\mathrm{Card}\,(A) = +\infty.$$

Properties of union and intersection

Theorem

Given A, B and C sets, the following properties hold trivially

- $A \subseteq A \cup B$.
- $A \cap B \subseteq A$.
- *If* $A \subseteq B$ *then* $A \cup B = B$ *and* $A \cap B = A$.
- $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$.
- $(A \cup B) \cup C = A \cup (B \cup C)$ *and* $(A \cap B) \cap C = A \cap (B \cap C)$.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Examples. Consider the following set of subscripts

$$E = \{1, 2, 3\},$$

and the following family of sets associated to E

$$A_1 = \{\spadesuit, \diamond, \dagger\}, \; A_2 = \{\spadesuit, \diamond, \maltese\}, \; A_3 = \{\Bell, \leftmoon\},$$

so

$$\bigcup_{\alpha \in E} A_\alpha = A_1 \cup A_2 \cup A_3 = \{\spadesuit, \diamond, \dagger, \maltese, \Bell, \leftmoon\}$$

and

$$\bigcap_{\alpha \in E} A_\alpha = A_1 \cap A_2 \cap A_3 = \emptyset$$

Examples. For all $n \in \mathbb{N} = \{0, 1, 2, 3, 4, 5, ...\}$ consider the set

$$A_n = \left[0, 1 + \frac{1}{n+1}\right] = \left\{\text{All numbers between 0 and } 1 + \frac{1}{n+1}\right\}.$$

For example...

$$A_0 = [0, 2]\,,\ A_1 = \left[0, \frac{3}{2}\right]\,,\ A_2 = \left[0, \frac{4}{3}\right]\,, \cdots$$

and

$$\bigcap_{n \in \mathbb{N}} A_n = A_1 \cap A_2 \cap A_3 \cap \cdots = [0, 1]\,.$$

The set of integer and rational numbers

We indicate with $\mathbb{Z}$ the set

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} = \mathbb{N} \bigcup \{-n \mid n \in \mathbb{N},\ \ n > 0\}\,.$$

We indicate with $\mathbb{Q}$ the set

$$\mathbb{Q} = \left\{ \frac{n}{m} \;\middle|\; n \in \mathbb{Z}, m \in \mathbb{Z} \setminus \{0\} \right\}.$$

- **Sum**:

$$\frac{n}{m} + \frac{k}{q} = \frac{n \cdot q + k \cdot m}{m \cdot q}. \quad \text{Example:} \quad \frac{1}{3} + \frac{7}{4} = \frac{1 \cdot 4 + 7 \cdot 3}{3 \cdot 4} = \frac{25}{12}.$$

- **Product**:

$$\frac{n}{m} \cdot \frac{k}{q} = \frac{n \cdot k}{m \cdot q}. \quad \text{Example:} \quad \frac{1}{3} \cdot \frac{7}{4} = \frac{1 \cdot 7}{3 \cdot 4} = \frac{25}{12}.$$

- **Inverse**:

$$\frac{1}{\frac{n}{m}} = \frac{m}{n}. \quad \text{Also indicated with:} \quad \frac{1}{\frac{n}{m}} = \left(\frac{n}{m}\right)^{-1}.$$

- k**-th power**:

$$\left(\frac{m}{n}\right)^k = \underbrace{\frac{m}{n} \cdot \frac{m}{n} \cdots \frac{m}{n}}_{k-\text{times}} = \frac{m^k}{n^k}. \quad \text{Example:} \quad \left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}}.$$

and

$$\left(\frac{m}{n}\right)^{-k} = \frac{n^k}{m^k} \quad \text{Example:} \quad \left(\frac{1}{2}\right)^{-10} = 2^{10}.$$

Moreover for any $q \in \mathbb{Q}$ it's easy to prove that

$$q^{n+m} = q^n \cdot q^m$$

for any n and m in $\mathbb{Z}$.

For example

$$\left(\frac{1}{2}\right)^{3+2} = \frac{1}{2^{3+2}} = \frac{1}{2^5} = \left(\frac{1}{2}\right)^5$$

Theorem

For all $q \in \mathbb{Q}$ with $q \neq 0$ we have $q^0 = 1$.

Proof. Take an arbitrary integer number n, then it holds that

$$q^0 = q^{n-n} = q^n \cdot q^{-n} = q^n \cdot \frac{1}{q^n} = 1.$$

Decimal representation of $\mathbb{Q}$

Definition

For every $q \in \mathbb{Q}$ we use the notation

$$q = k_n k_{n-1} \ldots k_0, d_1 d_2 \ldots$$

to indicate the decimal representation of q, that is

$$q = k_n \cdot 10^n + k_{n-1} \cdot 10^{n-1} + k_{n-2} \cdot 10^{n-2} + \ldots + k_0 \cdot 10^0 + d_1 \cdot 10^{-1} + d_2 \cdot 10^{-2} + \ldots$$

Example

$$\begin{aligned}
\frac{3}{10} &= 0,3 = 0 \cdot 10^0 + 3 \cdot 10^{-1} \\
\frac{173}{2} &= 86,5 = 8 \cdot 10^1 + 6 \cdot 10^0 + 5 \cdot 10^{-1} \\
\frac{1}{3} &= 0,3333333\ldots = 0 \cdot 10^0 + 3 \cdot 10^{-1} + 3 \cdot 10^{-2} + 3 \cdot 10^{-3} + \ldots
\end{aligned}$$

Theorem

Let $q = \frac{m}{n}$ be a rational number. Then there are two mutually exclusive possibilities:

1. *The decimal representation of q is made by a finite number of digits.*
2. *The decimal representation of q is made by an infinite number of digits but it is periodic. In this case the period contains at most $n - 1$ different digits.*

Decimal representation of $\mathbb{Q}$

Number	Decimal Representation	Length of the period
$\frac{9}{11}$	$0,8181818181\ldots = 0,\overline{81}$	2
$\frac{1}{7}$	$0,14285714285714\ldots = 0,\overline{142857}$	6
$\frac{1}{81}$	$0,01234567901234679\ldots = 0,\overline{012345679}$	9
$\frac{1}{29}$	$0,\overline{0344827586206896551724137931}$	28

WARNING

The set $\mathbb{Q}$ is not sufficient for many purposes.

Suppose we want to solve

$$x^2 = 2.$$

We already know that $\nexists x \in \mathbb{Q}$ such that $x^2 = 2$.

Maximum and Minimum

Consider a subset $E \subseteq \mathbb{Q}$.

Definition

We say that the set E has a minimum point if it exists a $m \in E$ such that

$$\forall q \in E \Rightarrow q \geq m.$$

If this is the case, m is called the minimum point of E or

$$m = \min(E).$$

Definition

We say that the set E has a maximum point if it exists a $M \in E$ such that

$$\forall q \in E \Rightarrow q \leq M.$$

If this is the case, M is called the maximum point of E or

$$M = \max(E).$$

Example

Some very simple examples ...

$$E = \left\{ \frac{3}{5} \right\} \Rightarrow \quad m = \frac{3}{5} \quad \textit{and} \quad M = \frac{3}{5}$$

$$E = \{-1, 0, 1\} \Rightarrow \quad m = -1 \quad \textit{and} \quad M = 1$$

$$E = \{q \in \mathbb{Q} \mid q \geq 0\} \Rightarrow \quad m = 0 \quad \textit{and} \quad \nexists M.$$

$$E = \{q \in \mathbb{Q} \mid q \leq 0\} \Rightarrow \quad \nexists m \quad \textit{and} \quad M = 0.$$

$$E = \{q \in \mathbb{Q} \mid 0 \leq q \leq 1\} \Rightarrow \quad m = 0 \quad \textit{and} \quad M = 1.$$

Theorem

Let $E \subseteq \mathbb{Q}$. The minimum of E, provided that it exists, it is unique, that is there cannot exist two different maxima of the same set. The same is true for the minimum.

Proof. Let's assume, by contradiction, that there exists two maxima M_1 and M_2 of E, with $M_1 \neq M_2$.

1. Since, by definition of maximum, $M_1 \in E$ and since M_2 is a maximum, then $M_1 \leq M_2$.
2. Since, by definition of maximum, $M_2 \in E$ and since M_1 is a maximum, then $M_2 \leq M_1$.

$M_1 \leq M_2$ and $M_2 \leq M_1$ imply $M_1 = M_2$, which contradicts $M_1 \neq M_2$.

Consider the case

$$E = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N},\ n > 0 \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$$

Note that $1 \in E$ and $\frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$, so $M = 1$.

Probelm

Does the set E have a minimum point?

The answer is trivially no. Suppose that we have found a minimum point, and let $\frac{1}{n^\star}$ be such point. Trivially

$$\frac{1}{n^\star + 1} < \frac{1}{n^\star},$$

but $\frac{1}{n^\star+1} \in E$ so $\frac{1}{n^\star}$ is not a minimum point.


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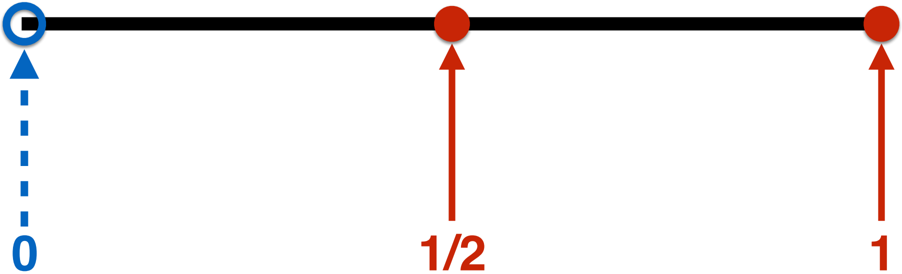
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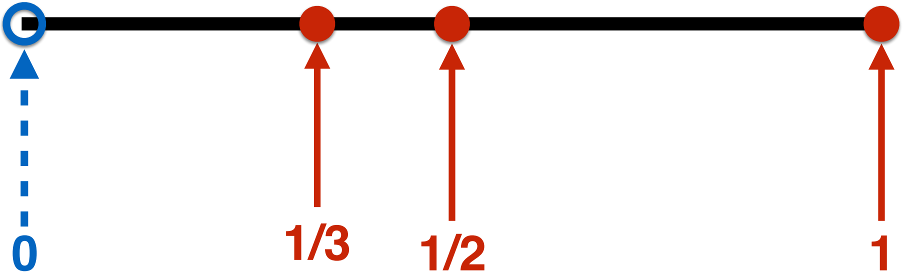
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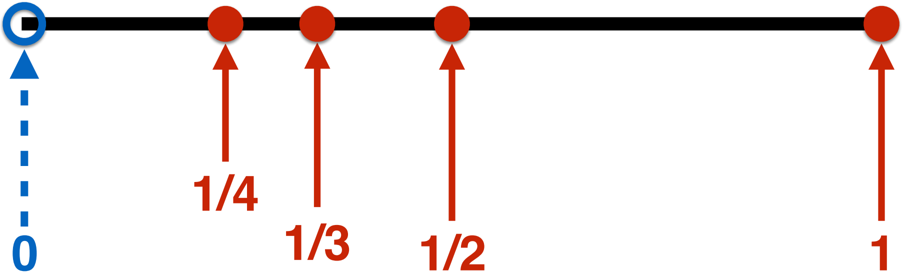
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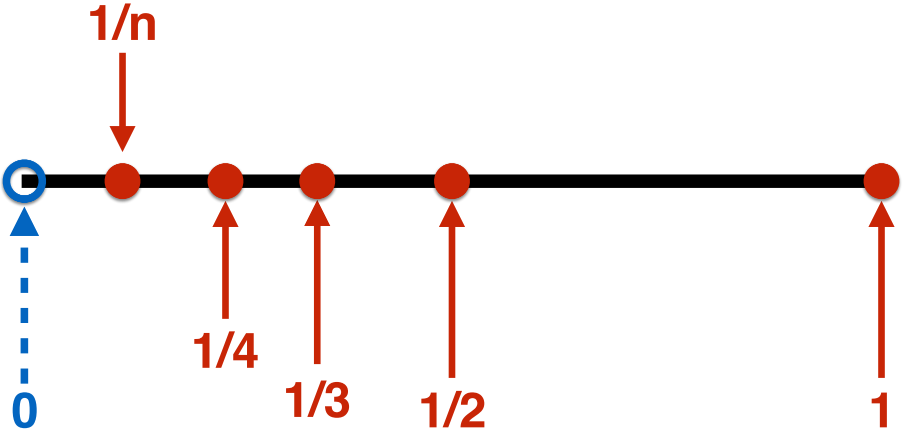
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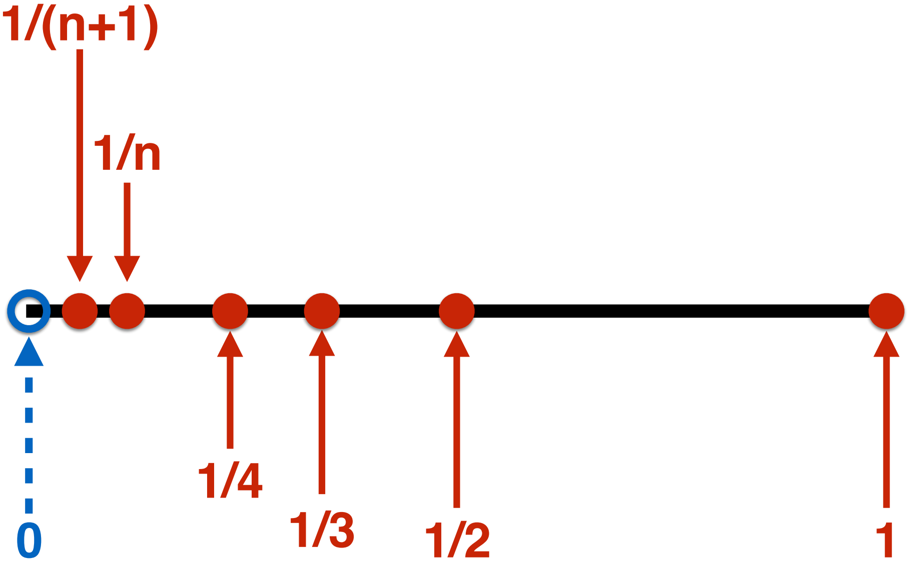
1/(n+1)
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1
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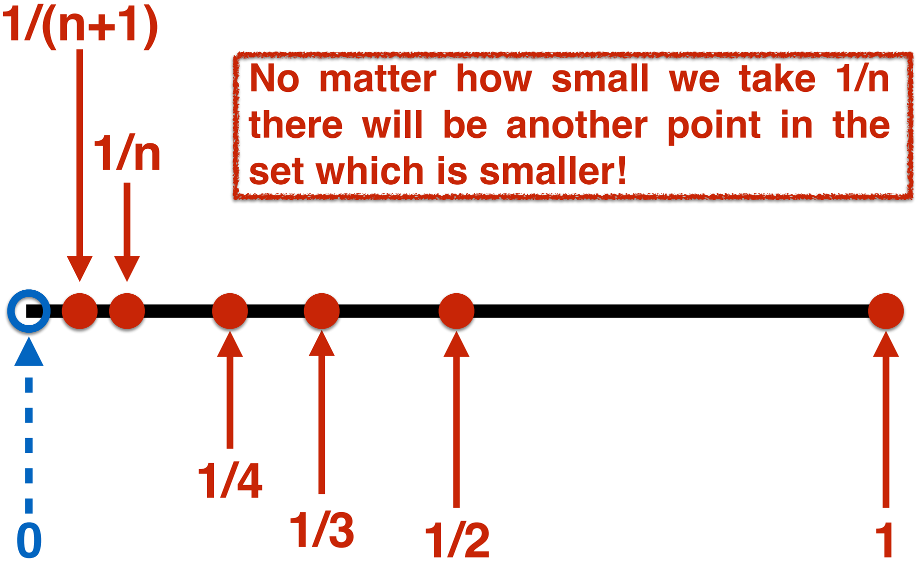

1/(n+1)
1/n
No matter how small we take 1/n there will be another point in the set which is smaller!
0
1/4
1/3
1/2
1

Summary

$$E = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N},\ n > 0 \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \right\}$$

then $\nexists$ the minimum of E, nevertheless

$$E' = E \bigcup \{0\} = \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \right\}$$

is such that 0 is the minimum point of E'. Similarly

$$F = \left\{ -\frac{1}{n} \,\middle|\, n \in \mathbb{N},\ n > 0 \right\} = \left\{ -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, \ldots \right\}$$

Then $\nexists$ the maximum of F, nevertheless

$$F' = F \bigcup \{0\} = \left\{ 0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, \ldots \right\}$$

is such that 0 is the maximum point of F'.

Remark

The set $E = \left\{ \frac{1}{n} \mid n \in \mathbb{N},\ n > 0 \right\}$ is clearly "limited from below", but...

it has no minimum!

$$\Downarrow$$

The definition of max and min appears to be too **tight**!

Nevertheless, before proceeding, we have to define two special symbols....

Definition

We introduce the symbols $+\infty$ and $-\infty$ through the following relationships:

- For all $q \in \mathbb{Q}$ then $q < +\infty$ and $q > -\infty$.
- For all $q \in \mathbb{Q}$ then $q + \infty = +\infty$ and $q - \infty = -\infty$.
- $(+\infty) + (+\infty) = +\infty$ and $(-\infty) + (-\infty) = -\infty$.
- $(+\infty) \times (+\infty) = +\infty$ and $(-\infty) \times (-\infty) = +\infty$.
- $(-\infty) \times (+\infty) = -\infty$ and $(+\infty) \times (-\infty) = -\infty$.
- For all $q \in \mathbb{Q}$ then $\frac{q}{+\infty} = \frac{q}{-\infty} = 0$.
- For all $q \in \mathbb{Q}$ with $q > 0$ then $q \times (+\infty) = +\infty$, $q \times (-\infty) = -\infty$.
- For all $q \in \mathbb{Q}$ with $q < 0$ then $q \times (+\infty) = -\infty$, $q \times (-\infty) = +\infty$.
- $(+\infty) - (+\infty)$, $(-\infty) + (+\infty)$, $0 \times (+\infty)$, $0 \times (-\infty)$ and $\frac{\infty}{\infty}$ are INDETERMINATE.

Definition

Let $E \subset \mathbb{Q}$. We say that a number $u \in \mathbb{Q}$ is an upper bound for E if

$$\forall x \in E \Rightarrow x \leq u$$

Definition

Let $E \subset \mathbb{Q}$. We say that a number $\ell \in \mathbb{Q}$ is a lower bound for E if

$$\forall x \in E \Rightarrow x \geq \ell$$

Exercize

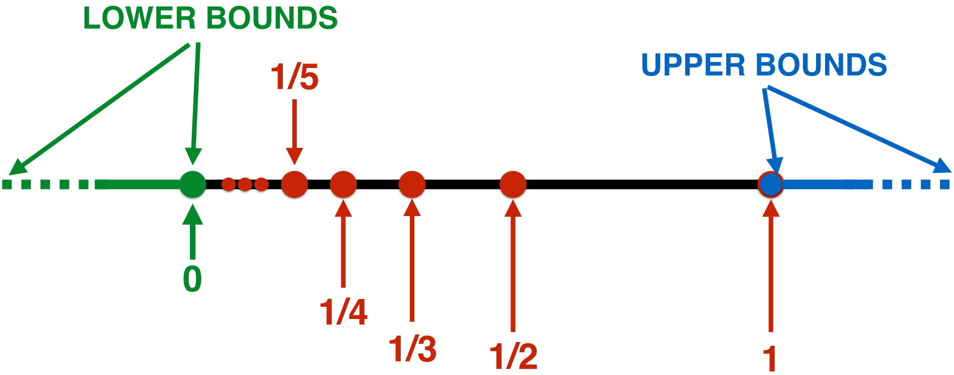
Which are the lower bounds and the upper bounds of the set

$$E = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N},\ n > 0 \right\} \quad ?$$

Solution. Clearly:

- For all $x \in E$ we have $x \leq 1$, hence all $u \in \mathbb{Q}$ such that $u \geq 1$ are upper bounds.
- For all $x \in E$ we have $x > 0$, hence all $\ell \in \mathbb{Q}$ such that $\ell \leq 0$ are lower bounds.

Any point 0<x<1 cannot be neither a lower bound nor an upper bound!



Definition

Let $E \subset \mathbb{Q}$. Consider the sets

$$U_E = \{u \in \mathbb{Q} \mid u \text{ is an upper bound of } E\}.$$

$$L_E = \{\ell \in \mathbb{Q} \mid \ell \text{ is a lower bound of } E\}.$$

Example

$E = \left\{\frac{1}{n} \mid n \in \mathbb{N},\ n > 0\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$ *then*

$$U_E = \{u \in \mathbb{Q} \mid u \geq 1\}, \quad L_E = \{\ell \in \mathbb{Q} \mid \ell \leq 0\}$$

Example

$E = \left\{\frac{1}{n^2} \mid n \in \mathbb{N},\ n > 0\right\} = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots\right\}$ *then*

$$U_E = \{u \in \mathbb{Q} \mid u \geq 1\}, \quad L_E = \{\ell \in \mathbb{Q} \mid \ell \leq 0\}$$

Definition

Let $E \subseteq \mathbb{Q}$. Recall that

$$U_E = \{u \in \mathbb{Q} \mid u \text{ is an upper bound of } E\}.$$

If $U_E = \emptyset$ we set

$$\sup_{\mathbb{Q}}(E) = +\infty.$$

If $U_E \neq \emptyset$ it can be proved that the minimum of U_E exists and we set

$$\sup_{\mathbb{Q}}(E) = \min(U_E).$$

Definition

Let $E \subseteq \mathbb{Q}$. Recall that

$$L_E = \{\ell \in \mathbb{Q} \mid \ell \text{ is a lower bound of } E\}.$$

If $L_E = \emptyset$ we set

$$\inf_{\mathbb{Q}}(E) = -\infty.$$

If $L_E \neq \emptyset$ it can be proved that the maximum of L_E exists and we set

$$\inf_{\mathbb{Q}}(E) = \max(U_E).$$

Example

Consider the set

$$E = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N},\ n > 0 \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \right\}$$

then

$$U_E = \{u \in \mathbb{Q} \mid u \geq 1\}, \quad L_E = \{\ell \in \mathbb{Q} \mid \ell \leq 0\}$$

*and so, although E **has no minimum element**, we have*

$$\inf_{\mathbb{Q}} (E) = \max (L_E) = 0,$$

while, straightforwardly, the maximum element and the supremum coincide

$$\sup_{\mathbb{Q}} (E) = \min (U_E) = 1.$$

Theorem

Let $E \subseteq \mathbb{Q}$ be a subset of $\mathbb{Q}$. The following statements hold

- *If E has a maximum point M then* $\sup_{\mathbb{Q}}(E) = M$.
- *If E has a minimum point m then* $\inf_{\mathbb{Q}}(E) = m$.

Remark

For every set $E \subseteq \mathbb{Q}$

(provided that, at least $E \neq \emptyset$)

the max and the min may not exist ...

... however, supremum and infimum always will exist!

Although they could be $+\infty$ or $-\infty$ ...

Exercize

Set	max	min	sup	inf
$\{n \in \mathbb{N} \mid n \leq 10\}$	10	0	10	0
$\{q \in \mathbb{Q} \mid q > -1\}$	$\nexists$	$\nexists$	$+\infty$	-1
$\{q \in \mathbb{Q} \mid q \geq -1\}$	$\nexists$	-1	$+\infty$	-1
$\{q \in \mathbb{Q} \mid q < 0\}$	$\nexists$	$\nexists$	0	$-\infty$
$\{q \in \mathbb{Q} \mid q \leq 0\}$	0	$\nexists$	0	$-\infty$
$\left\{1 - \frac{1}{n} \mid n \in \mathbb{N},\ n > 0\right\}$	$\nexists$	0	1	0
$\left\{1 - \frac{1}{n^3} \mid n \in \mathbb{N},\ n > 0\right\}$	$\nexists$	0	1	0
$\{q \in \mathbb{Q} \mid 0 \leq q < 2\}$	$\nexists$	0	2	0
$\{q \in \mathbb{Q} \mid 0 < q \leq 2\}$	2	$\nexists$	2	0
$\{q \in \mathbb{Q} \mid 0 \leq q \leq 2\}$	2	0	2	0

$\mathbb{Q}$ is far to be complete.

Problem. Consider the set

$$A = \left\{ q \in \mathbb{Q} \,\middle|\, q^2 < 2 \right\}.$$

We note that

- $A \neq \emptyset$, for example $\frac{1}{2} \in A$ since $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < 2$.
- The number 2 is an upper bound for A.

Does the set A has a supremum? The answer is yes, but the supremum does not exists in $\mathbb{Q}$! This means that $\mathbb{Q}$ has "holes" here and there...

Can we fill them?

Theorem

There exists a set $\mathbb{R}$ (called: the real line or the set of real numbers) such that $\mathbb{Q} \subset \mathbb{R}$ and such that all the subset $E \subset \mathbb{R}$ that are not empty and bounded from above have a supremum in $\mathbb{R}$. Similarly, all the subset $E \subset \mathbb{R}$ that are not empty and bounded from below have an infimum in $\mathbb{R}$

Irrational numbers

The sets of numbers are

$$\mathbb{N} = \{0, 1, 2, \ldots\} \subset \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \;\middle|\; m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}$$

A natural question

Which are the numbers of $\mathbb{R}$ that are not in $\mathbb{Q}$? That is, which elements does the set

$$\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$$

contain? Remember that $\mathbb{Q}$ contains all the number whose decimal representation is either finite or periodic, for example

$$\frac{1}{2} = 0,5 \quad \frac{1}{7} = 0,142857142857142857142857\ldots$$

The set $\mathbb{I}$ is called the set of irrational numbers and contains all the number whose decimal representation is neither finite nor periodic!

The number $p \in \mathbb{R}$ such that $p^2 = 2$ is called $p = \sqrt{2}$ and ...

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694807317667973799073247846210703885038753432764157273501384623091229702 49\ldots$$

We will never know the true value of $\sqrt{2}$! We can only construct algorithms to compute which are the next digits ...

Example

$$e = 2.718281828459045235360287471352662497757247093699959574966967627724076630353547594571382178525166427427466391932003059 9\ldots$$

$$\pi = 3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117067982148086513282306 6\ldots$$

The absolute value

Definition

For any real number $x \in \mathbb{R}$ we indicate with $|x|$ the number

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

and we call $|x|$ the the absolute value of x. Trivially $|x| = |-x|$.

Definition

Given any two real numbers $x \in \mathbb{R}$ and $y \in \mathbb{R}$ we call

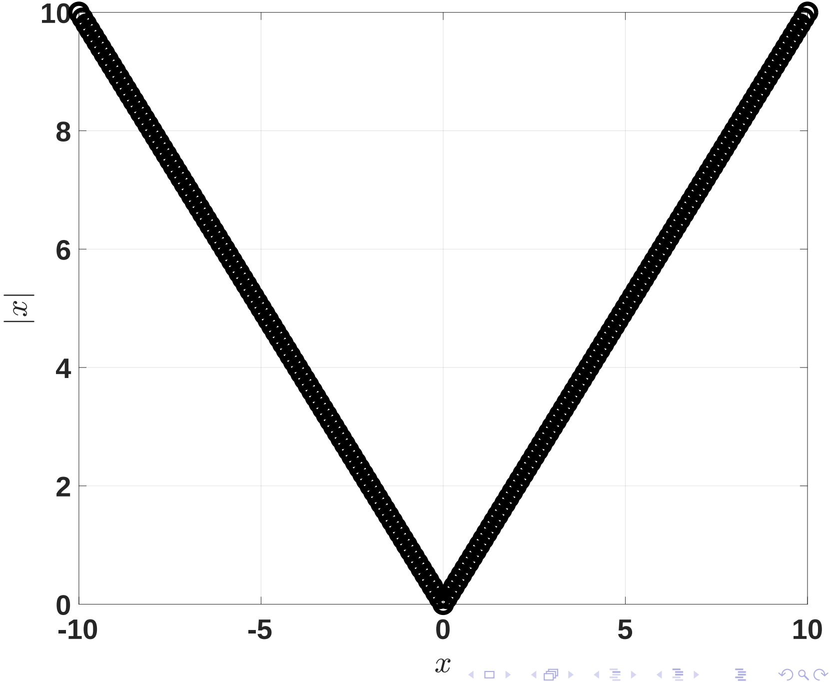
$$d(x, y) = |x - y|$$

the distance between x and y. Trivially $d(x, y) = d(y, x)$.

Example

$$|2| = 2, \quad \left|-\frac{1}{2}\right| = \frac{1}{2}, \quad |0| = 0 \quad |-1| = 1 \quad \text{etc..etc..}$$

$$d(1, -10) = |1 - (-10)| = 11 = d(-10, 1), \quad d(2, 0) = |2| = d(0, 2).$$


10
8
6
4
2
0
|x|
-10
-5
0
5
10
x

Theorem

For any two number $a \in \mathbb{R}$ and $b \in \mathbb{R}$ it holds that

$$|a+b| \leq |a| + |b|$$

Proof. Consider that

$$-|a| \leq a \leq |a|\,, \quad (\triangle)$$

and

$$-|b| \leq b \leq |b|\,, \quad (\square)\,.$$

Now sum $(\triangle)$ with $(\square)$, obtaining

$$-|a| - |b| \leq a + b \leq |a| + |b|$$

which means

$$\underbrace{-(|a|+|b|)}_{-c} \leq a + b \leq \underbrace{|a|+|b|}_{c}$$

which means

$$|a+b| \leq |c| = c = |a| + |b|\,.$$

Definition

Let a and b be two real numbers. We will adopt the following notations

$$
\begin{aligned}
(a,b) &= \{x \in \mathbb{R} \mid a < x < b\} \\
[a,b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\
(a,b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\
[a,b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\
(-\infty,a) &= \{x \in \mathbb{R} \mid x < a\} \\
(-\infty,a] &= \{x \in \mathbb{R} \mid x \leq a\} \\
(a,+\infty) &= \{x \in \mathbb{R} \mid x > a\} \\
[a,+\infty) &= \{x \in \mathbb{R} \mid x \geq a\}
\end{aligned}
$$

Neighborhoods

Definition

Consider a number $x_0 \in \mathbb{R}$ and another number $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$. The set

$$N_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon)$$

is called a neighborhood with center x_0 and radius ε.

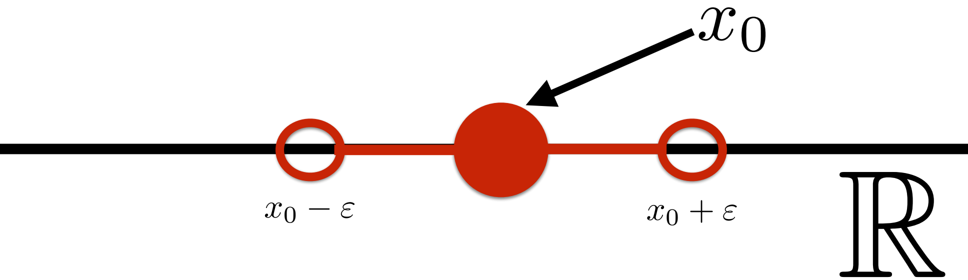
Remark

Since, by definition,

$$N_\varepsilon(x_0) = \{x \in \mathbb{R} \mid |x - x_0| < \varepsilon\}$$

the neighborhood $N_\varepsilon(x_0)$ contains all the points of $\mathbb{R}$ whose distance from the center x_0 is **strictly less** than the radius ε.

The two extrema are not included!!!



Neighborhoods

Definition

Consider a number $x_0 \in \mathbb{R}$ and another number $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$. The sets

$$N_\varepsilon\left(x_0^+\right) = (x_0, x_0 + \varepsilon) \text{ and } N_\varepsilon\left(x_0^-\right) = (x_0 - \varepsilon, x_0)$$

are called, respectively, a right-neighborhood and a left-neighborhood with center x_0 and radius ε.

$$\textbf{Remark}: \quad N_\varepsilon(x_0) = \{x_0\} \cup N_\varepsilon\left(x_0^+\right) \cup N_\varepsilon\left(x_0^-\right)$$

Definition

Given a set $E \subseteq \mathbb{R}$ a point $x_0 \in \mathbb{R}$ is called a limit point of E if every neighborhood of x_0 contains at least one element of E which is different from x_0:

$$\forall \varepsilon > 0 \Rightarrow N_\varepsilon(x_0) \cap E \setminus \{x_0\} \neq \emptyset$$

The set made by all the limit points of E is called the derivative set and it is indicated with E'.

Exercize

Find the limit points of the set $E = (0, 1)$.

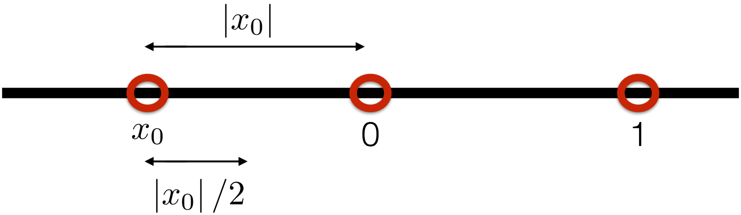
Solution. Consider a point $x_0 \in \mathbb{R}$. There can be five cases. Case one:

- $x_0 < 0$. In this case consider the neighborhood of x_0 with radius $\varepsilon = |x_0|/2 = -x_0/2$. Clearly

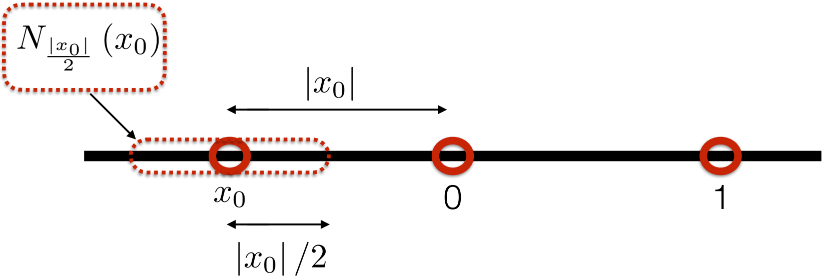
$$N_{|x_0|/2}(x_0) \cap (0, 1) \setminus \{x_0\} = \emptyset.$$

CASE 1: $x_0 < 0$

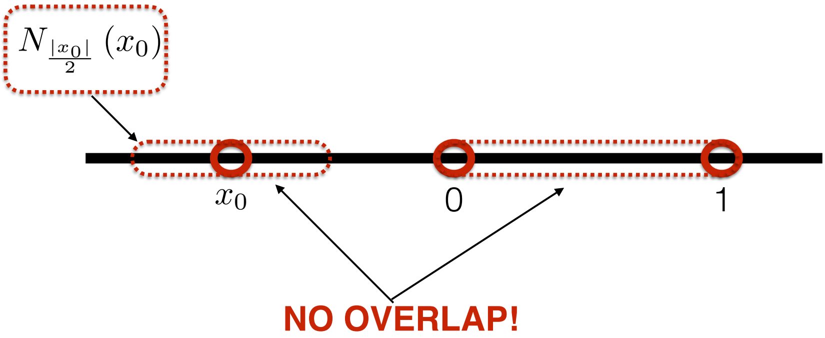
CASE 1: $x_0 < 0$



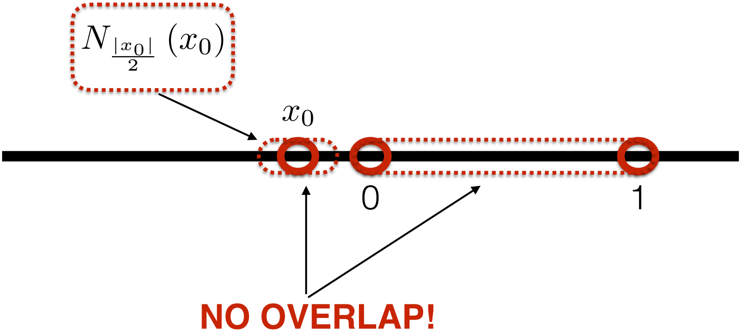
CASE 1: $x_0 < 0$



CASE 1: $x_0 < 0$



CASE 1: $x_0 < 0$



No matter how close x_0 is to 0 provided that $x_0 < 0$

Exercize

Find the limit points of the set $E = (0, 1)$.

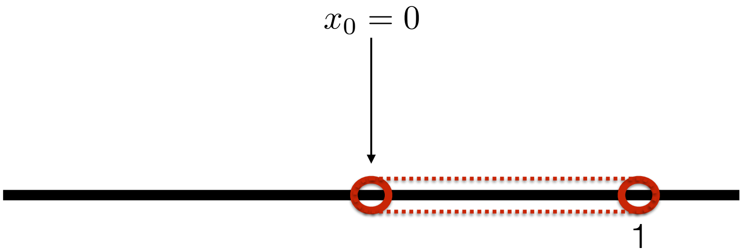
Solution. Consider a point $x_0 \in \mathbb{R}$. There can be five cases. Case two:

- $x_0 = 0$. In this case, no matter how small is $\varepsilon > 0$, we have

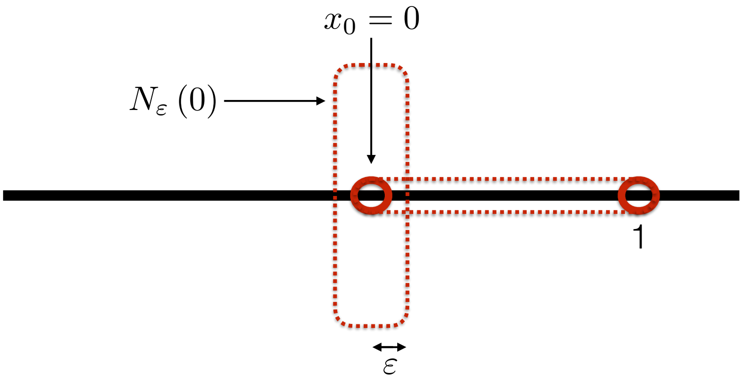
$$N_\varepsilon(0) \cap (0, 1) \setminus \{0\} = (0, \varepsilon) \neq \emptyset.$$

  i.e. $x_0 = 0$ is a limit point.

CASE 2: $x_0 = 0$



CASE 2: $x_0 = 0$



Exercize

Find the limit points of the set $E = (0, 1)$.

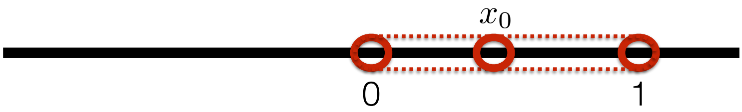
Solution. Consider a point $x_0 \in \mathbb{R}$. There can be five cases. Case three:

- $0 < x_0 < 1$. In this case, no matter how small is $\varepsilon > 0$, we have

$$N_\varepsilon(x_0) \cap (0, 1) \setminus \{x_0\} = (0, \varepsilon) \neq \emptyset.$$

i.e. x_0 is a limit point.

CASE 3: $0 < x_0 < 1$



Exercize

Find the limit points of the set $E = (0, 1)$.

Solution. Consider a point $x_0 \in \mathbb{R}$. There can be five cases.

- Case four: $x_0 = 1$. Exactly the same reasoning of $x_0 = 0$, whence $x_0 = 1$ is a limit point.
- Case five: $x_0 > 1$. Exactly the same reasoning of $x_0 < 0$, whence $x_0 > 1$ is not a limit point.

Conclusions

If $E = (0, 1)$ then $E' = [0, 1]$.

Definition

Let $E \subseteq \mathbb{R}$. A point $x_0 \in E$ which is not a limit point of E is called an isolated point.

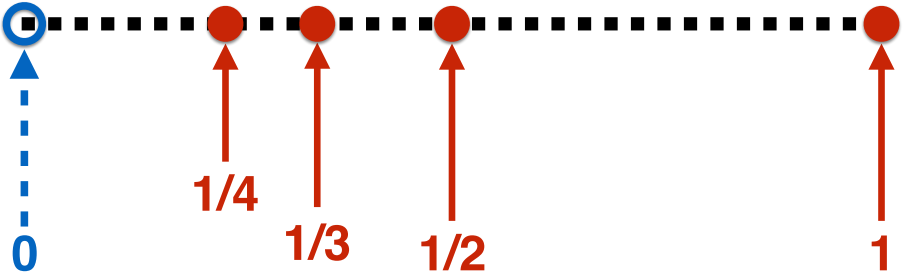
Example

- $E = \{0\}$. Trivially,

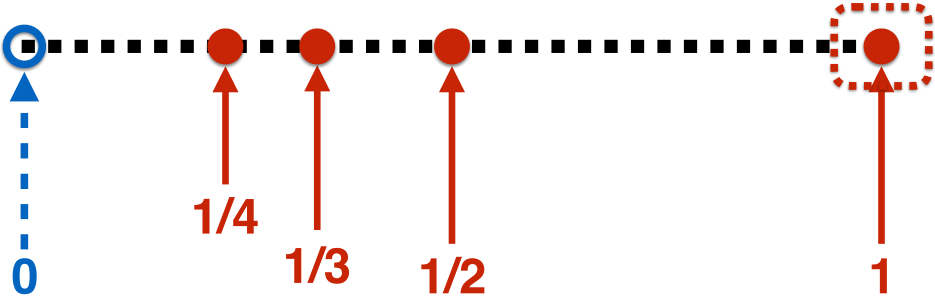
  $$\forall \varepsilon > 0 \Rightarrow N_\varepsilon(0) \cap E \setminus \{0\} = \emptyset$$

  Whence x_0 **is not** a limit point of $E \implies x_0$ is an isolated point of E.
- $E = \left\{\frac{1}{n} \;\middle|\; n \in \mathbb{N}, n > 0\right\}$. In this case, all the points of E are isolated point.

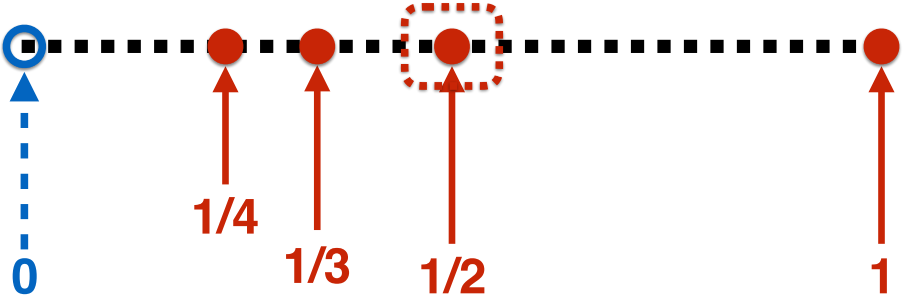
  The point 0 (which is not in E) is the unique limit point of E.


0
1/4
1/3
1/2
1

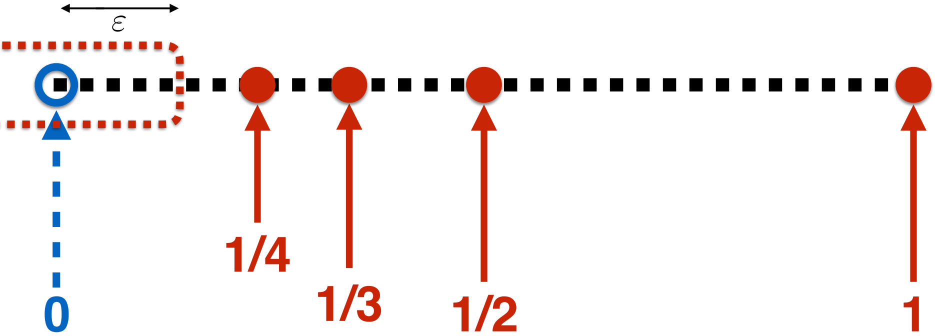
$$N_\varepsilon(1) \cap E \setminus \{1\} = \emptyset$$



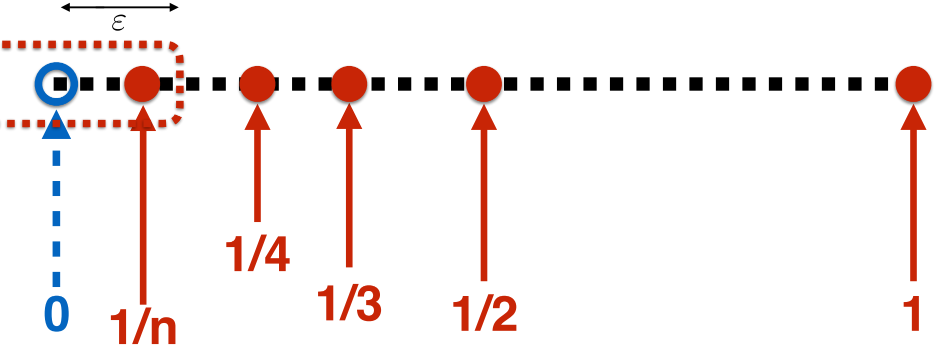
$$N_\varepsilon(1/2) \cap E \setminus \{1/2\} = \emptyset$$



Take an arbitrarily small ε ...

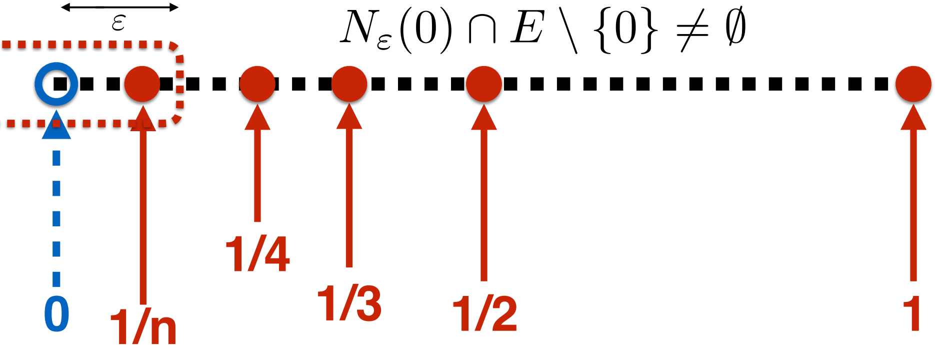


Take an arbitrarily small ε ... and a sufficiently large n.



Take an arbitrarily small ε ... and a sufficiently large n.

$$N_\varepsilon(0) \cap E \setminus \{0\} \neq \emptyset$$



Definition

Let $E \subseteq \mathbb{R}$. A point $x_0 \in E$ is an interior point of E if

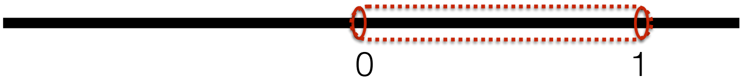
$$\exists \varepsilon : N_\varepsilon(x_0) \subset E.$$

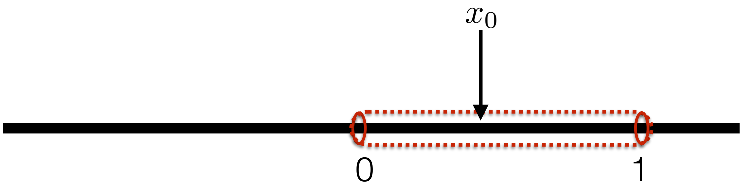
Example

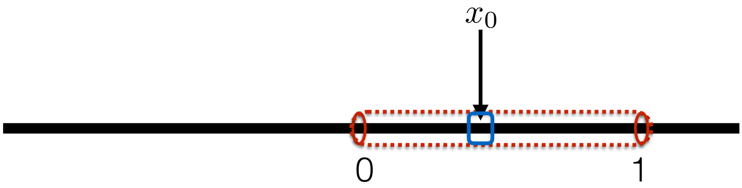
- $E = \{0\}$. The only possible interior point is 0 (it is the unique element of E!), but

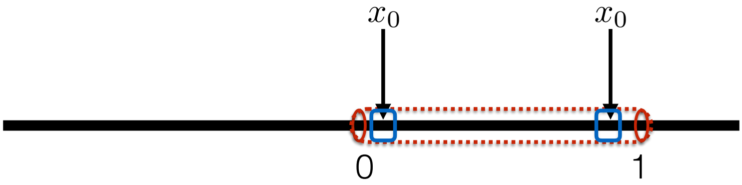
$$\forall \varepsilon > 0 \Rightarrow N_\varepsilon(0) \supset E = \{0\},$$

  whence E has no interior points.
- $E = (0, 1)$. In this case all the points of E are interior points of E.
- $E = [0, 1]$. In this case 0 and 1 are not interior points of E, while all the points in $(0, 1)$ are interior points of E.


0
1


x_0
0
1


x_0
0
1


x_0
x_0
0
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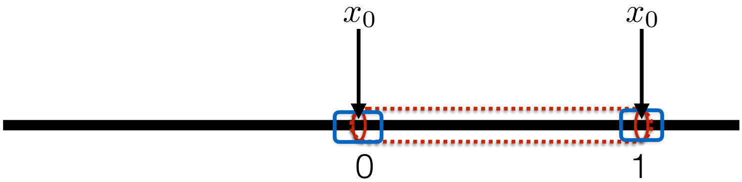


If the center is either in zero or one,
no matter how small is the radius, a part of the neighborhood will be outside of the set.

Definition

A set $E \subset \mathbb{R}$ is said to be open if every point of E is an interior point of E.

Remark

All the sets of the form

$$(a, b)$$

are open. For example $(0, 1)$ is open, as well as

$$(-1, 1)$$

$$(-1, 0) \cup (0, 1)$$

etc..etc.. While the set

$$[0, 1]$$

is not open.

Definition

A set $E \subset \mathbb{R}$ is said to be closed if every limit point of E is a point of E

Remark

All the sets of the form

$$[a, b]$$

are closed. For example $[0, 1]$ is closed, as well as

$$[-1, 1]$$

$$[-10, -5] \cup [0, 1]$$

etc..etc.. While the set

$$(0, 1)$$

is not closed.

Recall that E' indicates the set that contains all the limit points of a set E.

Definition

Given a set E the set $\overline{E} = E \cup E'$ is called the closure of E.

Example

Consider $E = (0,1)$, we know that $E' = [0,1]$ so

$$\overline{(0,1)} = [0,1]\,.$$

Trivially

$$\overline{[0,1]} = [0,1]\,.$$

Remark

A set E is closed if and only if $E = \overline{E}$.

Definition

Given a set $E \subseteq \mathbb{R}$ a point $x_0 \in \mathbb{R}$ is said an exterior point of E if

$$\exists \varepsilon > 0 : N_\varepsilon(x_0) \subset E^c = \mathbb{R} \setminus E.$$

Example

Consider $E = (0, 1)$, then all the points of the set

$$(-\infty, 0) \cup (1, \infty)$$

are exterior points. Note that 0 and 1 are not exterior points.

Consider $E = [0, 1]$, then all the points of the set

$$(-\infty, 0) \cup (1, \infty)$$

are exterior points. Note that 0 and 1 are not exterior points.

Definition

Given a set $E \subseteq \mathbb{R}$ a point $x_0 \in \mathbb{R}$ is said an boundary point of E if

$$\forall \varepsilon > 0 \Rightarrow N_\varepsilon(x_0) \cap E \neq \emptyset \text{ and } N_\varepsilon(x_0) \cap E^c \neq \emptyset.$$

Example

- Consider $E = (0, 1)$, then the two points

$$\{0, 1\}$$

  are the boundary points. The same is true for $E = [0, 1]$.
- Consider $E = (1, +\infty]$ then $\{1\}$ is the unique boundary point.

Some exercises

$$\underline{\mathbb{N}} = \mathbb{N} \setminus \{0\}\,.$$

	Type of Point					
Set	Limit	Isolated	Interior	Exterior	Boundary	
$(0,1)$	$[0,1]$	$\nexists$	$(0,1)$	$(-\infty,0)\cup(1,\infty)$	$\{0,1\}$	Open
$\{0,1\}$	$\nexists$	$\{0,1\}$	$\nexists$	$\mathbb{R}\setminus\{0,1\}$	$\{0,1\}$	Closed
$\{\frac{1}{n} \mid n\in\underline{\mathbb{N}}\}$	$\{0\}$	$\{\frac{1}{n} \mid n\in\underline{\mathbb{N}}\}$	$\nexists$	$\mathbb{R}\setminus\{0,\frac{1}{n} \mid n\in\underline{\mathbb{N}}\}$	$\{0,\frac{1}{n} \mid n\in\underline{\mathbb{N}}\}$	–
$[0,1]\cup\{2\}$	$[0,1]$	$\{2\}$	$(0,1)$	$\mathbb{R}\setminus\{[0,1]\cup\{2\}\}$	$\{0,1,2\}$	Closed
$(0,1)\cup\{2\}$	$[0,1]$	$\{2\}$	$(0,1)$	$\mathbb{R}\setminus\{[0,1]\cup\{2\}\}$	$\{0,1,2\}$	–
$(-\infty,1)$	$(-\infty,1]$	$\nexists$	$(-\infty,1)$	$(1,\infty)$	$\{1\}$	Open
$(-\infty,1]$	$(-\infty,1]$	$\nexists$	$(-\infty,1)$	$(1,\infty)$	$\{1\}$	Closed

Definition

Let $D \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. A function is a correspondence that associate to each element of D **one and only one** element of $\mathbb{R}$.

$$
\begin{aligned}
f \quad : \quad & D \subseteq \mathbb{R} \to \mathbb{R} \\
& x \in D \to f(x) \in \mathbb{R}
\end{aligned}
$$

Standard nomenclature

$D =$ Domain of the function.

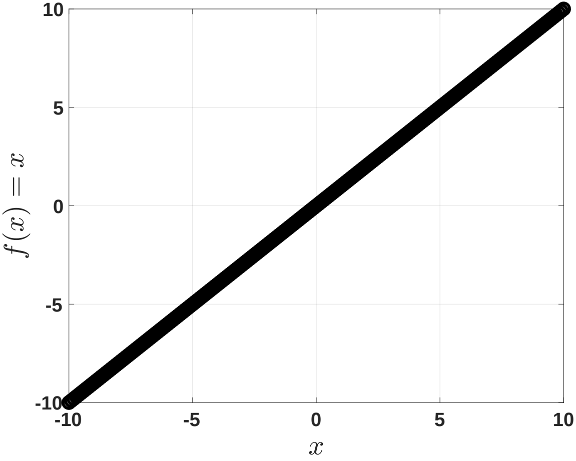
$x =$ Independent variable, it must always belong to the domain!.

$f(x) =$ The image of x through f or also the dependent variable.

Functions: examples

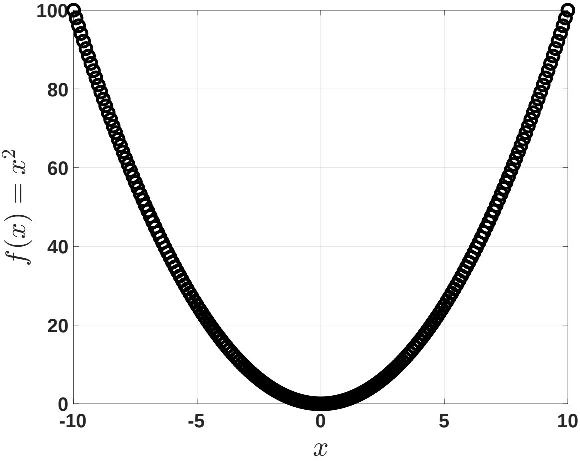
Example

$$f(x) = x$$



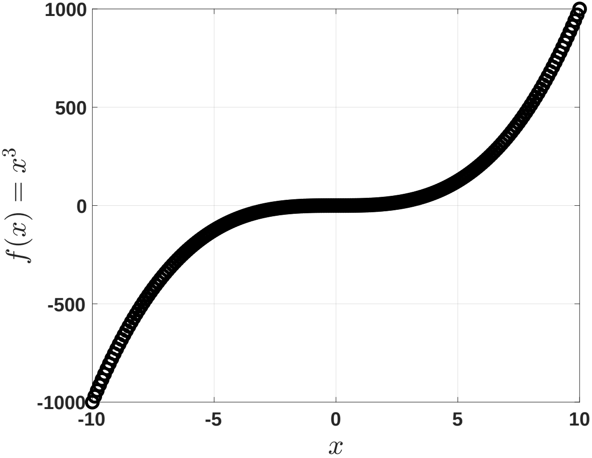
Example

$$f(x) = x^2 = x \cdot x$$



Example

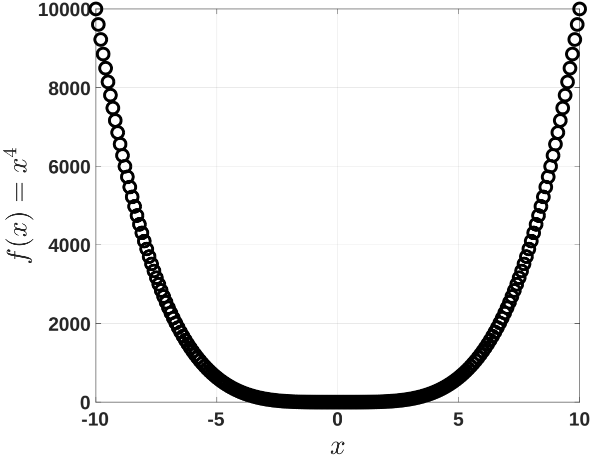
$$f(x) = x^3 = x \cdot x \cdot x$$



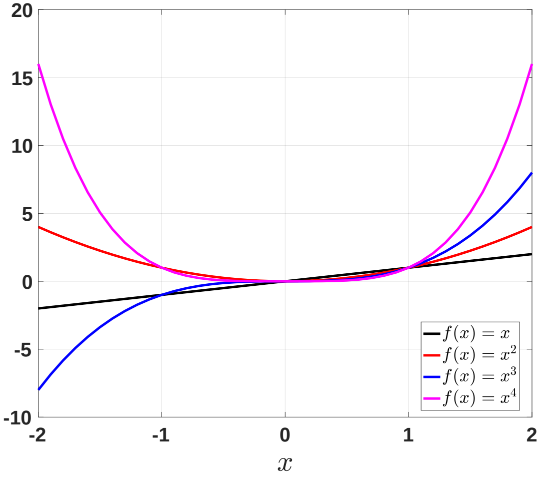
Functions: examples

Example

$$f(x) = x^4 = x \cdot x \cdot x \cdot x$$

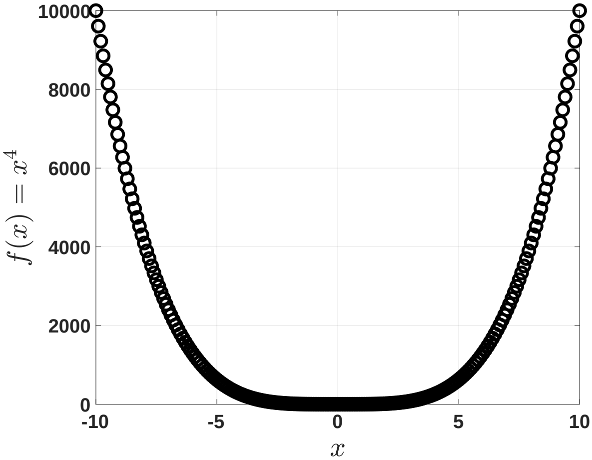


Example



Example

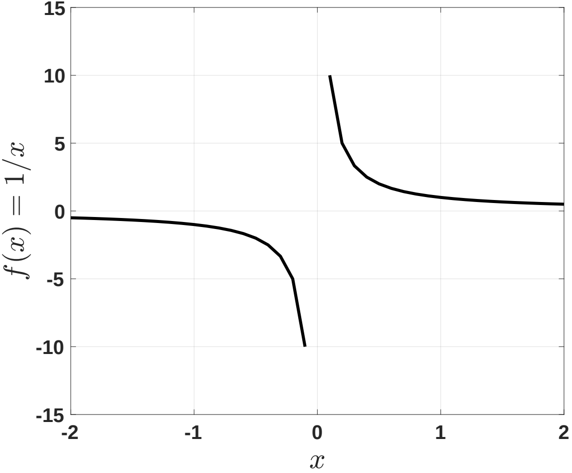
$$f(x) = \frac{1}{x} \Rightarrow D = \mathbb{R} \setminus \{0\}$$



Functions: examples

Example

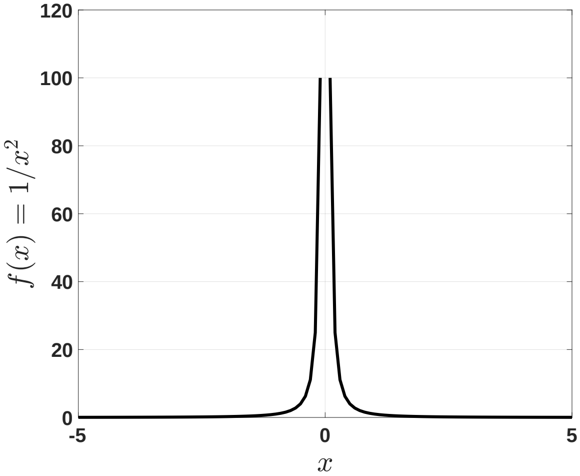
$$f(x) = \frac{1}{x} \Rightarrow D = \mathbb{R} \setminus \{0\}$$



Functions: examples

Example

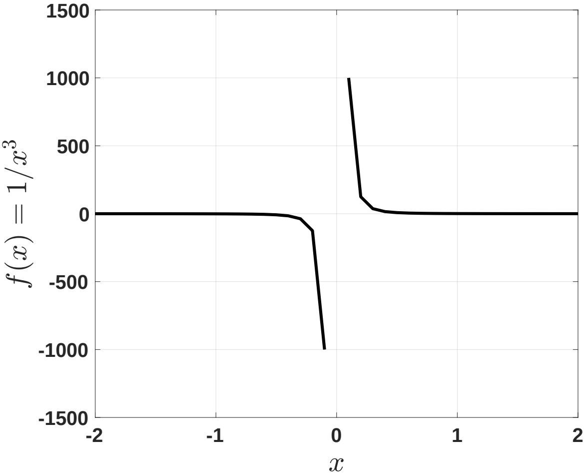
$$f(x) = \frac{1}{x^2} \Rightarrow D = \mathbb{R} \setminus \{0\}$$



Functions: examples

Example

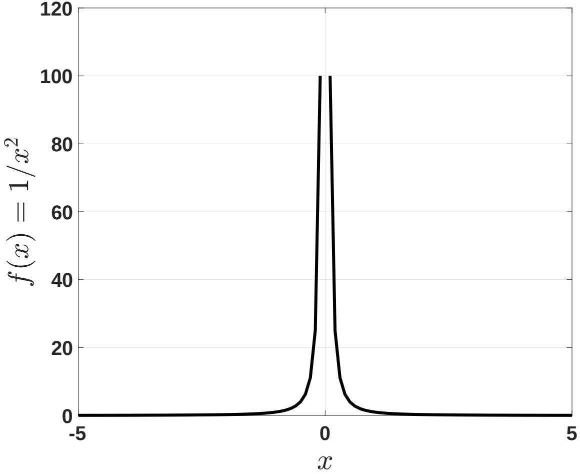
$$f(x) = \frac{1}{x^3} \Rightarrow D = \mathbb{R} \setminus \{0\}$$



Functions: examples

Example

$$f(x) = \frac{1}{x^2} \Rightarrow D = \mathbb{R} \setminus \{0\}$$



Functions: examples

Example

$$f(x) = \frac{1}{x-1} \Rightarrow D = \mathbb{R} \setminus \{1\}$$

$$f(x) = \frac{1}{(x-1)^2} \Rightarrow D = \mathbb{R} \setminus \{1\}$$

$$f(x) = \frac{1}{(x-1)^2\,(x-2)} \Rightarrow D = \mathbb{R} \setminus \{1, 2\}$$

$$f(x) = \frac{1}{(x+1)^2\,(x+2)^7\,x} \Rightarrow D = \mathbb{R} \setminus \{-2, -1, 0\}$$

The image of a function

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. We call the image of f the set

$$I_f = \{y \in \mathbb{R} \mid \exists x \in D : f(x) = y\},$$

that is: the sets of points in $\mathbb{R}$ which are the images, through f, of at least one point of the domain.

Problem

How do I find the image of a function?

I have to find all the y such that the equation

$$y = f(x)$$

has **at least** one x that solves it.

The image of a function

Exercize

Find the image of $f(x) = x$.

Solution. The equation $y = f(x) = x$ is solvable for every y! It is enough to take $x = y$. Hence

$$I_f = \mathbb{R}.$$

Exercize

Find the image of $f(x) = \frac{1}{x}$.

Solution. The equation $y = f(x) = \frac{1}{x}$ is for sure solvable if either $y > 0$ or $y < 0$, if this is the case it is enough to consider

$$x = \frac{1}{y}.$$

However if $y = 0$ we get $\frac{1}{x} = 0$ which is impossible to be solved. Hence

$$I_f = \mathbb{R} \setminus \{0\}.$$

Definition

Given any two sets X and Y a function is a correspondence that associate to each element $x \in X$ one and only one element $y \in Y$. In formula

$$\begin{aligned} f : X &\rightarrow Y \\ x \in X &\rightarrow f(y) \in Y. \end{aligned}$$

The sets X and Y are called, resp., domain and co-domain of the function. The image is defined, as usual, as

$$I_f = \{y \in Y \mid \exists x \in X : f(x) = y\}.$$

Definition

Let $D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $I \subset D$ be an open interval $I = (a, b)$, subset of the domain. We say that the function f is strictly increasing in I if

$$\forall x_1, x_2 \in I : x_1 < x_2 \Rightarrow f(x_1) < f(x_2),$$

we say that the function f is increasing in I if

$$\forall x_1, x_2 \in I : x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2).$$

Definition

Let $D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $I \subset D$ be an open interval $I = (a, b)$, subset of the domain. We say that the function f is strictly decreasing in I if

$$\forall x_1, x_2 \in I : x_1 < x_2 \Rightarrow f(x_1) > f(x_2),$$

we say that the function f is decreasing in I if

$$\forall x_1, x_2 \in I : x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2).$$

Example

- Consider two coefficients m and b and the function $f(x) = m\,x + b$. Take $x_1 < x_2$, then

$$f(x_2) - f(x_1) = m\,x_2 + b - m\,x_1 - b = m\,(x_2 - x_1)\,,$$

  whence $f(x) = m\,x + b$ is strictly increasing if $m > 0$ and strictly decreasing if $m < 0$.

- The function $f(x) = x^2$ is strictly increasing in $(0, +\infty)$ and strictly decreasing in $(-\infty, 0)$. Take $x_1 < x_2$, then

$$f(x_2) - f(x_1) = x_2^2 - x_1^2 = \underbrace{(x_1 + x_2)}_{A}\,(x_2 - x_1)\,.$$

  The factor A is positive if both x_2 and x_1 are in $(0, +\infty)$ and negative if both x_2 and x_1 are in $(-\infty, 0)$.

Definition

A function $f : X \to Y$ is said to injective if

$$\forall x_1, x_2 \in X \text{ if } x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$

that is, any two **distinguished** elements of the domain are mapped into two **distinguished** element of the image. Equivalently:

$$\forall x_1, x_2 \in X \text{ if } f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

Example

The function $f(x) = x^2$ is not injective:

$$x = -2 \neq y = 2 \text{ but } f(2) = 2^2 = 4 = (-2)^2 = f(-2).$$

Definition

A function $f : X \to Y$ is said to surjective if

$$\forall y \in Y \; \exists x \in X : \; f(x) = y,$$

that is, all the elements of the co-domain are images of at least one-element of the domain.

Example

The function $f(x) = x^2$ defined as $f : \mathbb{R} \to \mathbb{R}^+$ where $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$ is surjective: the equation $y = x^2$ has always two solutions if $y \geq 0$. The same function defined as $f : \mathbb{R} \to \mathbb{R}$ is not surjective since for example

$$-1 = x^2$$

has no solution.

Definition

A function $f : X \to Y$ is said to bijective if it is both surjective and injective.

Definition

Suppose that f is a function $f : X \to Y$ and that g is another function $g : W \to Z$ such that $I_f \subseteq W$. The composite function $g \circ f$ is defined as

$$\begin{aligned} g \circ f : X &\to Z \\ x &\to (g \circ f)(x) = g(f(x)) \end{aligned}$$

Remark

The condition $I_f \subseteq W$ is fundamental, it guarantees that $f(x) \in W$ and so $g(f(x))$ is well-defined.

Example

- $f(x) = x + 1$, $g(x) = x - 1$ then
$$(g \circ f)(x) = g(f(x)) = g(x+1) = x + 1 - 1 = x.$$
- $f(x) = x^2$, $g(x) = 1/x$ then
$$(g \circ f)(x) = g(f(x)) = g\left(x^2\right) = 1/x^2.$$
- $f(x) = x^2$, $g(x) = x^3$ then
$$(g \circ f)(x) = g(f(x)) = g\left(x^2\right) = \left(x^2\right)^3 = x^5.$$
- $f(x) = -x$, $g(x) = |x|$ then
$$(g \circ f)(x) = g(f(x)) = g(-x) = |-x| = |x|.$$

The inverse function

Definition

The identity function is indicated with the greek letter ι and is defined as

$$\forall x \in \mathbb{R}, \quad \iota(x) = x$$

Definition

A function $f : X \to Y$ is said to be invertible if there exists a second function $g : Y \to X$ such that

$$g \circ f = f \circ g = \iota,$$

if this function g exists it is called the inverse function and it is indicated with $f^{(-1)}$.

The inverse function

WARNING

The notation $f^{(-1)}$ it is just to remind that the function operates "backward", but $f^{(-1)}(x)$ may not coincide with $(f(x))^{-1} = \frac{1}{f(x)}$.

Theorem

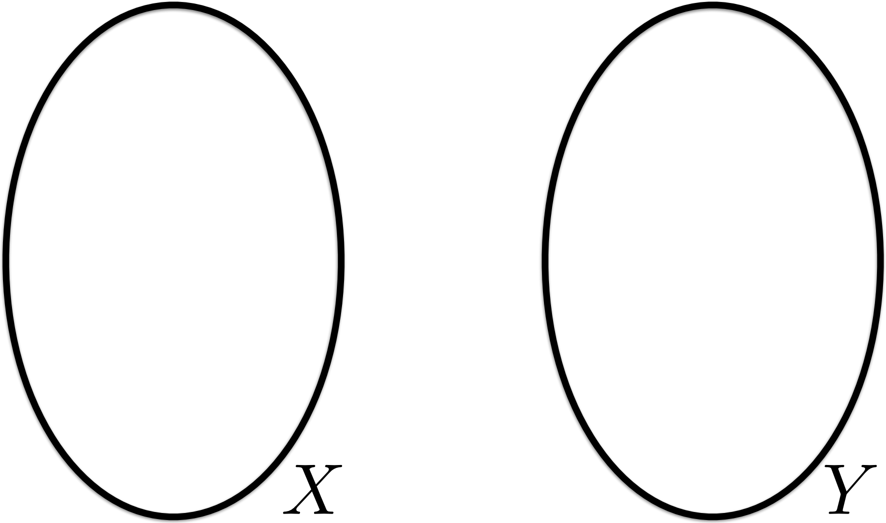
A function $f : X \to Y$ is invertible if and only if it is a bijection, that is if and only if is both surjective and injective.

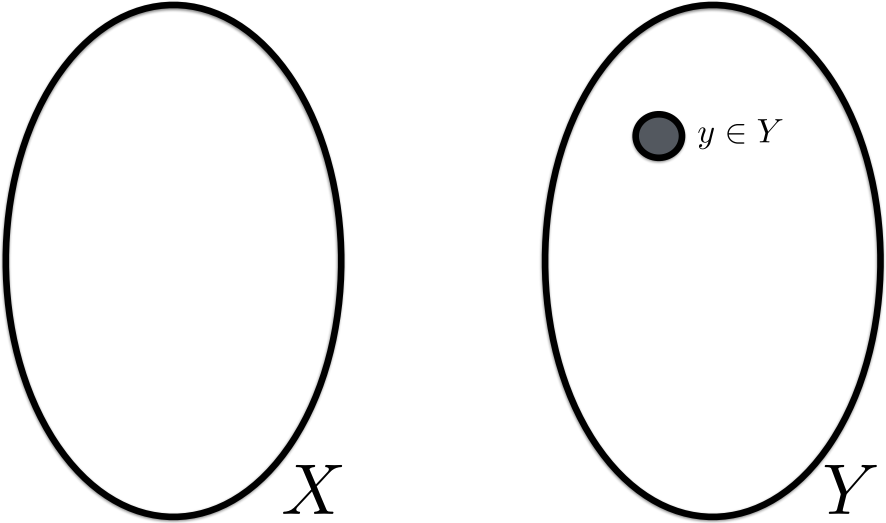
Existence and construction of the inverse

- Take a $y \in Y$. Question: is it possible to put y in correspondence to **ONE AND ONLY ONE** $x \in X$ such that $f(x) = y$?
- In other words...does the equation

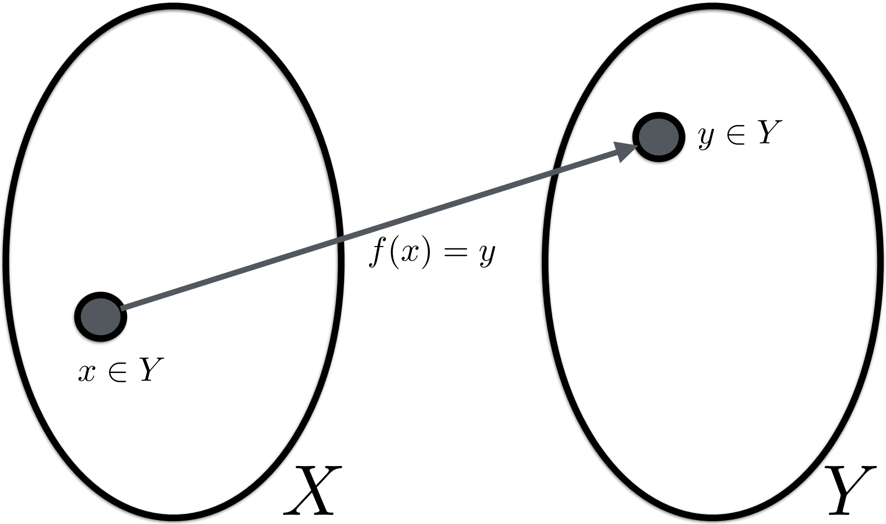
$$y = f(x)$$

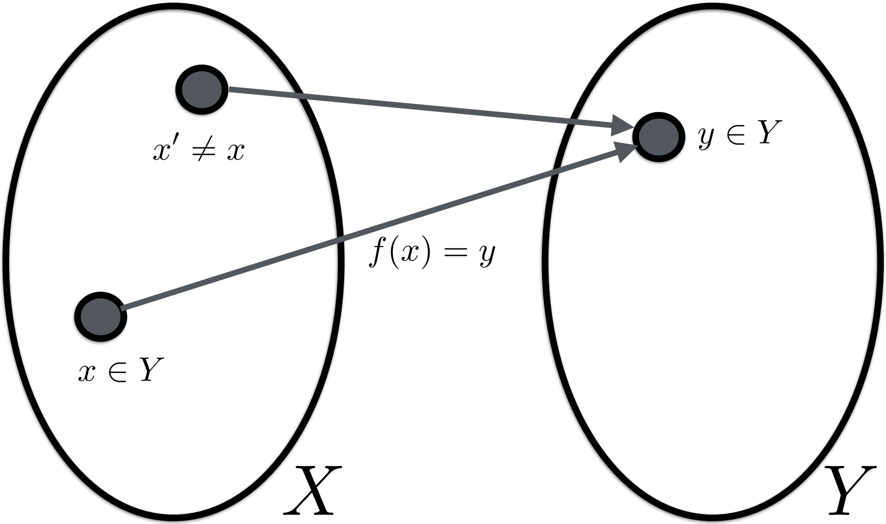
has a **unique** solution $x \in X$ for all $y \in Y$?
- Injective $\Rightarrow$ the solution exists. Surjective $\Rightarrow$ it is unique.


X
Y

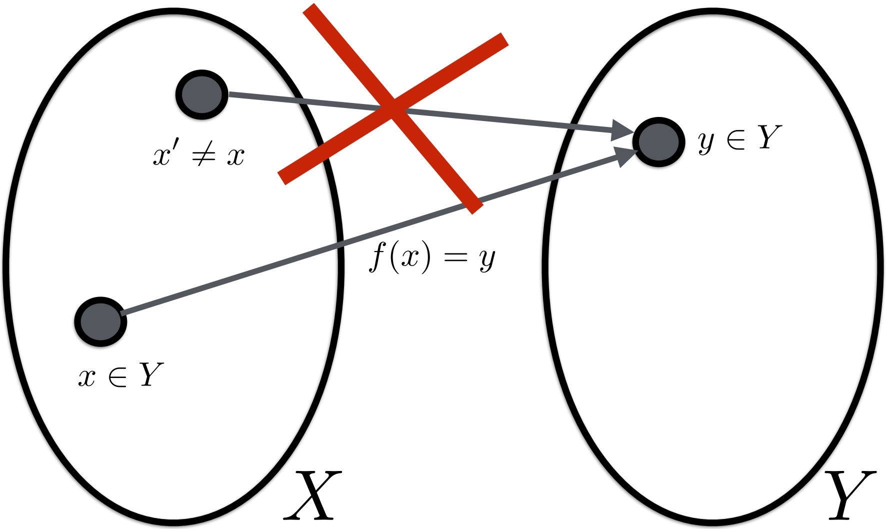

$y \in Y$
X
Y

It exists thanks to the surjectivity

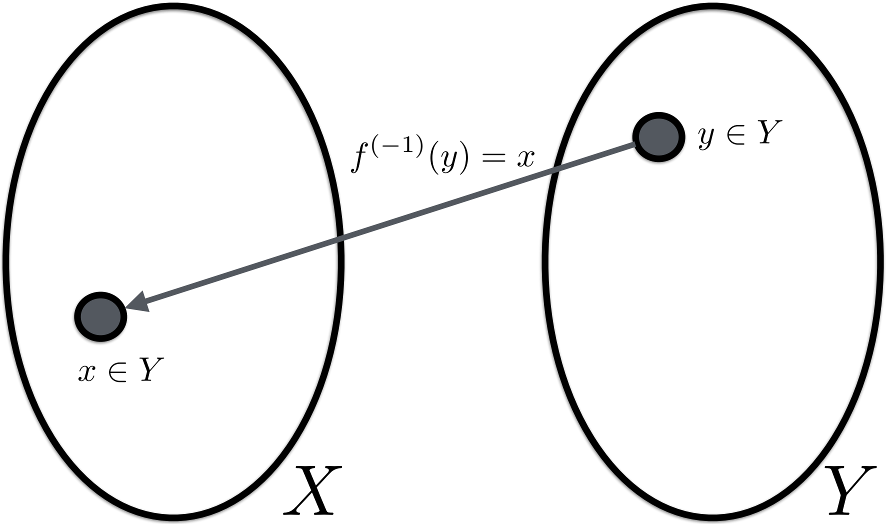


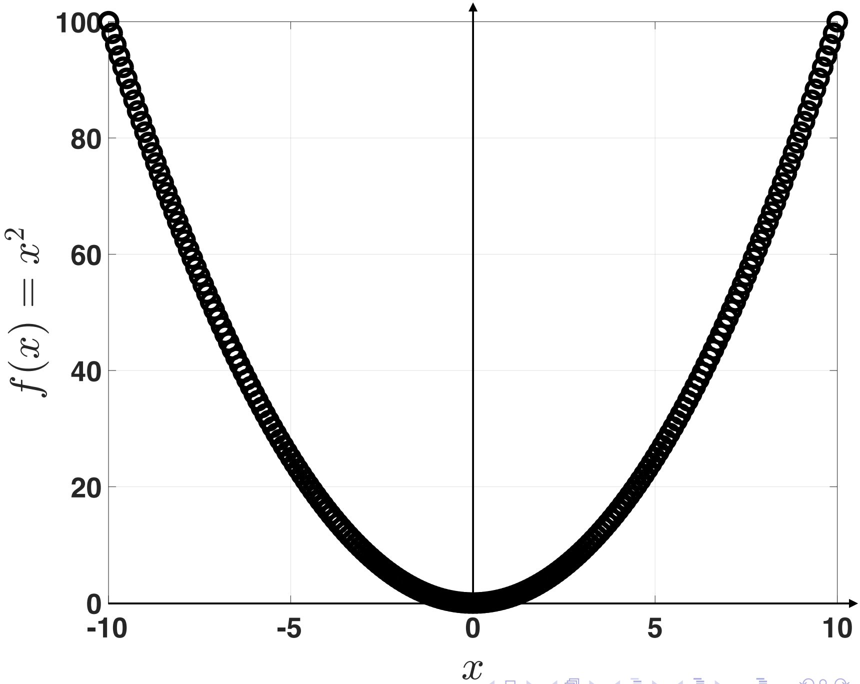

$x' \neq x$
$x \in Y$
$f(x) = y$
$y \in Y$
X
Y

IMPOSSIBLE! The function is injective!



So now do the following association...




100
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20
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-10
-5
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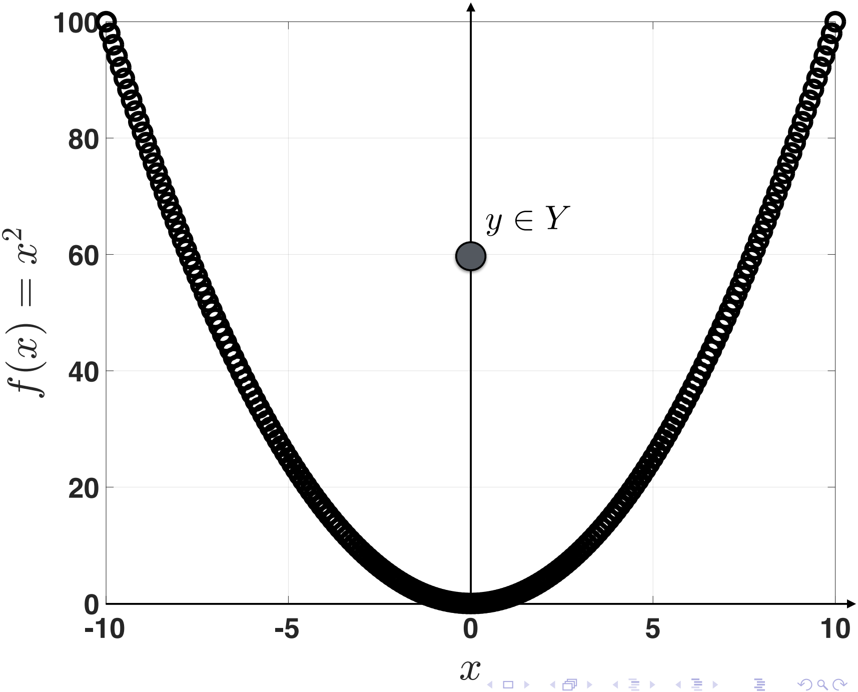
5
10
$f(x) = x^2$
x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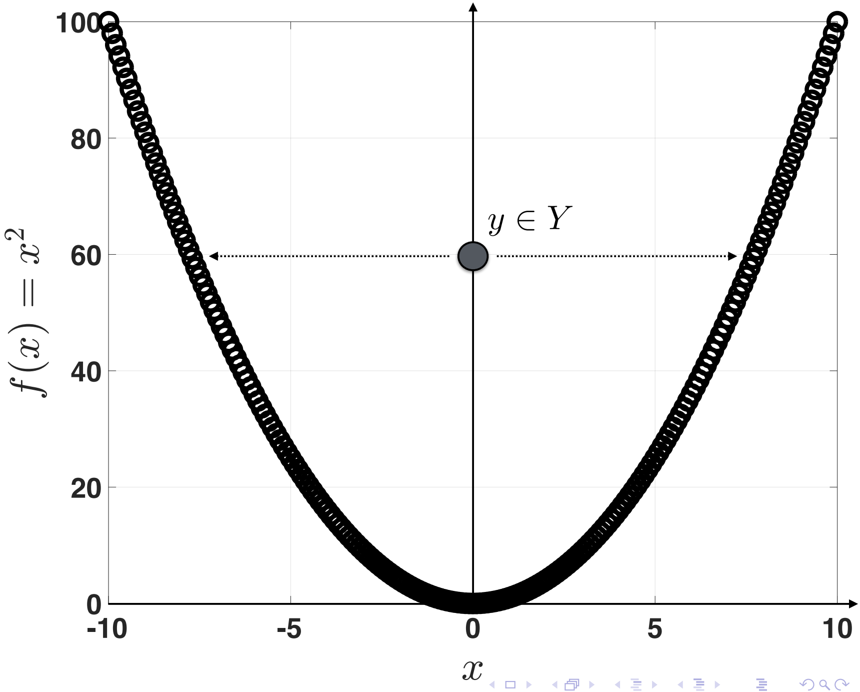
-10
-5
0
5
10
$f(x) = x^2$
x
$y \in Y$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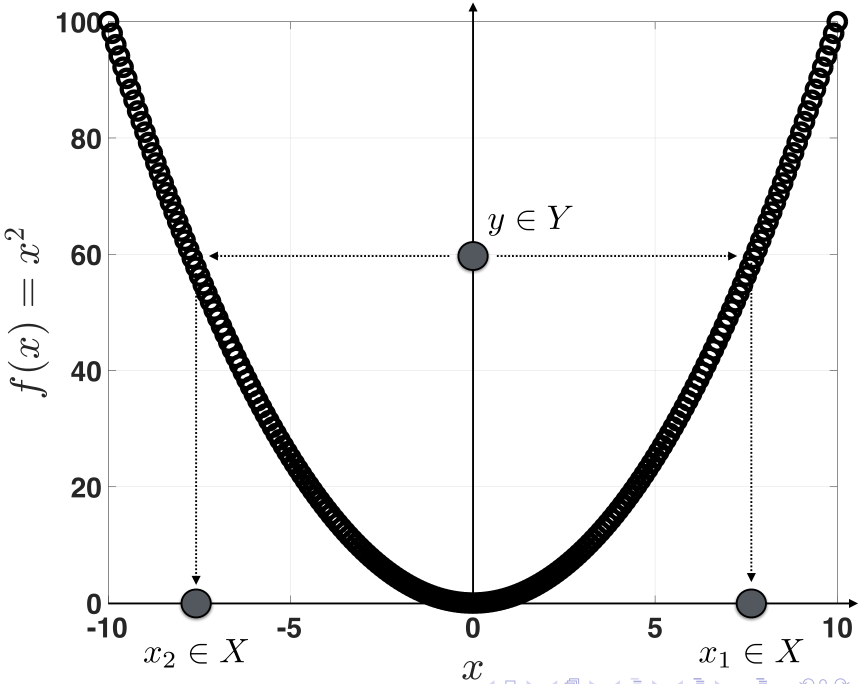
10
$f(x) = x^2$
$y \in Y$
x



100
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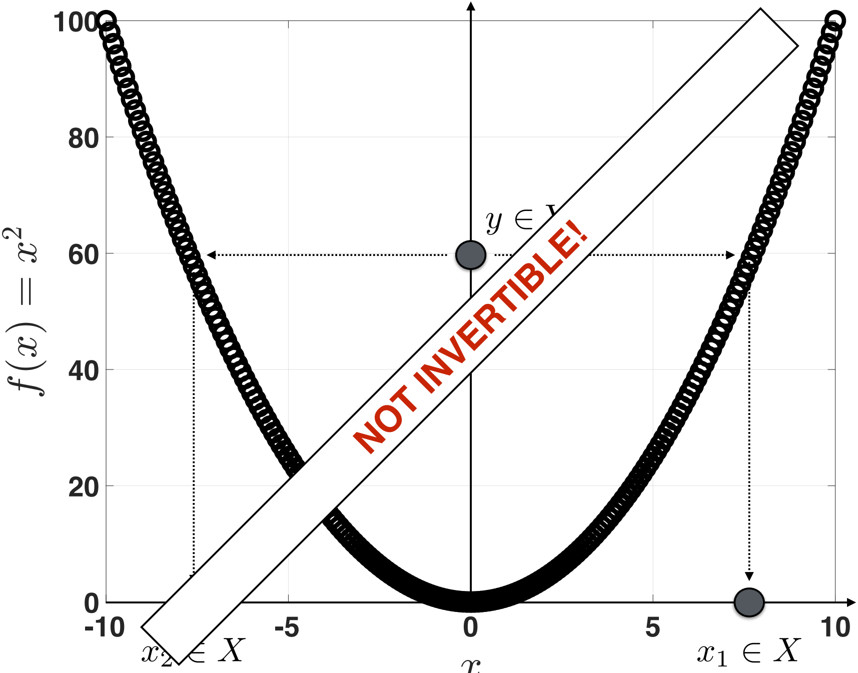
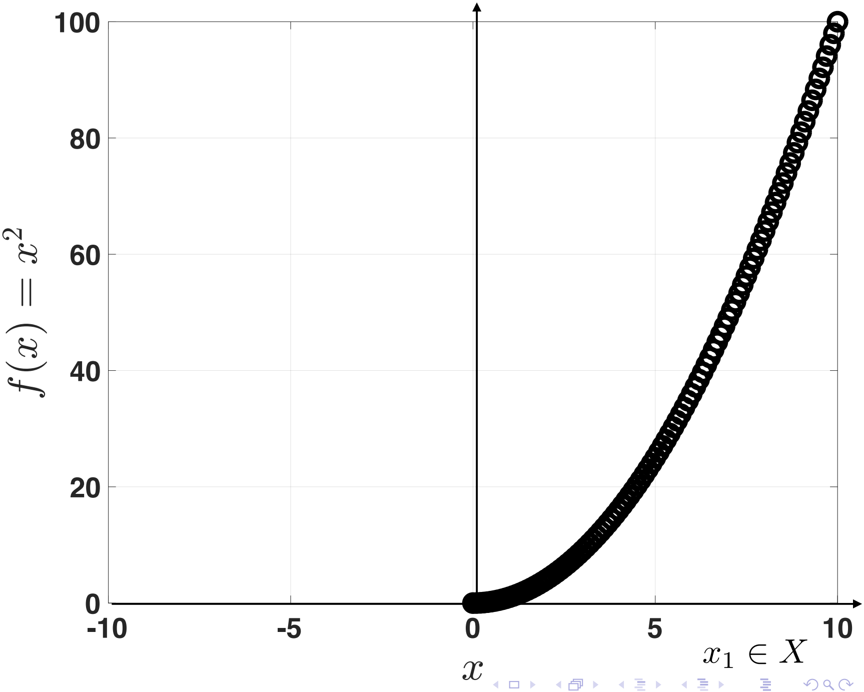
-10
-5
0
5
10
$f(x) = x^2$
$y \in Y$
$x_2 \in X$
$x_1 \in X$
x



NOT INVERTIBLE!
$y \in Y$
$f(x) = x^2$
x
$x_2 \in X$
$x_1 \in X$
100
80
60
40
20
0
-10
-5
0
5
10


100
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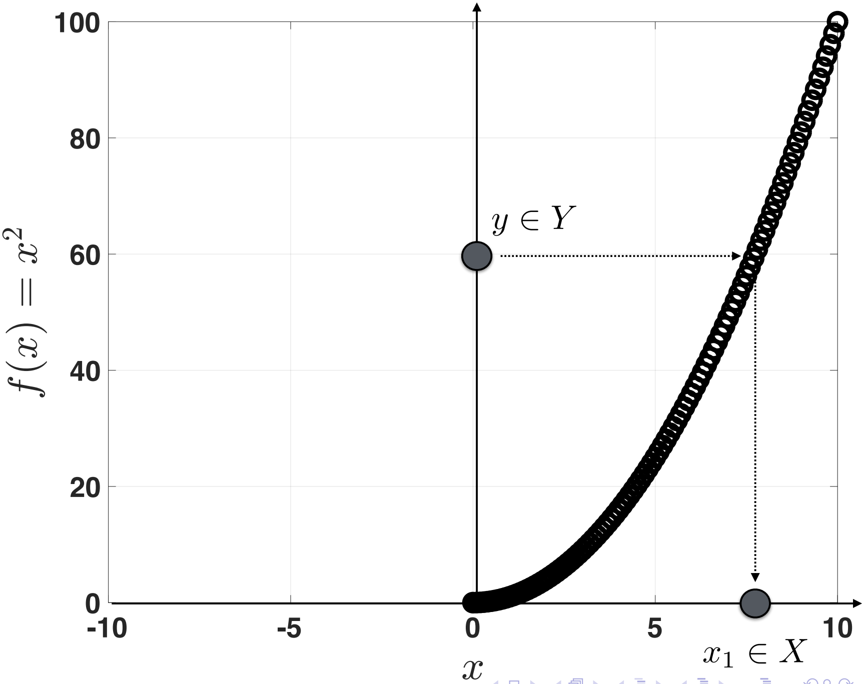
0
5
10
$f(x) = x^2$
$x_1 \in X$
x


100
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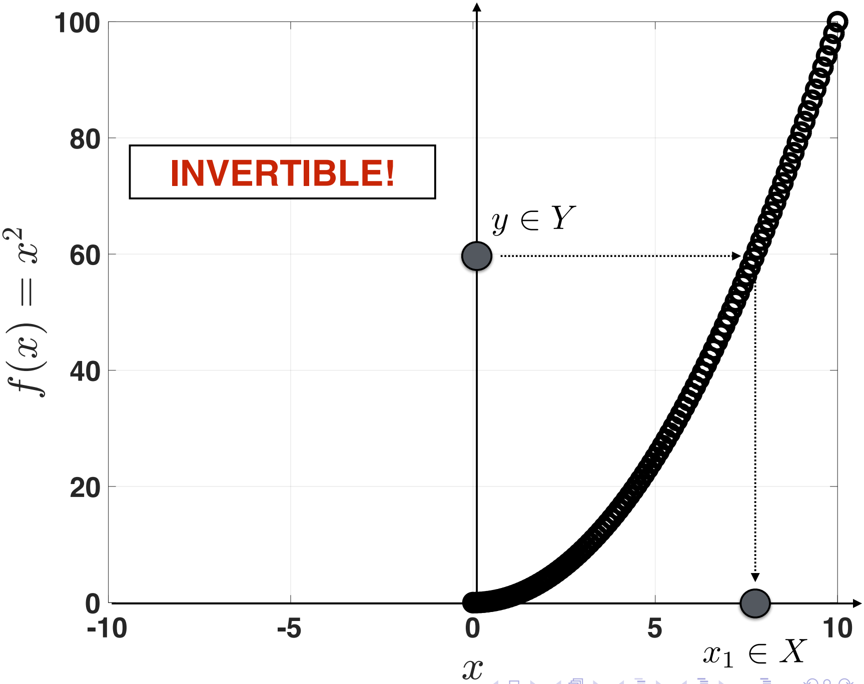
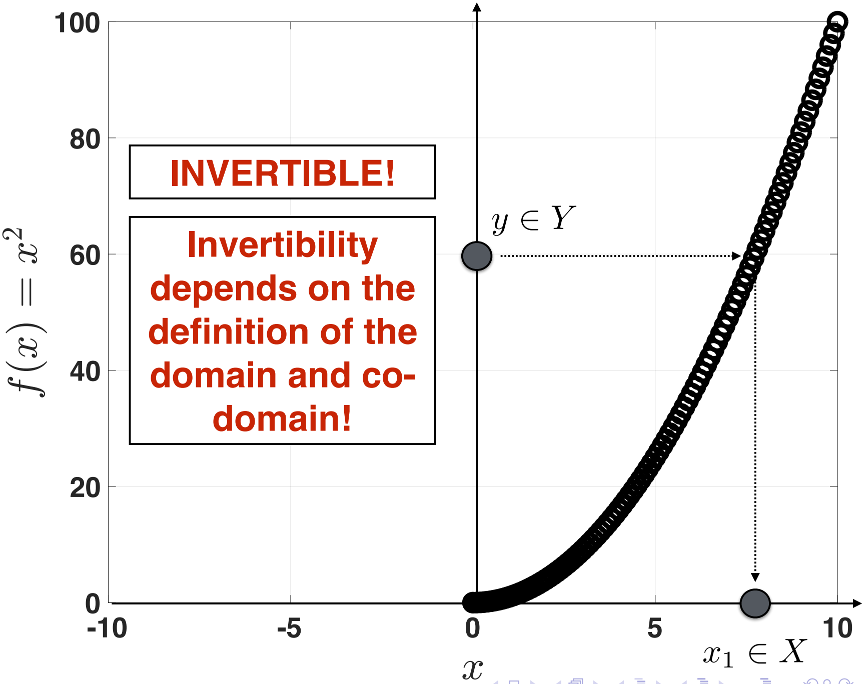
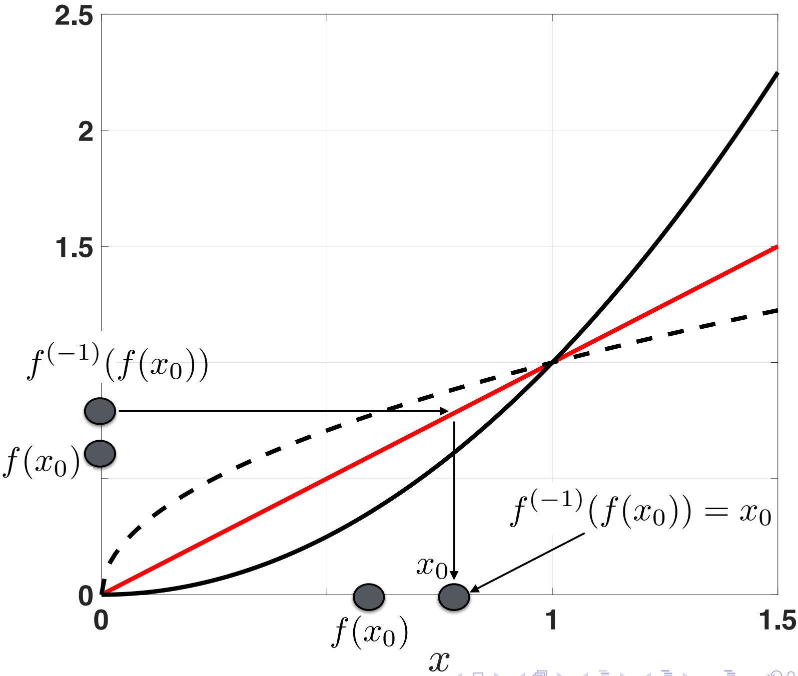
0
-10
-5
0
5
10
$f(x) = x^2$
$y \in Y$
$x_1 \in X$
x



INVERTIBLE!
$y \in Y$
$x_1 \in X$
$f(x) = x^2$
x
100
80
60
40
20
0
-10
-5
0
5
10


INVERTIBLE!
Invertibility depends on the definition of the domain and co-domain!
$y \in Y$
$x_1 \in X$
$f(x) = x^2$
x
100
80
60
40
20
0
-10
-5
0
5
10



2.5
2
1.5
0
0
1
1.5
$f^{(-1)}(f(x_0))$
$f(x_0)$
$f^{(-1)}(f(x_0)) = x_0$
x_0
$f(x_0)$
x

Theorem

For all $n \in \mathbb{N}$ and for all $y \in \mathbb{R}$ with $y > 0$ the equation

$$y = x^n$$

has a unique solution x which is called the n-th root of y and it is indicated with

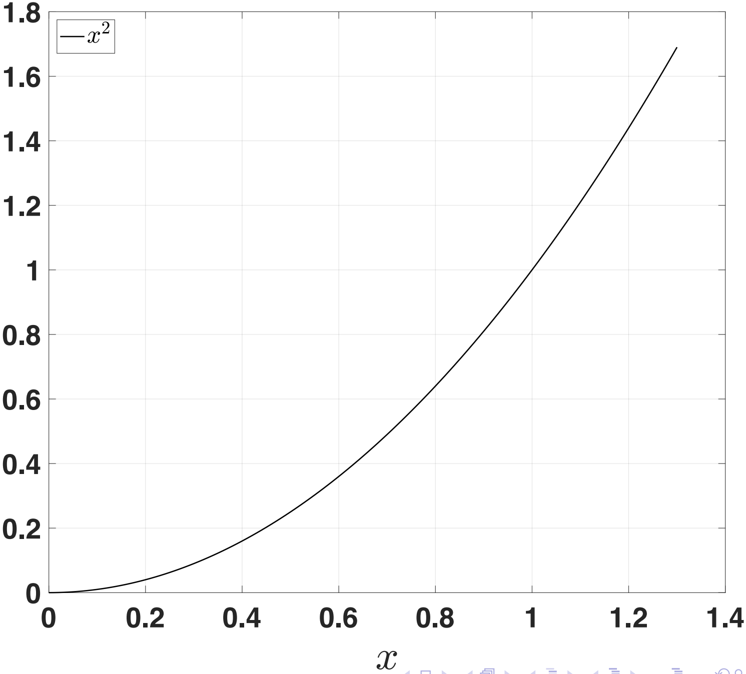
$$x = y^{\frac{1}{n}}.$$

Definition

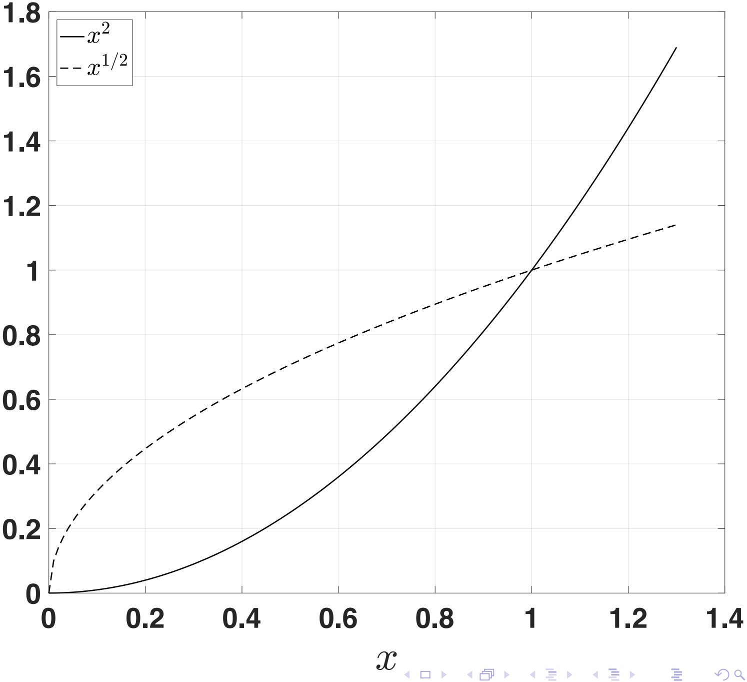
The function $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$f(x) = x^{\frac{1}{n}}$$

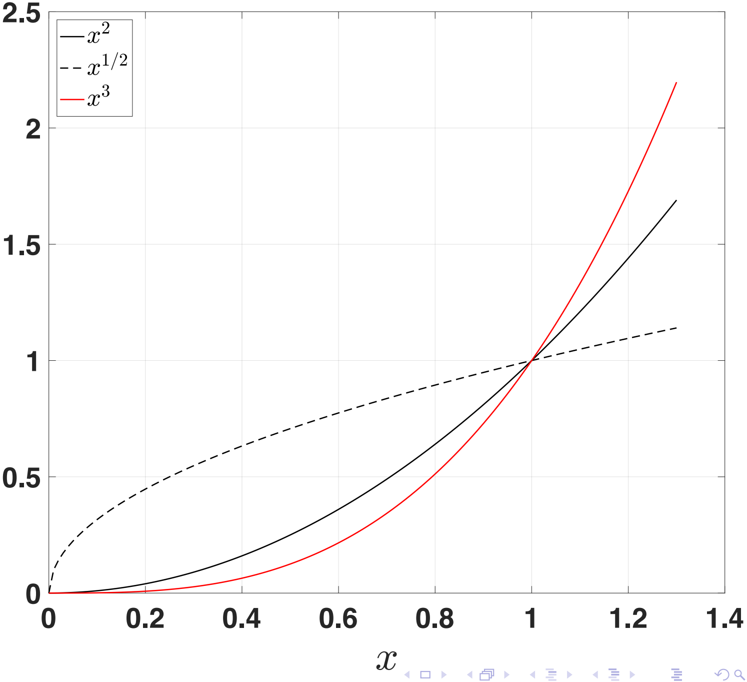
is called the radical function.

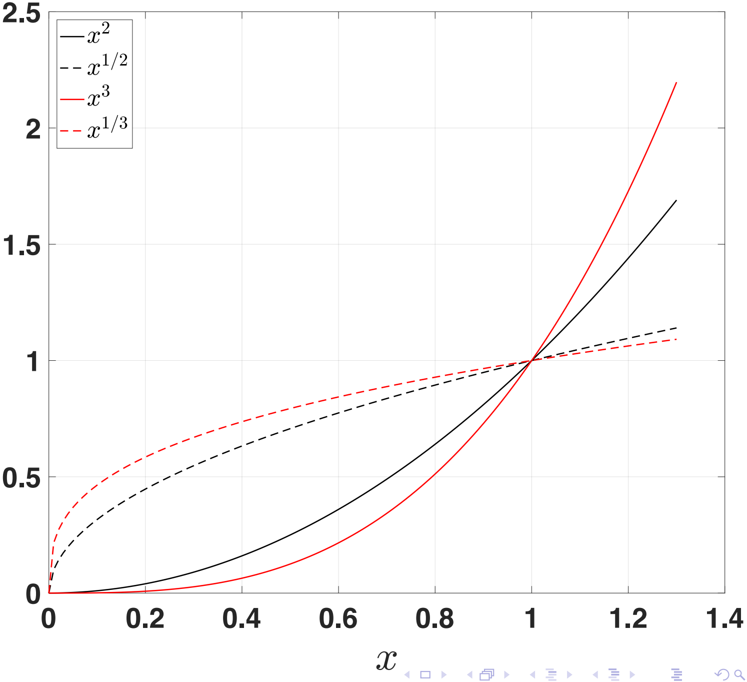


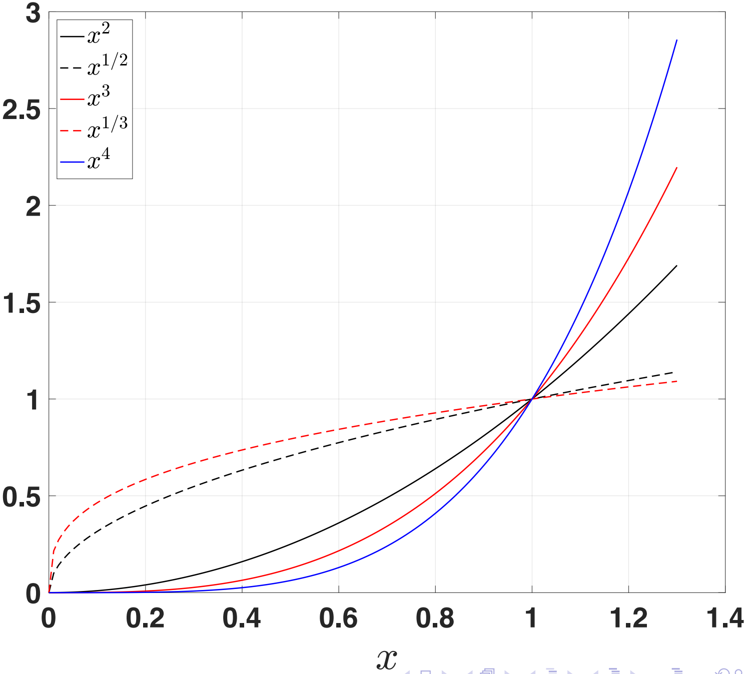
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
x^2
x



x^2
$x^{1/2}$
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
x


2.5
2
1.5
1
0.5
0
0.2
0.4
0.6
0.8
1.2
1.4
x^2
$x^{1/2}$
x^3
x


2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
x^2
$x^{1/2}$
x^3
$x^{1/3}$
x


x^2
$x^{1/2}$
x^3
$x^{1/3}$
x^4
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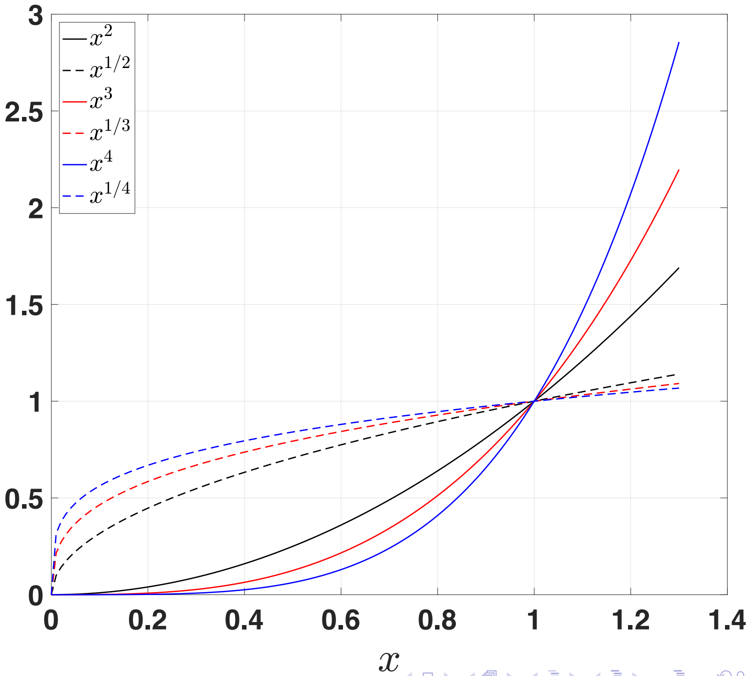
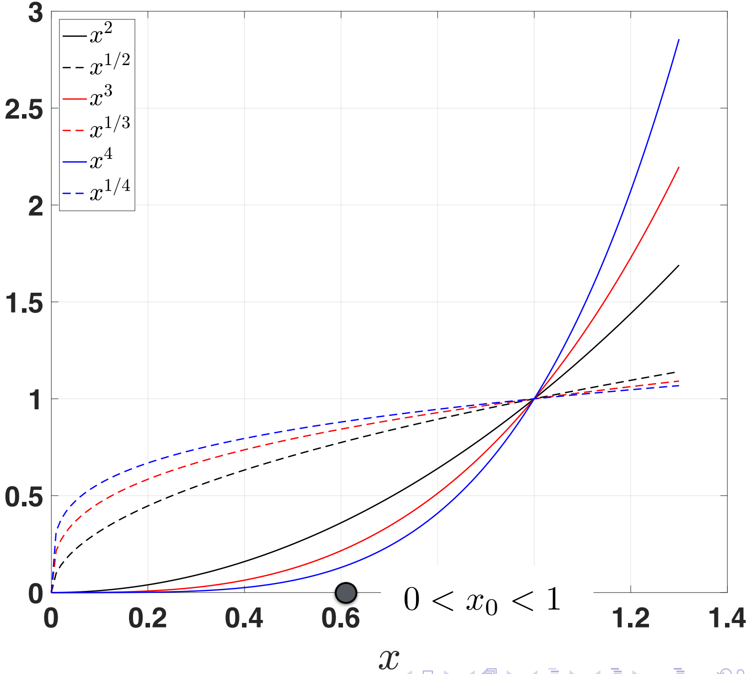
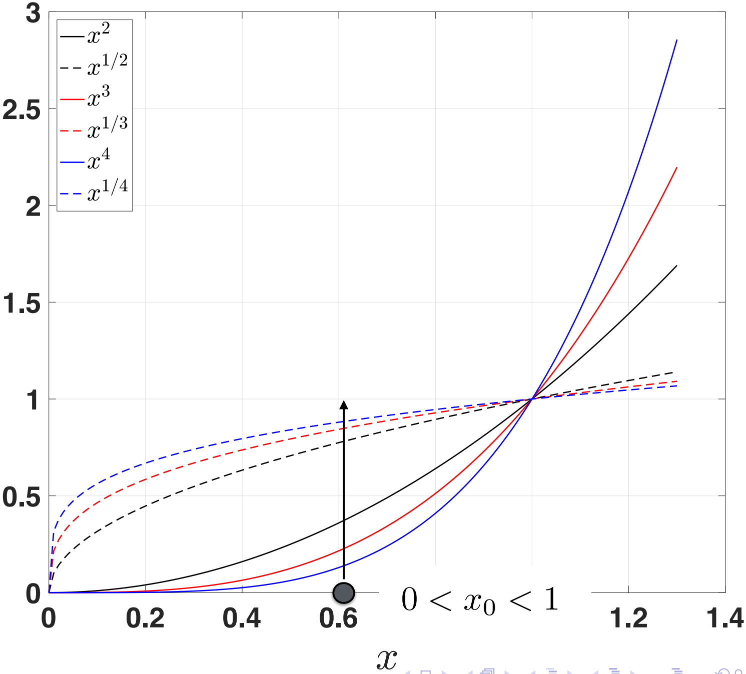
0.5
1
1.5
2
2.5
3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
x

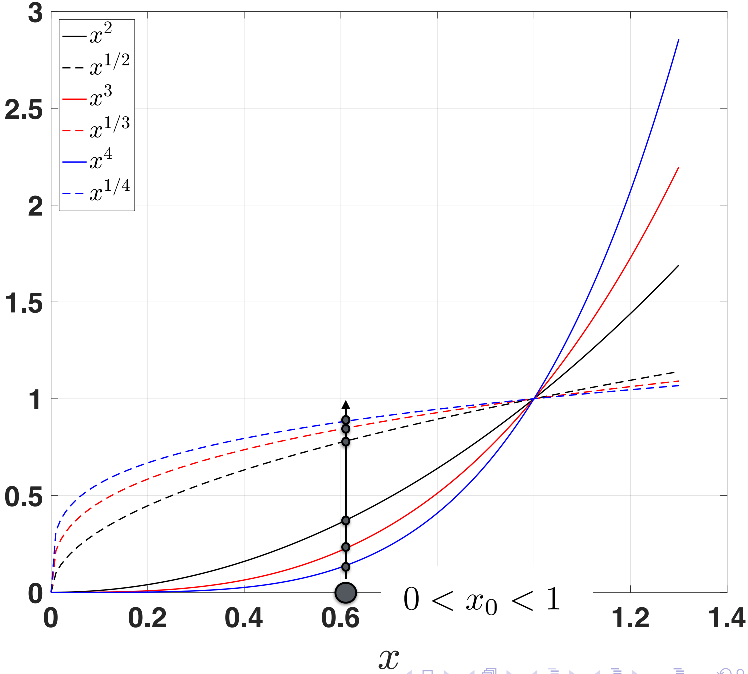


x^2
$x^{1/2}$
x^3
$x^{1/3}$
x^4
$x^{1/4}$
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
x

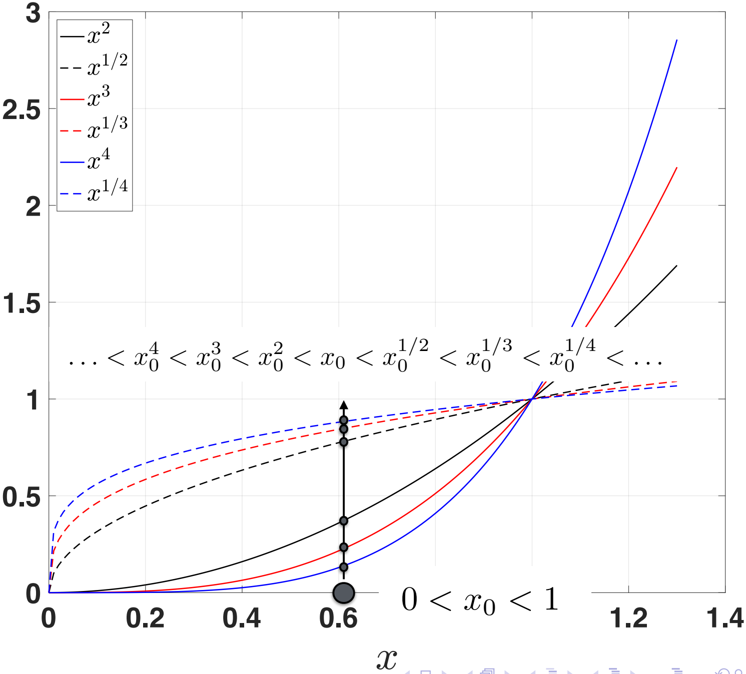

x^2
$x^{1/2}$
x^3
$x^{1/3}$
x^4
$x^{1/4}$
$0 < x_0 < 1$
x
0
0.5
1
1.5
2
2.5
3
0
0.2
0.4
0.6
1.2
1.4

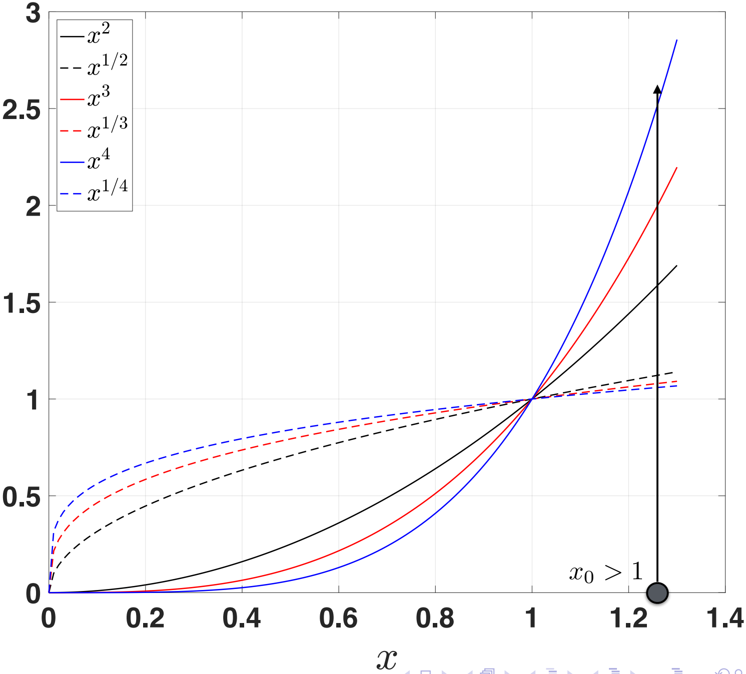


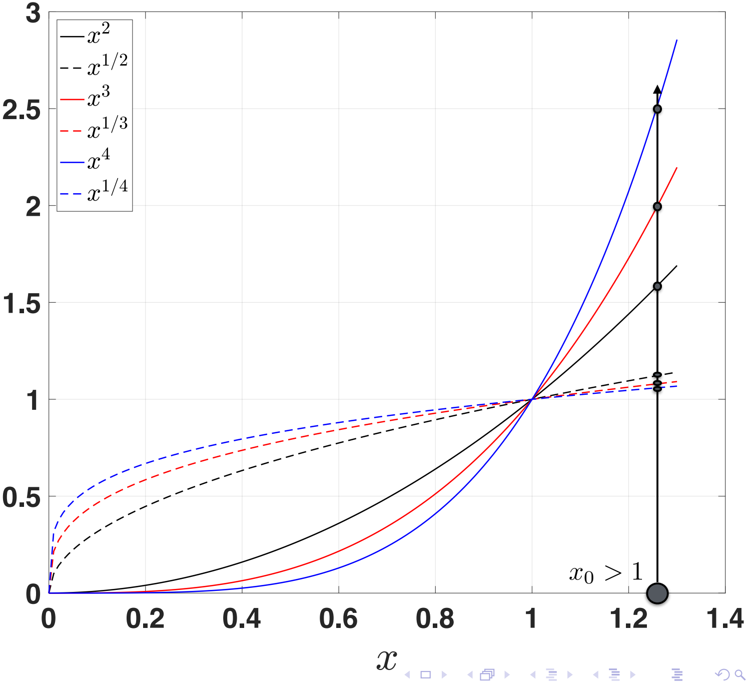
x^2
$x^{1/2}$
x^3
$x^{1/3}$
x^4
$x^{1/4}$
$0 < x_0 < 1$
0
0.2
0.4
0.6
1.2
1.4
0.5
1
1.5
2
2.5
3
x



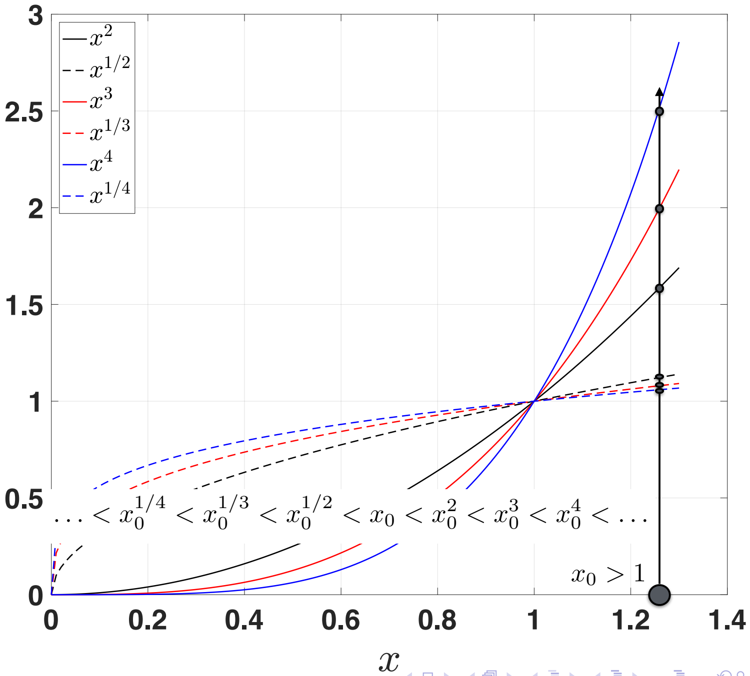
x^2
$x^{1/2}$
x^3
$x^{1/3}$
x^4
$x^{1/4}$
$0 < x_0 < 1$
x
0
0.5
1
1.5
2
2.5
3
0.2
0.4
0.6
1.2
1.4


x^2
$x^{1/2}$
x^3
$x^{1/3}$
x^4
$x^{1/4}$
$\ldots < x_0^4 < x_0^3 < x_0^2 < x_0 < x_0^{1/2} < x_0^{1/3} < x_0^{1/4} < \ldots$
$0 < x_0 < 1$
x
0
0.5
1
1.5
2
2.5
3
0.2
0.4
0.6
1.2
1.4


x^2
$x^{1/2}$
x^3
$x^{1/3}$
x^4
$x^{1/4}$
$x_0 > 1$
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
x



x^2
$x^{1/2}$
x^3
$x^{1/3}$
x^4
$x^{1/4}$
$x_0 > 1$
0
0.5
1
1.5
2
2.5
3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
x


x^2
$x^{1/2}$
x^3
$x^{1/3}$
x^4
$x^{1/4}$
$\ldots < x_0^{1/4} < x_0^{1/3} < x_0^{1/2} < x_0 < x_0^2 < x_0^3 < x_0^4 < \ldots$
$x_0 > 1$
x

The radical function

The following definition is now well-posed:

Definition

Let $q = \frac{n}{m}$ with $n \in \mathbb{N}$ and $m \in \mathbb{N}$, $m > 0$, be a rational number. For all $x > 0$ we define

$$x^{\frac{n}{m}} = \left(x^{\frac{1}{m}}\right)^n,$$

that is, the m-th root of x to the power n. It can be proved that this definition is equivalent to

$$x^{\frac{n}{m}} = (x^n)^{\frac{1}{m}},$$

that is, the m-th root of x to the power n. If $q < 0$ we set $x^q = \frac{1}{x^{-q}}$.

Example

$$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8, \quad 4^{\left(-\frac{3}{2}\right)} = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{8}.$$

Up to now we have defined ...

$$x^n \stackrel{\text{def}}{=} \underbrace{x \cdots x}_{n-\text{ times}}, \quad x^{-n} \stackrel{\text{def}}{=} \frac{1}{x^n},$$

and

$$x^{\frac{n}{m}} \stackrel{\text{def}}{=} (x^n)^{\frac{1}{m}}, \quad x^{-\frac{n}{m}} \stackrel{\text{def}}{=} \frac{1}{x^{\frac{n}{m}}}.$$

Now consider an irrational number, for example

$$\sqrt{2} = 1.\underbrace{41421356237309504880168872420 9\ldots}_{\text{Infinite, non-periodic, number of digits.}}$$

what does it mean

$$3^{\sqrt{2}} \quad ??$$

Is it possible to define 3^x for **ALL** $x \in \mathbb{R}$? We need the concept of **SEQUENCES**!

Sequences

Definition

A sequence of real number $(s_n)_{n\in\mathbb{N}}$ is any function

$$\begin{aligned} s:\mathbb{N} &\to \mathbb{R} \\ n &\to s(n)=s_n. \end{aligned}$$

The number s_n is also called the n-th element of the sequence.

Example

Consider the sequence $s_n=\frac{1}{n}$, hence

$$s_1=1,\quad s_2=\frac{1}{2},\quad s_3=\frac{1}{3},\quad s_4=\frac{1}{4},\ldots$$

or for example $s_n=\frac{n}{n+1}$, hence

$$s_1=\frac{1}{2},\quad s_2=\frac{2}{3},\quad s_3=\frac{3}{4},\quad s_4=\frac{4}{5},\ldots$$

Limit of sequences

What happens to a sequence s_n when n is "very large" ?

n	s_n		
	$\frac{1}{n}$	$\frac{n}{n+1}$	$(-1)^n$
1	1	$\frac{1}{2} = 0.5$	-1
2	$\frac{1}{2} = 0.5$	$\frac{2}{3} = 0.6667$	1
3	$\frac{1}{3} = 0.333$	$\frac{3}{4} = 0.75$	-1
4	$\frac{1}{4} = 0.25$	$\frac{4}{5} = 0.8$	1
5	$\frac{1}{5} = 0.2$	$\frac{5}{6} = 0.8333$	-1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- The elements of $s_n = 1/n$ seem to approach 0 as n grows.
- The elements of $s_n = n/(n+1)$ seem to approach 1 as n grows.
- The elements of $s_n = (-1)^n$ swings between 1 and -1.

The idea

For a sequence of real numbers s_n, if it is possible to find a number $\ell \in \mathbb{R}$, such that s_n is arbitrarily close to ℓ provided that n is sufficiently large...then I say that ℓ is the limit of the sequnce.

Definition

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If it exists $\ell \in \mathbb{R}$ such that

$$\forall \varepsilon > 0, \exists n^* : \forall n > n^* \Rightarrow |s_n - \ell| < \varepsilon.$$

then ℓ is called the limit of the sequence and we use the notation

$$\ell = \lim_{n\to\infty} s_n$$

or we say $s_n \to \ell$ as $n \to \infty$.

Exercize

Prove that, according to the definition of limit,

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Solution. Take an arbitrarily small $\varepsilon > 0$. We have to prove that there exists a $n^* \in \mathbb{N}$ such that for all $n > n^*$ it holds that $\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon$. It is enough to take

$$n^* = \min\left\{ n \in \mathbb{N} \;\middle|\; n \geq \frac{1}{\varepsilon} \right\}$$

because if $n > n^*$ then

$$\frac{1}{n} < \frac{1}{n^*} < \varepsilon.$$

Exercize

Prove that, according to the definition of limit,

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Solution. Take an arbitrarily small $\varepsilon > 0$. We have to prove that there exists a $n^* \in \mathbb{N}$ such that for all $n > n^*$ it holds that

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-n-1}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \varepsilon,$$

but this is exactly what we have done in the previous exercise.

Toward the exponential function...

Consider the following iterative rule:

$$\begin{cases} q_0 & = 1 \\ q_{n+1} & = \frac{q_n}{2} + \frac{1}{q_n}. \end{cases}$$

that is

$$\begin{aligned} q_1 &= \frac{q_0}{2} + \frac{1}{q_0} = \frac{1}{2} + 1 = \frac{3}{2} = \mathbf{1}.5 \in \mathbb{Q} \\ q_2 &= \frac{q_1}{2} + \frac{1}{q_1} = \frac{\frac{3}{2}}{2} + \frac{2}{3} = \frac{17}{12} = \mathbf{1.41}6 \in \mathbb{Q} \\ q_3 &= \frac{q_2}{2} + \frac{1}{q_2} = \frac{\frac{17}{12}}{2} + \frac{12}{17} = \frac{577}{408} = \mathbf{1.41421}5 \in \mathbb{Q}. \end{aligned}$$

Note that $q_n \in \mathbb{Q}$ for all n so 3^{q_n} is well-defined for all n. Besides, we will prove that $q_n \to \sqrt{2}$ as $n \to +\infty$.

The idea

I define

$$3^{\sqrt{2}} \stackrel{\text{def}}{=} \lim_{n\to\infty} 3^{q_n}.$$

Definition

Let a be a positive number with either $0 < a < 1$ or $a > 1$.

For all $x \in \mathbb{R}$ let q_n be a sequence of points of $\mathbb{Q}$, i.e. $q_n \in \mathbb{Q}$ for all n, such that

$$x = \lim_{n\to\infty} q_n.$$

Then the exponential function $f(x) = a^x$ is defined as

$$a^x = \lim_{n\to\infty} a^{q_n}.$$

If $x < 0$ we define $a^x = \frac{1}{a^{-x}}$.

Theorem

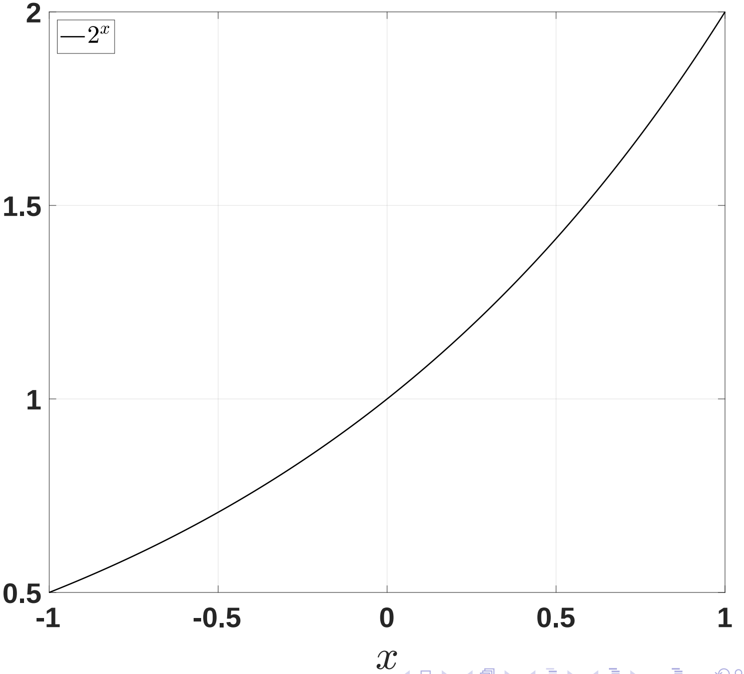
It can be proved that the definition above ***does not depend*** *on the chosen sequence* q_n.

Properties of the exponential function

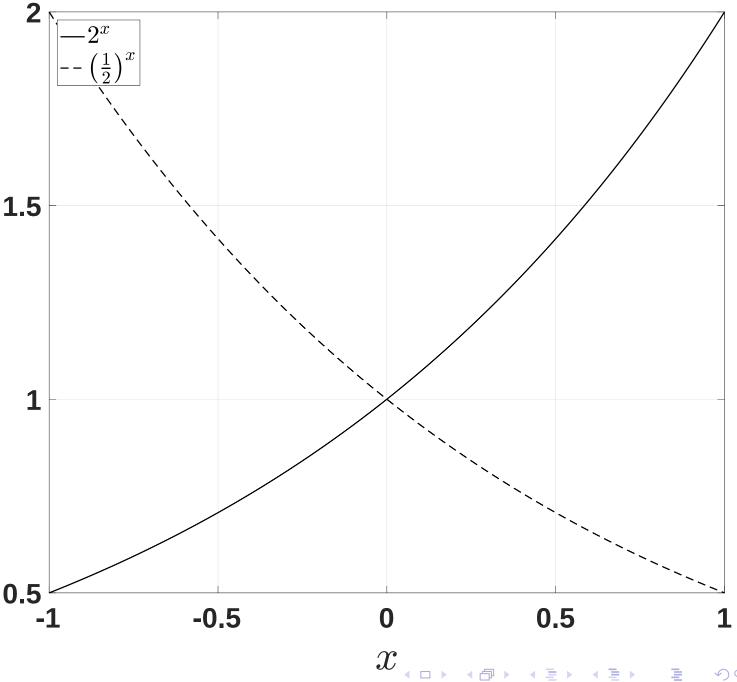
Let a be a positive real number with either $0 < a < 1$ or $a > 1$.

- The function $f(x) = a^x$ is defined for all $x \in \mathbb{R}$ with values in $\mathbb{R}^+ \setminus \{0\}$, that is
$$\forall x \in \mathbb{R} \Rightarrow a^x > 0.$$
- $a^0 = 1$.
- Given x and y it holds that
$$a^{x+y} = a^x\, a^y, \quad (a^x)^y = (a^y)^x = a^{x\,y}.$$
- If $0 < a < 1$ the function is strictly decreasing.
- If $a > 1$ the function is strictly increasing.

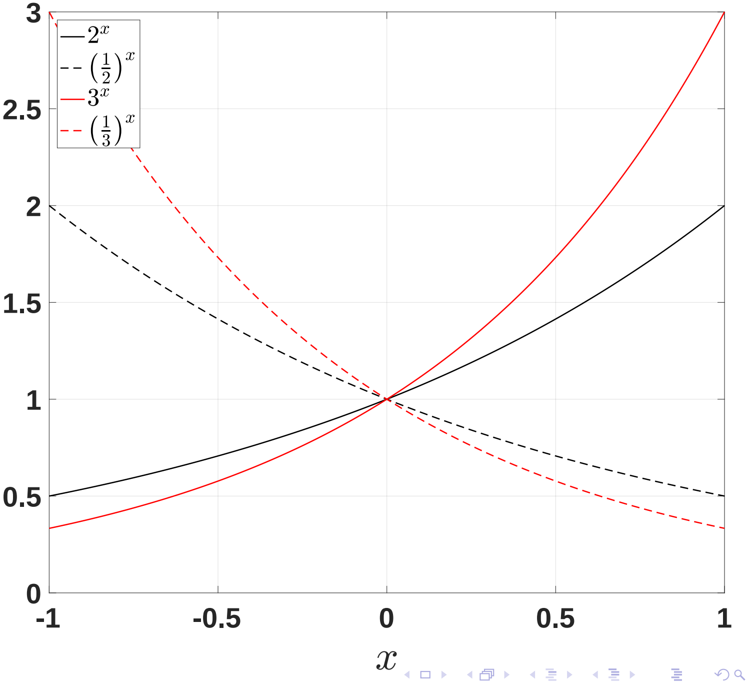
x	2^x	$\left(\frac{1}{2}\right)^x$
-100	$2^{-100} = \frac{1}{2^{100}} = 0.000....$	$\left(\frac{1}{2}\right)^{-100} = 2^{100} = \text{Huge number!}$
$\vdots$	$\vdots$	$\vdots$
-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	$\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$
-1	$2^{-1} = \frac{1}{2}$	$\left(\frac{1}{2}\right)^{-1} = 2$
0	$2^0 = 1$	$\left(\frac{1}{2}\right)^0 = 1$
1	$2^1 = 2$	$\left(\frac{1}{2}\right)^1 = \frac{1}{2}$
2	$2^2 = 4$	$\left(\frac{1}{2}\right)^2 = \frac{1}{2^2} = \frac{1}{4}$
3	$2^3 = 8$	$\left(\frac{1}{2}\right)^3 = \frac{1}{2^3} = \frac{1}{8}$
$\vdots$	$\vdots$	$\vdots$
100	$2^{100} = \text{Huge number!}$	$\left(\frac{1}{2}\right)^{100} = 2^{100} = 0.000...$

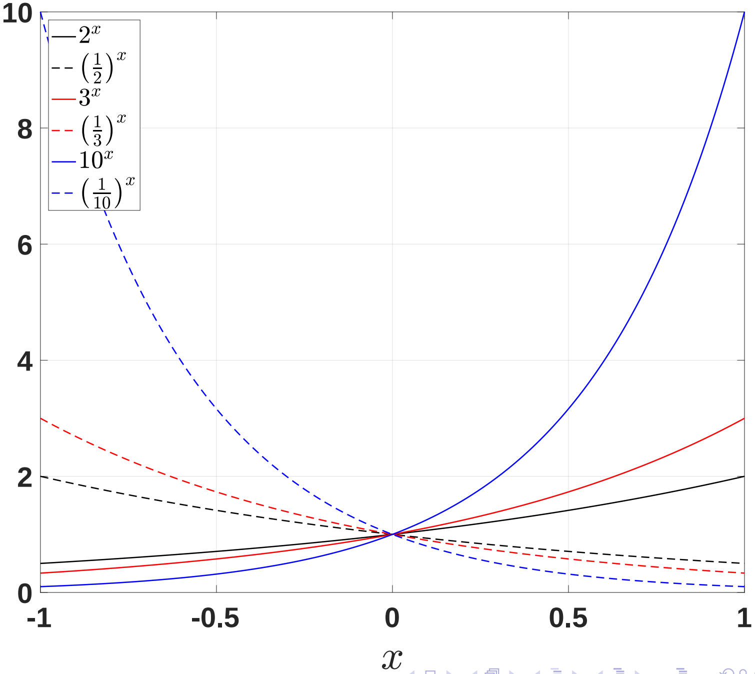


2
1.5
1
0.5
-1
-0.5
0
0.5
1
2^x
x

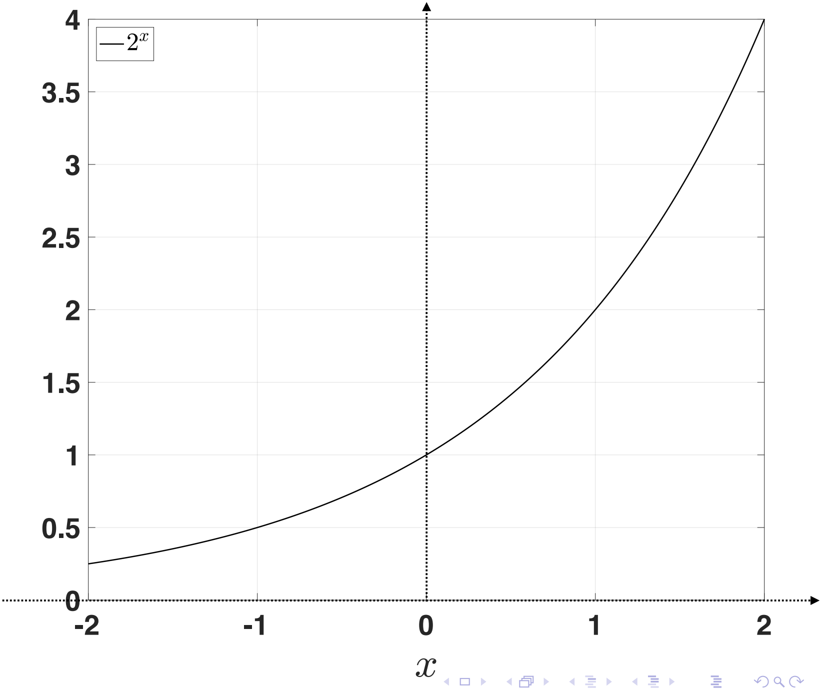


2^x
$\left(\frac{1}{2}\right)^x$
2
1.5
1
0.5
-1
-0.5
0
0.5
1
x

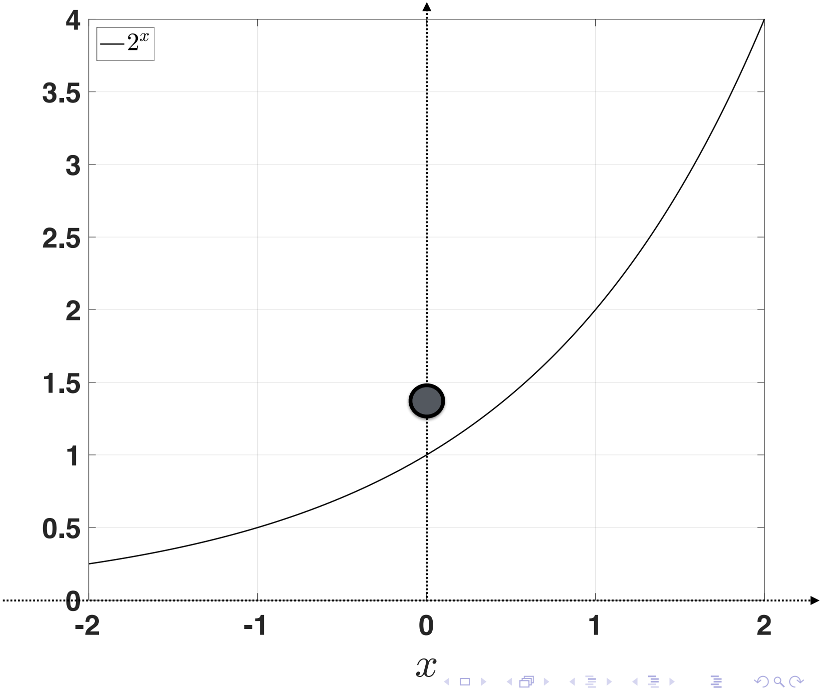

2^x
$\left(\frac{1}{2}\right)^x$
3^x
$\left(\frac{1}{3}\right)^x$
3
2.5
2
1.5
1
0.5
0
-1
-0.5
0
0.5
1
x



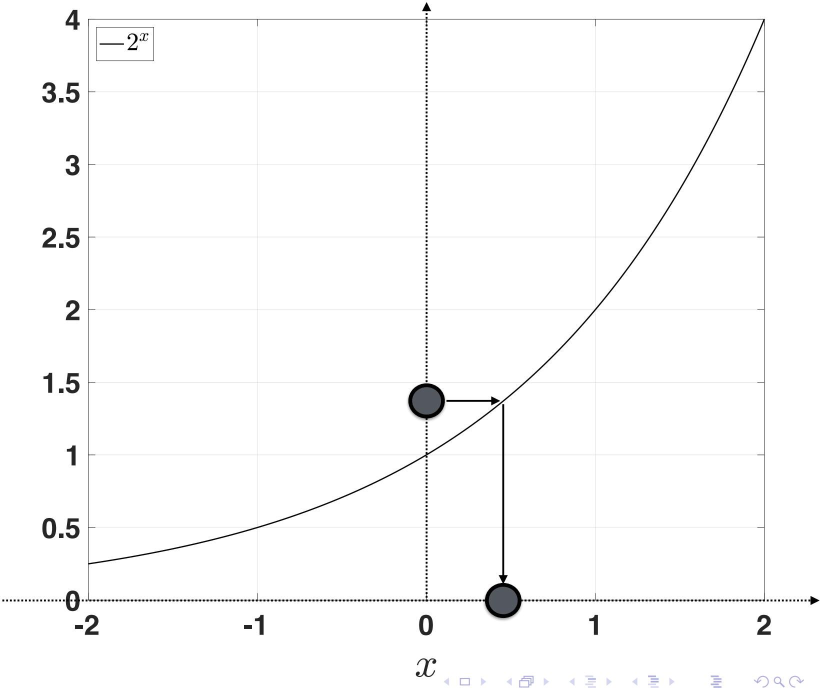
10
8
6
4
2
0
-1
-0.5
0
0.5
1
x
2^x
$\left(\frac{1}{2}\right)^x$
3^x
$\left(\frac{1}{3}\right)^x$
10^x
$\left(\frac{1}{10}\right)^x$



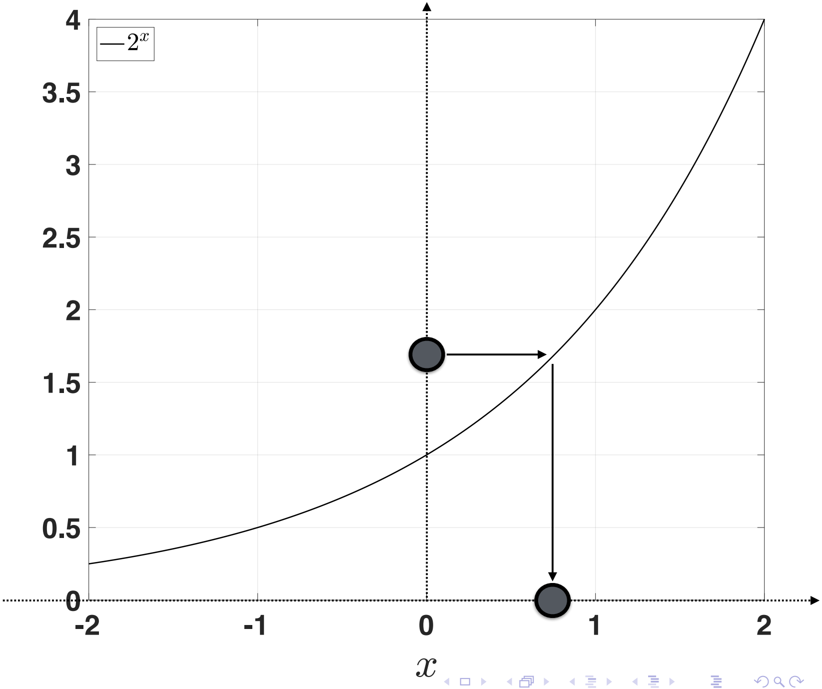
—2^x
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
x



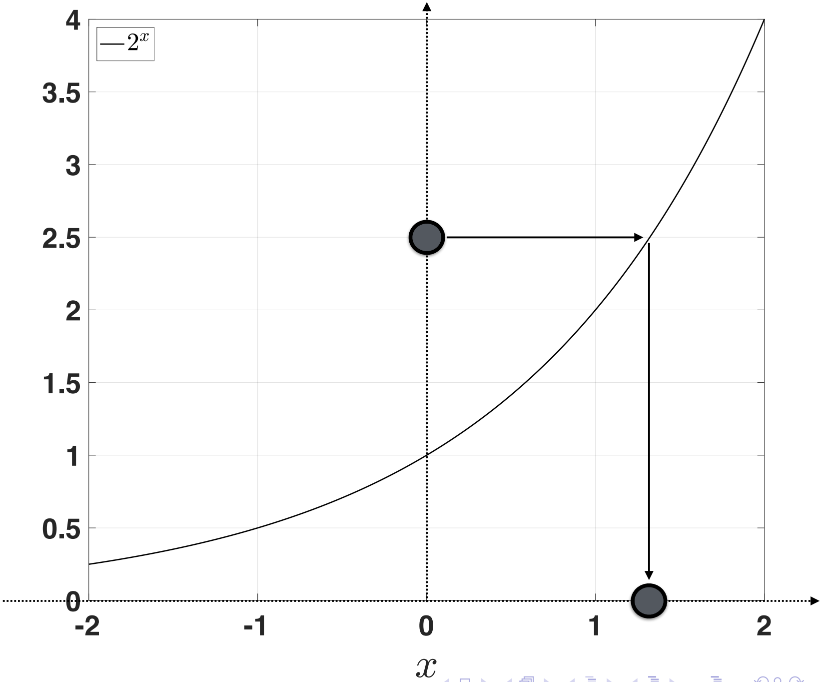
2^x
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
x



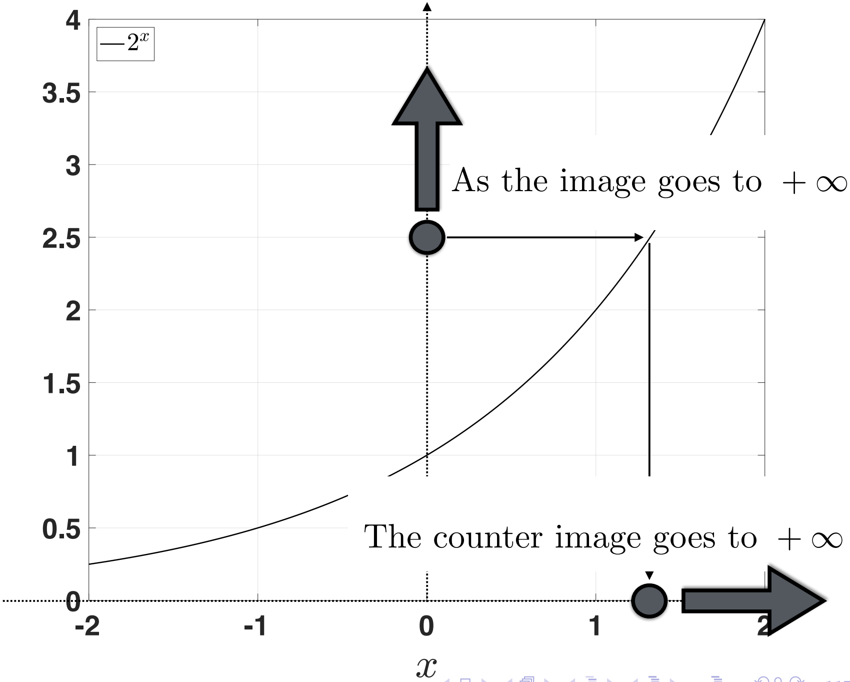
2^x
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
x



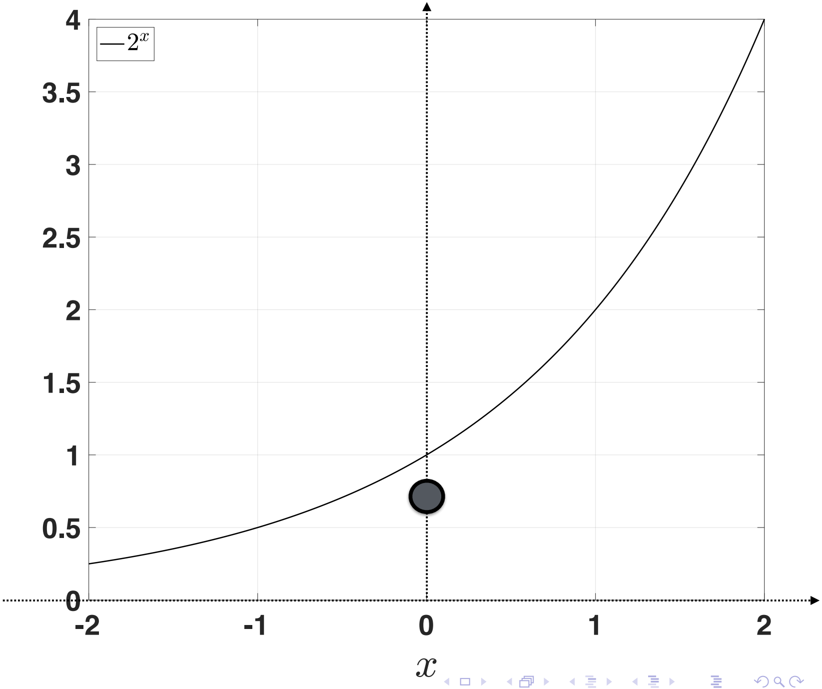
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
2^x
x



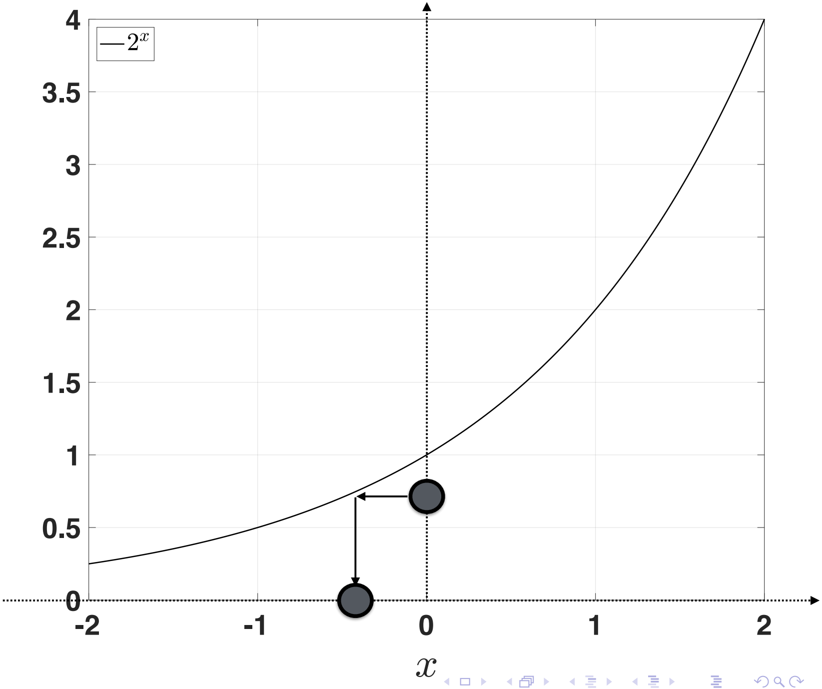
2^x
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
x



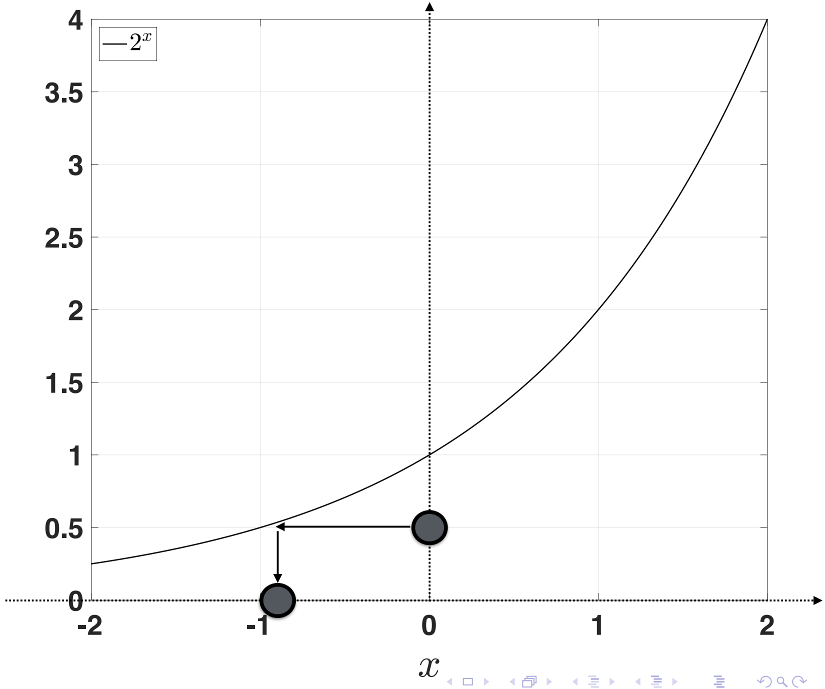
2^x
As the image goes to $+\infty$
The counter image goes to $+\infty$
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
x



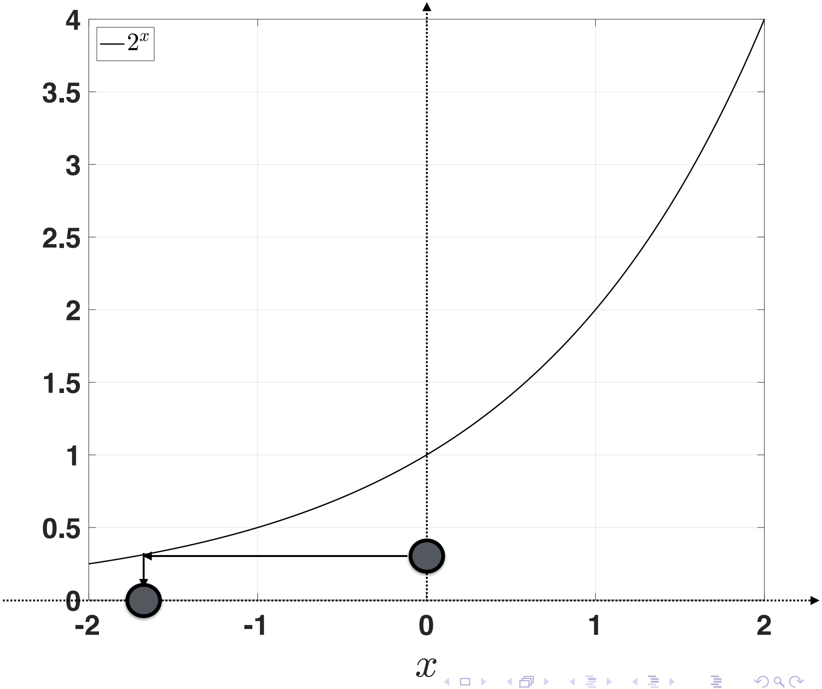
2^x
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
x



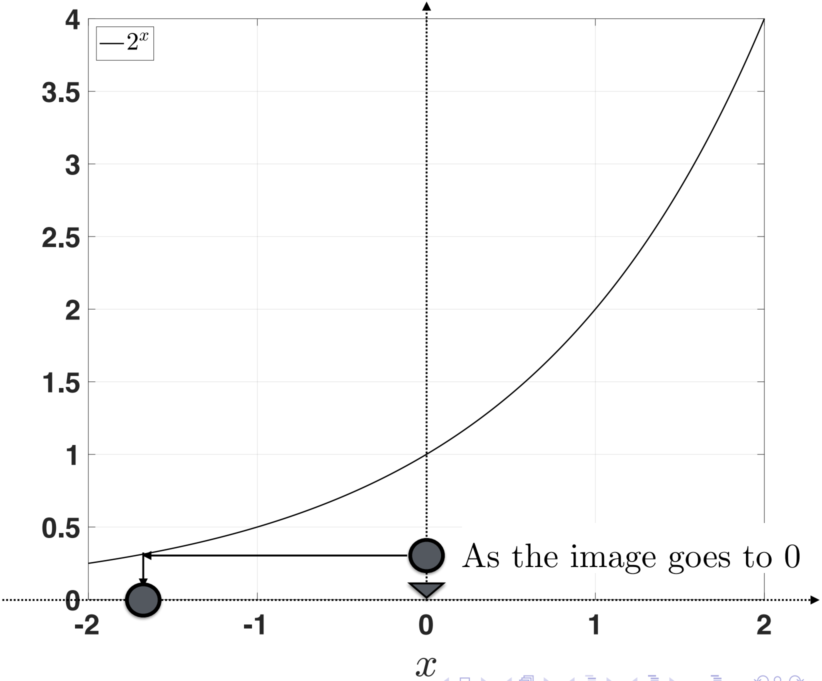
2^x
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
x

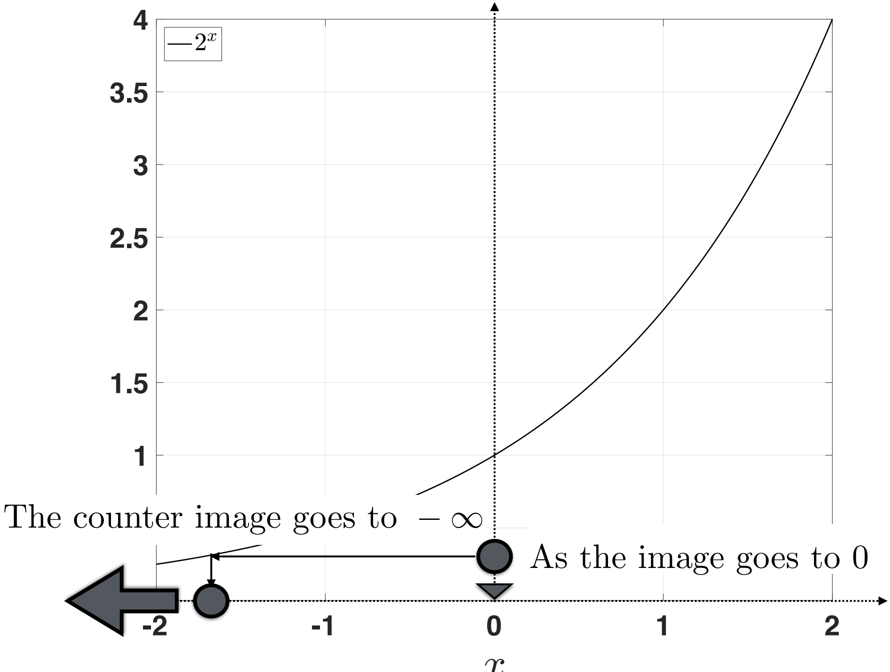


2^x
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
x

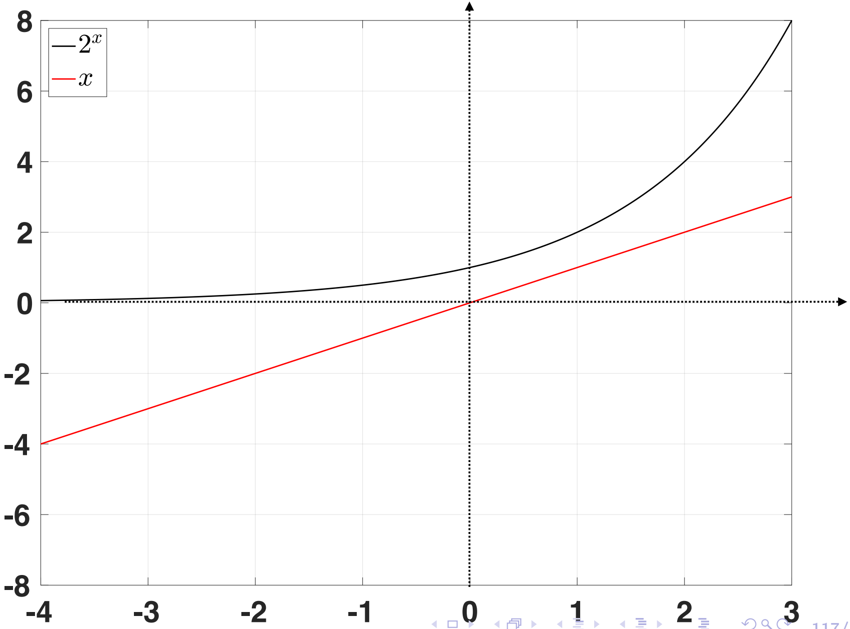


2^x
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
x

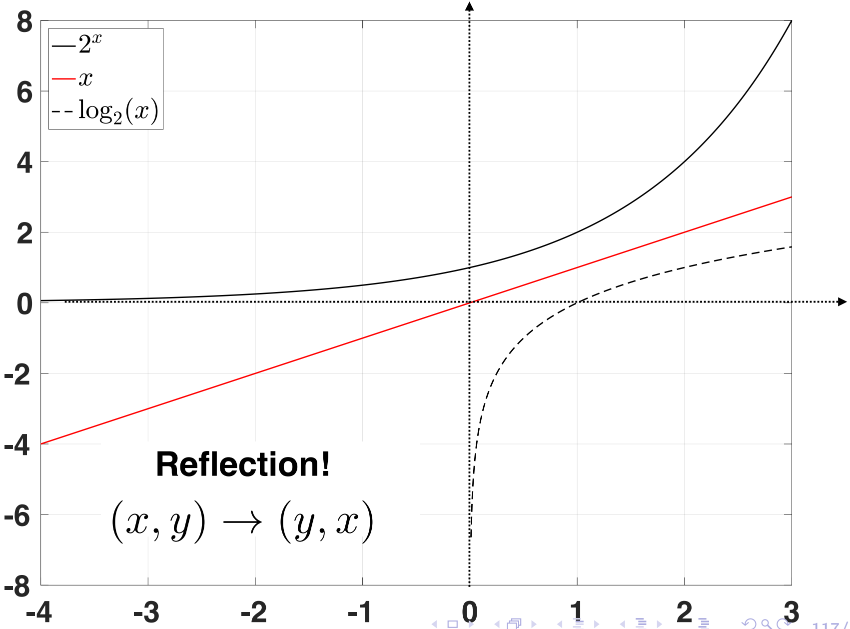

2^x
4
3.5
3
2.5
2
1.5
1
0.5
0
-2
-1
0
1
2
x
As the image goes to 0



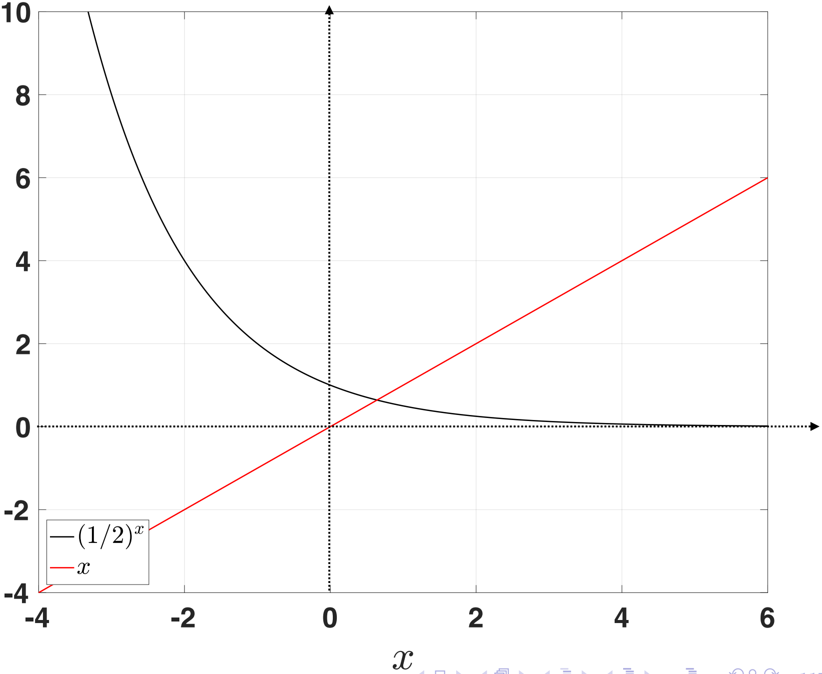
4
3.5
3
2.5
2
1.5
1
2^x
The counter image goes to $-\infty$
As the image goes to 0
-2
-1
0
1
2
x



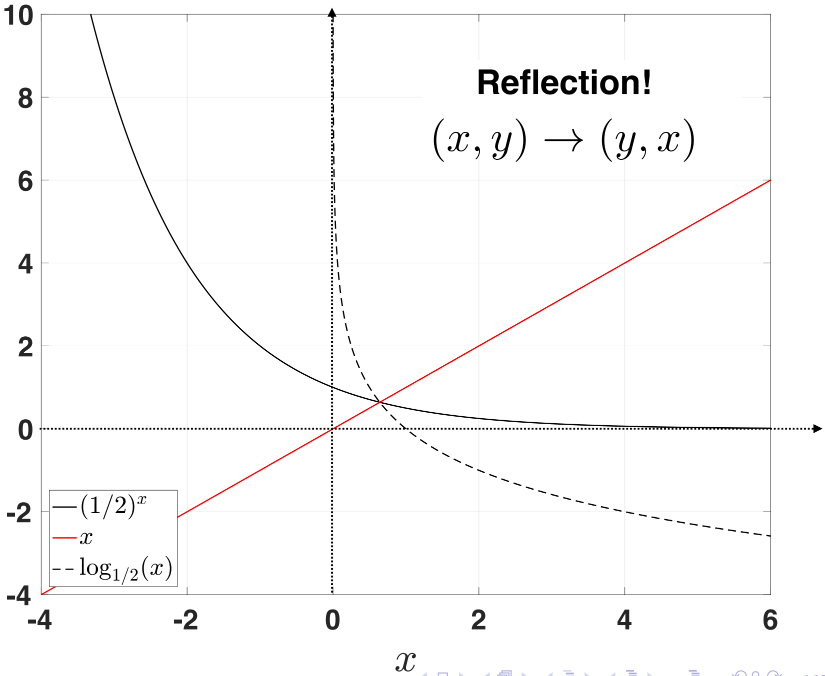
2^x
x
8
6
4
2
0
-2
-4
-6
-8
-4
-3
-2
-1
0
1
2
3



2^x
x
$\log_2(x)$
Reflection!
$(x, y) \to (y, x)$
8
6
4
2
0
-2
-4
-6
-8
-4
-3
-2
-1
0
1
2
3



10
8
6
4
2
0
-2
-4
-4
-2
0
2
4
6
x
$(1/2)^x$
x


Reflection!
$(x, y) \to (y, x)$
10
8
6
4
2
0
-2
-4
-4
-2
0
2
4
6
x
$(1/2)^x$
x
$\log_{1/2}(x)$

Theorem

Let a be a positive number with either $0 < a < 1$ *or* $a > 1$.

For all $x \in \mathbb{R}$, $x > 0$, *there exists a unique* y *such that*

$$a^y = x \quad (\maltese)$$

The number y *is called the logarithm to base* a *of* x *and it is indicated as*

$$y = \log_a(x).$$

The function $f(x) = \log_a(x)$ *is called the logarithmic function.*

Remark

Remember that $a^y > 0$ for all y, so the equation $(\maltese)$ cannot be solved if $x \leq 0$. In other words the **domain of the logarithmic function is** $(0, +\infty)$.

Properties of the exponential function

Let a be a positive real number with either $0 < a < 1$ or $a > 1$.

- The function $f(x) = \log_a(x)$ is defined for all $x \in (0, \infty)$ with values in $\mathbb{R}$.
- Given $x > 0$ and $y > 0$ it holds that

$$\log_a(x\,y) = \log_a(x) + \log_a(y)\,, \quad \log_a(x^y) = y\,\log_a(x)\,.$$

- $\log_a(1) = 0$.
- If $0 < a < 1$ the function is strictly decreasing, $\log_a(x) > 0$ for $x \in (0, 1)$ and $\log_a(x) < 0$ for $x \in (1, +\infty)$.
- If $a > 1$ the function is strictly increasing, $\log_a(x) < 0$ for $x \in (0, 1)$ and $\log_a(x) > 0$ for $x \in (1, +\infty)$.

Exercize

Find the domain of the function

$$f(x) = \log_3\left(1 - x^2\right)$$

and find for which values $f(x) > 0$, $f(x) > 0$ *and* $f(x) < 0$.

Solution. The argument of the logarithm must be **STRICTLY POSITIVE**, hence the domain is

$$D = \left\{x \in \mathbb{R} \;\middle|\; 1 - x^2 > 0\right\} = (-1, 1).$$

Since the base is $3 > 1$ then $f(x) > 0$ if $1 - x^2 > 1$ that is $-x^2 > 0$ which is impossible, hence $f(x) \leq 0$ for all x. In particular

$$f(x) = 0 \Leftrightarrow 1 - x^2 = 1 \Leftrightarrow x = 0.$$

Exercize

Find the domain of the function

$$f(x) = \log_2(\log_2(\log_2(x))).$$

Solution. The most external logarithm requires

$$\log_2(\log_2(x)) > 0$$

which is true if and only if

$$\log_2(x) > 1$$

which is true if and only if

$$2^{\log_2(x)} > 2^1 \Leftrightarrow x > 2.$$

Whence the domain is $D = (2, \infty)$.

Exercize

For each $n \in \mathbb{N}$, $n > 0$, consider the function

$$f_n(x) = \log_2\left(x - \frac{1}{n}\right)$$

Find the domain D_n of f_n and find the domain D where all the f_n's are defined.

Solution. For a generic n it must be

$$x - \frac{1}{n} > 0 \Leftrightarrow x > \frac{1}{n}.$$

Whence

$$D_n = \left(\frac{1}{n}, \infty\right)$$

and

$$D = \bigcap_{n=1}^{\infty} D_n = D_1 = (1, \infty).$$

Exercize

Find the domain of the function

$$f(x) = \log_2\left(\frac{x}{1+2^x}\right).$$

Solution. The condition to be imposed is

$$\frac{x}{1+2^x} > 0,$$

Nevertheless the denominator $1 + 2^x > 0$ for all x, whence

$$D = (0, \infty).$$

Declaration of the axiom

Let $\mathcal{P}_n$ be a proposition defined on the set of integer numbers $\mathbb{N}$. Assume $\mathcal{P}_{n^*}$ is true for some $n^* \in \mathbb{N}$. If for all $n \geq n^*$ it holds that

$$\mathcal{P}_n \Rightarrow \mathcal{P}_{n+1}$$

then $\mathcal{P}_n$ is true for all $n \geq n^*$.

Exercize

Find a closed-form formula for the sum of the first n integers:

$$S_n = 1 + 2 + 3 + 4 + 5 + \ldots + n = ?$$

Solution. Let's write down the first terms:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + 2 = 3 = \frac{2 \cdot 3}{2} \\ S_3 &= 1 + 2 + 3 = 6 = \frac{3 \cdot 4}{2} \\ S_4 &= 1 + 2 + 3 + 4 = 10 = \frac{4 \cdot 5}{2} \end{aligned} \tag{0.1}$$

The guess is $S_n = n \cdot (n+1)/2$. Let's prove it.

Exercize

Find a closed-form formula for the sum of the first n integers:

$$S_n = 1 + 2 + 3 + 4 + 5 + \ldots + n = ?$$

Solution. We claim that

$$S_n = \frac{n \cdot (n+1)}{2} \qquad (✠)\,.$$

We know that the proposition is true for $n = 1, 2, 3, 4$. Let's assume that (✠) is true. Then

$$\begin{aligned} S_{n+1} &= \underbrace{1 + 2 + 3 + \ldots + n}_{S_n} + n + 1 \\ &= S_n + n + 1 = \frac{n \cdot (n+1)}{2} + n + 1 = \frac{n \cdot (n+1) + 2\,(n+1)}{2} \\ &= \frac{(n+1) \cdot (n+2)}{2}, \end{aligned} \tag{0.2}$$

which means that the claim (✠) is true also for $n + 1$.

Exercize

Show that, $\forall x \geq -1$, it holds that

$$(1+x)^n \geq 1 + n\,x \qquad (\square)$$

Solution. Let's start with $n = 0$. The inequality ($\square$) gives

$$(1+x)^0 = 1 \geq 1,$$

which is trivially true. Let's assume that ($\square$) is true for a generic n. At $n+1$ we get

$$\begin{aligned}
(1+x)^{n+1} &= \underbrace{(1+x)}_{\geq 0}\underbrace{(1+x)^n}_{\text{Use } (\square)} \\
&\geq (1+x)\,(1+n\,x) = 1 + n\,x + x + \underbrace{n\,x^2}_{\geq 0} \\
&\geq 1 + n\,x + x = 1 + (n+1)\,x, \qquad (0.3)
\end{aligned}$$

which means that the claim ($\square$) is true also for $n+1$.

Exercize

For all $x \in \mathbb{R}$ prove that

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}, \quad (\triangle)$$

Solution. Let's start with $n = 0$. The equality $(\triangle)$ gives

$$1 = \frac{1 - x}{1 - x} = 1,$$

which is trivially true. Let's assume that $(\triangle)$ is true for a generic n. At $n + 1$ we get

$$\begin{aligned}
\underbrace{1 + x + x^2 + \cdots + x^n}_{\text{Use } (\triangle)} + x^{n+1} &= \frac{1 - x^{n+1}}{1 - x} + x^{n+1} \\
&= \frac{1 - x^{n+1} + x^{n+1} - x^{n+2}}{1 - x} = \frac{1 - x^{n+2}}{1 - x}.
\end{aligned}$$

which means that the claim $(\triangle)$ is true also for $n + 1$.

Example

The formulas derived so far allow to skip a lot of computation...

$$1 + 2 + 3 + 4 + 5 + \cdots + 100 = \frac{100 \cdot 101}{2} = 50 \cdot 101 = 5050.$$

$$1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 = \frac{1 - 2^6}{1 - 2} = \frac{1 - 64}{-1} = 63.$$

Definition

A sequence of real number $(s_n)_{n\in\mathbb{N}}$ is any function

$$\begin{aligned} s:\mathbb{N} &\to \mathbb{R} \\ n &\to s(n)=s_n. \end{aligned}$$

The number s_n is also called the n-th element of the sequence.

Definition

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If it exists $\ell\in\mathbb{R}$ such that

$$\forall\varepsilon>0,\exists n^*:\forall n>n^*\Rightarrow 0<|s_n-\ell|<\varepsilon.$$

then ℓ is called the limit of the sequence and we use the notation

$$\ell=\lim_{n\to\infty}s_n$$

or we say $s_n\to\ell$ as $n\to\infty$.

Definition

A sequence of real number $(s_n)_{n\in\mathbb{N}}$ is any function

$$\begin{aligned} s:\mathbb{N} &\rightarrow \mathbb{R} \\ n &\rightarrow s(n) = s_n. \end{aligned}$$

The number s_n is also called the n-th element of the sequence.

Definition

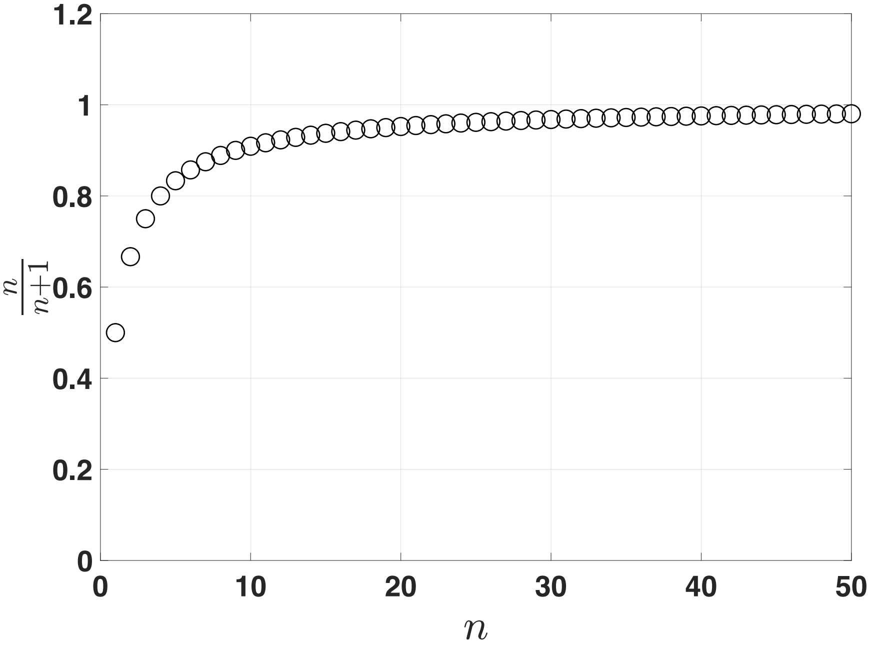
Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If

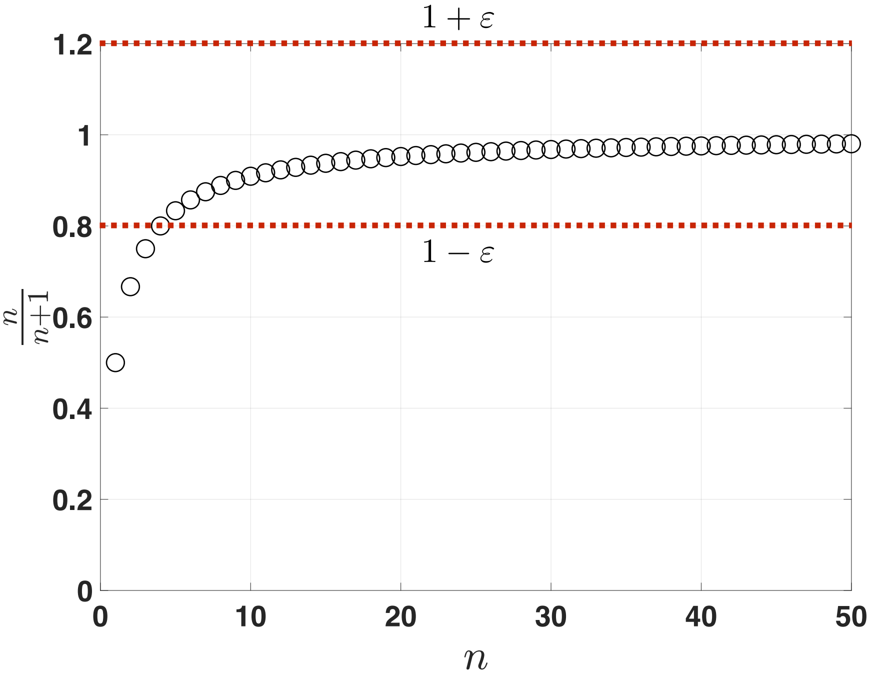
$$\forall M > 0, \exists n^* : \forall n > n^* \Rightarrow s_n > M.$$

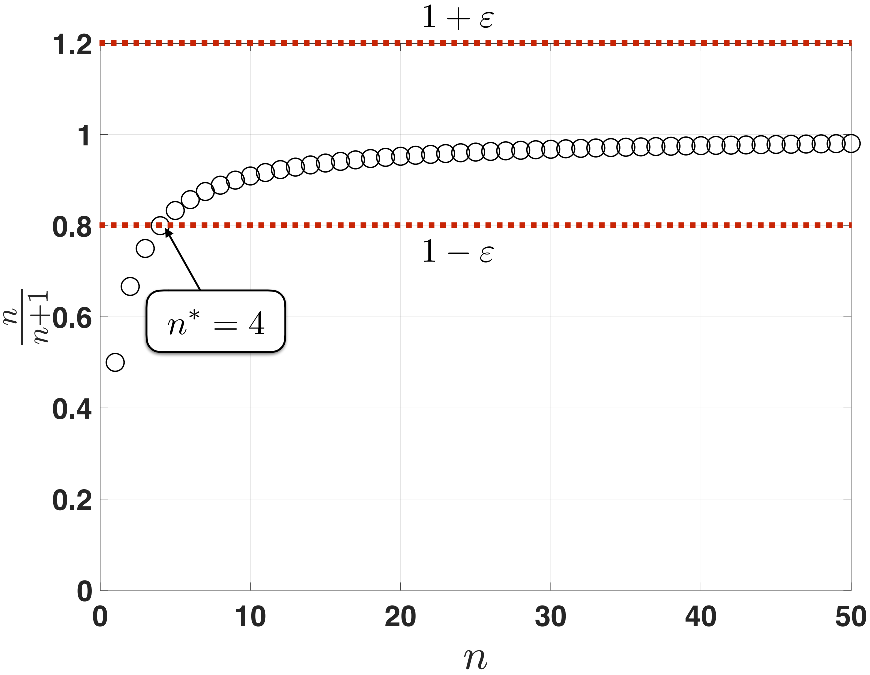
we say that

$$\lim_{n\to\infty} s_n = +\infty$$

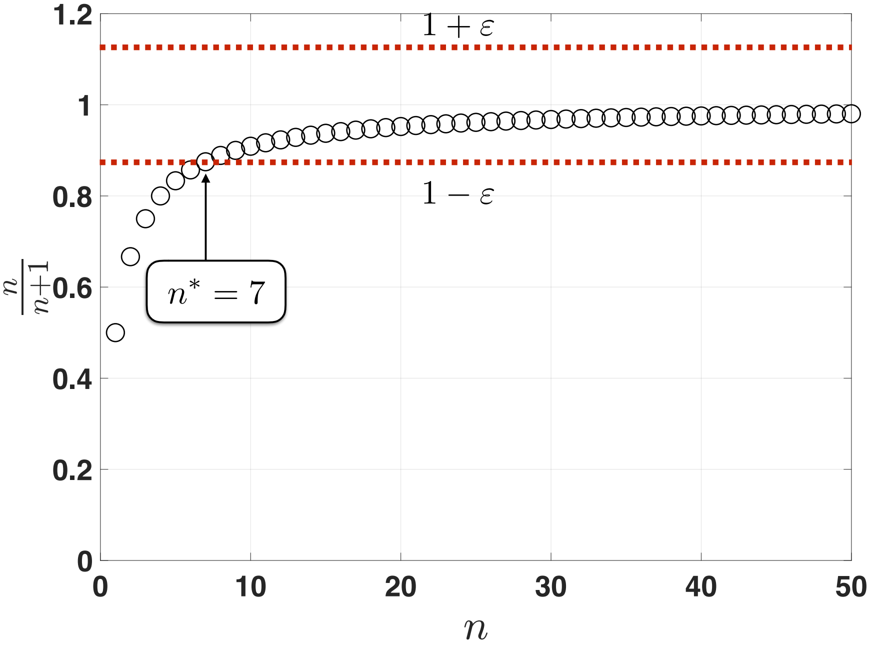
or we say $s_n \rightarrow +\infty$ as $n \rightarrow \infty$.

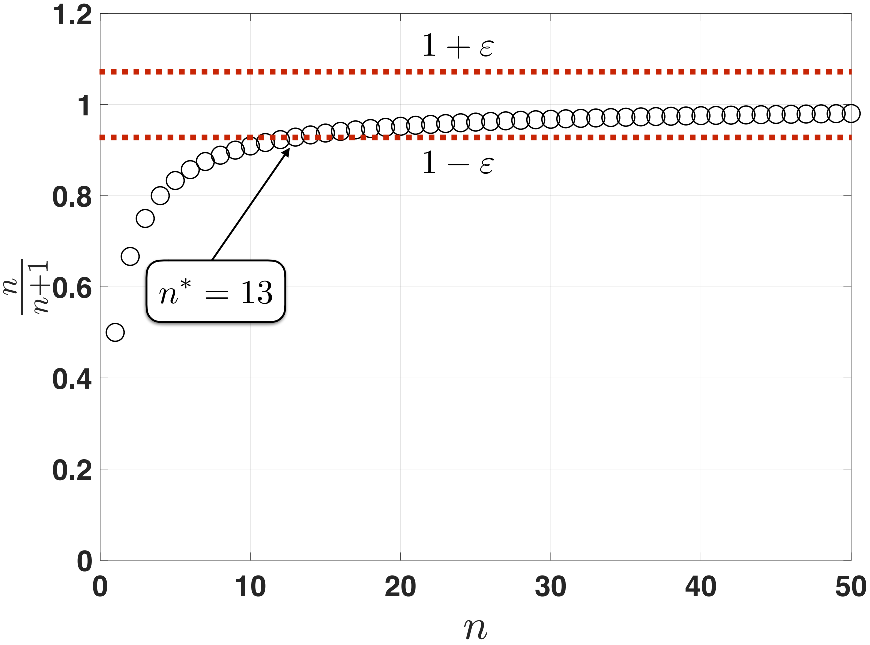

1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
$\frac{n}{n+1}$
n

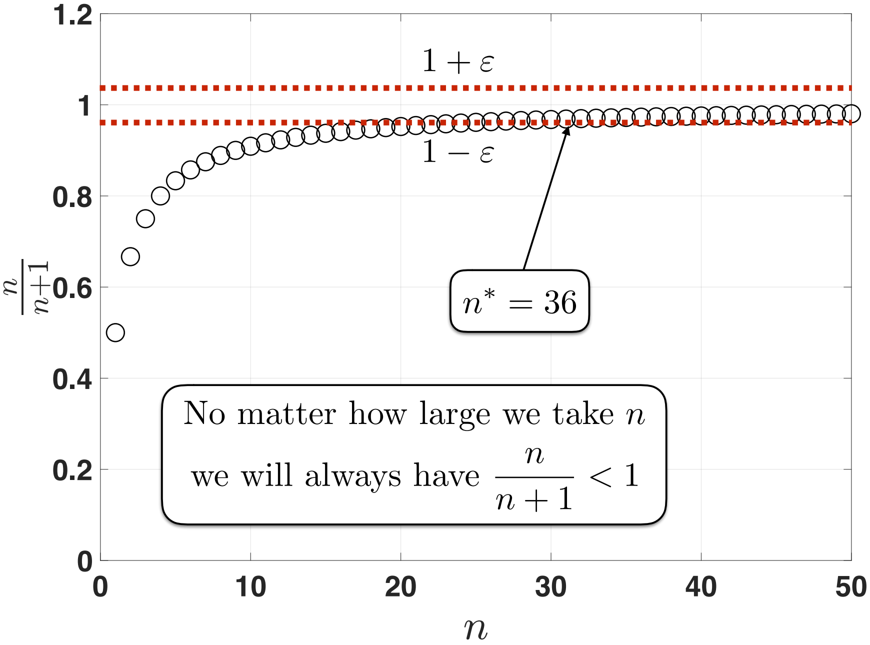

$1+\varepsilon$
$1-\varepsilon$
1.2
1
0.8
0.6
0.4
0.2
0
$\frac{n}{n+1}$
0
10
20
30
40
50
n



$1+\varepsilon$
1.2
1
0.8
0.6
0.4
0.2
0
$1-\varepsilon$
$n^* = 4$
$\frac{n}{n+1}$
0
10
20
30
40
50
n


1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
$1+\varepsilon$
$1-\varepsilon$
$n^* = 7$
$\frac{n}{n+1}$
n


1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
$1+\varepsilon$
$1-\varepsilon$
$n^* = 13$
$\frac{n}{n+1}$
n


1.2
1
0.8
0.6
0.4
0.2
0
$\frac{n}{n+1}$
$1+\varepsilon$
$1-\varepsilon$
$n^* = 36$
No matter how large we take n
we will always have $\frac{n}{n+1} < 1$
0
10
20
30
40
50
n

Definition

A sequence of real number $(s_n)_{n\in\mathbb{N}}$ is any function

$$\begin{aligned} s:\mathbb{N} &\to \mathbb{R} \\ n &\to s(n) = s_n. \end{aligned}$$

The number s_n is also called the n-th element of the sequence.

Definition

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If

$$\forall M>0, \exists n^* : \forall n > n^* \Rightarrow s_n < -M.$$

we say that

$$\lim_{n\to\infty} s_n = -\infty$$

or we say $s_n \to -\infty$ as $n \to \infty$.

Sequences: the limit may not exist

Warning

Pay attention! It is not guaranteed that a sequence $(s_n)_{n\in\mathbb{N}}$ falls in one of the three cases:

$$\lim_{n\to\infty} s_n = \begin{cases} \ell \\ +\infty \\ -\infty \end{cases}.$$

This theorem gives us a **necessary** condition for a sequence to have a finite limit ℓ.

Theorem

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. If $\exists \ell \in \mathbb{R}$ such that

$$\lim_{n\to\infty} s_n = \ell$$

then

$$\lim_{n\to\infty} s_{2\,n} = \ell \text{ and } \lim_{n\to\infty} s_{2\,n+1} = \ell.$$

Example

Consider the "alternating" sequence

$$s_n = (-1)^n \,.$$

The guess is that s_n does not have a limit!

$$s_{2\,n} = (-1)^{2\,n} = +1 \to +1.$$

$$s_{2\,n+1} = (-1)^{2\,n+1} = -1 \to -1.$$

whence, by the necessary condition, it is not possible that $s_n \to \ell$.

Theorem

The limit of a sequence, when it exists, is unique.

Proof. Suppose, by contradiction, that a sequence has two limits ℓ and ℓ' with $\ell \neq \ell'$. By definition of limit:

$$\forall \varepsilon > 0, \exists n_1 : \forall n > n_1 \Rightarrow 0 < |s_n - \ell| < \varepsilon.$$

and, simultaneously,

$$\forall \varepsilon > 0, \exists n_2 : \forall n > n_2 \Rightarrow 0 < |s_n - \ell'| < \varepsilon.$$

Take $n^* = \max(n_1, n_2)$, whence for $n > n^*$ we have ...

$$|\ell - \ell'| = |\ell - s_n + s_n - \ell'| \leq |\ell - s_n| + |s_n - \ell'| < \varepsilon + \varepsilon = 2\,\varepsilon \to 0$$

whence $\ell = \ell'$.

Theorem

Let $E \subset \mathbb{R}$ and let x_0 be a limit point of E. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in E$ for all n and $x_n \to x_0$ when $n \to \infty$.

Proof. Recall the definition of limit point

$$\forall \varepsilon > 0 \Rightarrow (x_0 - \varepsilon, x_0 + \varepsilon) \cap E \setminus \{x_0\} \neq \emptyset$$

Since ε can be taken arbitrarily small then, for all $n \in \mathbb{N}$, I take $\varepsilon_n = \frac{1}{n}$ and for each n I know that

$$\forall n > 0 \Rightarrow \left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right) \cap E \setminus \{x_0\} \neq \emptyset$$

which means that I can find, for all n, a point $x_n \in E$, $x_n \neq x_0$, such that

$$x_n \in \left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right)$$

or

$$0 < |x_n - x_0| < \frac{1}{n},$$

i.e. $x_n \to x_0$.

Exercize

Prove that, $\forall a \in \mathbb{R}$ *and* $\forall b \in \mathbb{R}$*, then*

$$|a-b| \geq ||a|-|b||\,.$$

Solution

$$a = a-b+b \Rightarrow |a| = |a-b+b| \leq |a-b|+|b| \Rightarrow |a|-|b| \leq |a-b|\,, \quad (\square)\,,$$

and, symmetrically,

$$b = b-a+a \Rightarrow |b| = |b-a+a| \leq |a-b|+|a| \Rightarrow |b|-|a| \leq |a-b|\,, \quad (\triangle)\,.$$

Summing up, ($\square$) plus ($\triangle$) yield

$$\begin{cases} |a|-|b| & \leq |a-b| \\ |b|-|a| & \leq |a-b| \end{cases} \Rightarrow ||a|-|b|| \leq |a-b|\,.$$

Theorem

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. As $n \to \infty$ if $s_n \to \ell$ then $|s_n| \to |\ell|$.

Proof. Since by hypothesis $s_n \to \ell$ we have that

$$\forall \varepsilon > 0, \exists n^* : \forall n > n^* \Rightarrow 0 < |s_n - \ell| < \varepsilon.$$

By the reverse triangular inequality we get

$$||s_n| - |\ell|| < |s_n - \ell| < \varepsilon$$

which means $|s_n| \to |\ell|$ as $n \to \infty$.

Is the converse true? That is...can we say that if $|s_n| \to |\ell|$ then $s_n \to \ell$?

Remark

When you think that a sentence is true you have to prove it!

When you think that a sentence is false you have to find a **counter-example**: that is at least one case in which the statement of the sentence is false!

Take

$$s_n = (-1)^n$$

then $|s_n| = 1$, whence

$$|s_n| \to 1$$

nevertheless we know that $\nexists \ell = \lim_{n\to\infty} (-1)^n$.

There is however an exception ...

Theorem

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. As $n \to \infty$ if $|s_n| \to 0$ then $s_n \to 0$.

Proof. The hypothesis, in formula, reads

$$\forall \varepsilon > 0, \exists n^* : \forall n > n^* \Rightarrow 0 < ||s_n|| < \varepsilon,$$

which trivially means

$$\forall \varepsilon > 0, \exists n^* : \forall n > n^* \Rightarrow 0 < |s_n| < \varepsilon,$$

which means

$$\lim_{n\to\infty} s_n = 0.$$

Functions and sequences

Suppose that $x_n \to x_0$ and that $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a function. What can I say about

$$x'_n \stackrel{\text{def}}{=} f(x_n) \quad ?$$

Does $x'_n \to \ell'$ for some ℓ' ? If yes, can I say that $\ell' = f(\ell)$?

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $x_0 \in D$ be a **point of the domain**. We say that the function f is continuous in x_0 if and only if **for all sequences** $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x_0$ it holds that

$$f(x_n) \to f(x_0).$$

We say that a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is continuous if it is continuous in all $x \in D$.

Theorem

The following functions are continuous:

$$\begin{aligned}
\cos(x) &: \quad \mathbb{R} \to [-1,1] \\
\sin(x) &: \quad \mathbb{R} \to [-1,1], \\
x^{2n} &: \quad \mathbb{R} \to [0,+\infty) \\
x^{2n+1} &: \quad \mathbb{R} \to \mathbb{R} \\
x^{1/n} &: \quad [0,+\infty) \to [0,+\infty) \\
a^{x} &: \quad \mathbb{R} \to (0,+\infty) \\
\log_a(x) &: \quad (0,+\infty) \to \mathbb{R}
\end{aligned}$$

Example

$$\begin{aligned} \frac{1}{n} &\to 0 \\ \cos\left(\frac{1}{n}\right) &\to \cos(0) = 1 \\ \sin\left(\frac{1}{n}\right) &\to \sin(0) = 0 \end{aligned}$$

What about, for example

$$\log_2\left(\frac{1}{n}\right) \to ?$$

Since $\frac{1}{n} \to 0$ and the domain of the log is $(0, +\infty)$ **we cannot use the continuity argument!!**.

Nevertheless

$$\log_2\left(\frac{1}{n}\right) = \log_2\left(n^{-1}\right) = -\log_2(n) \to -\infty.$$

Theorem

Let $(s_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ be two sequences.

1. *If $s_n \to \ell$ and $q_n \to \ell'$ then $s_n + q_n \to \ell + \ell'$ and $s_n \cdot q_n \to \ell \cdot \ell'$.*
2. *If $s_n \to \ell$ and $q_n \to \ell'$ with $\ell' \neq 0$ then $s_n/q_n \to \ell/\ell'$.*
3. *If $s_n \to +\infty$ and $q_n \to +\infty$ then $s_n + q_n \to +\infty$ and $s_n \cdot q_n \to +\infty$.*
4. *If $s_n \to -\infty$ and $q_n \to -\infty$ then $s_n + q_n \to -\infty$ and $s_n \cdot q_n \to +\infty$.*
5. *If $s_n \to +\infty$ and $q_n \to -\infty$ then $s_n + q_n$ is undetermined (i.e. the limit could be anything) and $s_n \cdot q_n \to -\infty$.*

Exercize

Compute, if exists, the following limit

$$\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right).$$

Solution. We have $\sqrt{n^2+n}\to+\infty$ so the limit is of the form $+\infty-\infty$ and it is undetermined! Nevertheless

$$\begin{aligned}\left(\sqrt{n^2+n}-n\right) &= \frac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n}\\ &= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n}=\frac{n}{\sqrt{n^2+n}+n}\\ &= \frac{1}{\sqrt{1+\frac{1}{n}}+1}\to\frac{1}{2}.\end{aligned}\tag{0.4}$$

Exercize

Compute, if exists, the following limit

$$\lim_{n\to\infty}\left(\sqrt{n^2+100\,n}-n\right).$$

Solution. As in the previous case $+\infty-\infty$ is undetermined!

$$\begin{aligned}\left(\sqrt{n^2+100\,n}-n\right) &= \frac{\left(\sqrt{n^2+100\,n}-n\right)\,\left(\sqrt{n^2+100\,n}+n\right)}{\sqrt{n^2+100\,n}+n}\\ &= \frac{n^2+100\,n-n^2}{\sqrt{n^2+100\,n}+n}=\frac{100\,n}{\sqrt{n^2+n}+n}\\ &= \frac{100}{\sqrt{1+\frac{1}{n}}+1}\to 50. \qquad (0.5)\end{aligned}$$

Warning!

The same undetermined form may corresponds to an infinite number of different results!

This is why is called **undetermined**.

Exercize

Compute, if exists, the following limit

$$\lim_{n\to\infty} \frac{3\,n^7 - 8\,n^6 + 15\,n^{3/2} - 10\,n^{1/2}}{12\,n - 4\,n^7}.$$

Solution. Trick: put in evidence the highest power at the numerator and at the denominator.

$$\frac{3\,n^7 - 8\,n^6 + 15\,n^{3/2} - 10\,n^{1/2}}{12\,n - 4\,n^7} = \frac{n^7\,\left(3 - 8\,n^{-1} + 15\,n^{-11/2} - 10\,n^{-13/2}\right)}{n^7\,(12\,n^{-6} - 4)}$$

$$= \frac{n^7\,\left(3 - 8\,\frac{1}{n} + 15\,\left(\frac{1}{n}\right)^{11/2} - 10\,\left(\frac{1}{n}\right)^{13/2}\right)}{n^7\,\left(12\,\left(\frac{1}{n}\right)^6 - 4\right)}$$

Since for $\alpha > 0$ we have $\left(\frac{1}{n}\right)^\alpha \to 0$ we get

$$\lim_{n\to\infty} \frac{3\,n^7 - 8\,n^6 + 15\,n^{3/2} - 10\,n^{1/2}}{12\,n - 4\,n^7} = -\frac{3}{4}.$$

Theorem

Let $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ be three sequences such that

$$q_n \leq p_n \leq r_n.$$

The following three results hold:

- *if $q_n \to \ell$ and $r_n \to \ell$ then also $p_n \to \ell$.*
- *If $q_n \to +\infty$ then also $p_n \to +\infty$.*
- *If $r_n \to -\infty$ then also $p_n \to -\infty$.*

An application of the comparison theorem

For all $p > 0$ it holds that

$$\lim_{n\to\infty} p^{1/n} = 1.$$

In fact, assume first $p > 1$ define $x_n = p^{1/n} - 1 \geq 0$ and note that

$$p = (1 + x_n)^n \geq 1 + n\,x_n$$

whence

$$0 \leq x_n \leq \frac{p-1}{n} \to 0 \quad \Longrightarrow x_n \to 0$$

whence $p^{1/n} \to 1$. If $0 < p < 1$ consider $q = 1/p > 1$ and note that

$$p^{1/n} = \frac{1}{q^{1/n}}$$

but we know that, since $q > 1$ then $q^{1/n} \to 1$, whence $p^{1/n} \to 1$.

An application of the comparison theorem

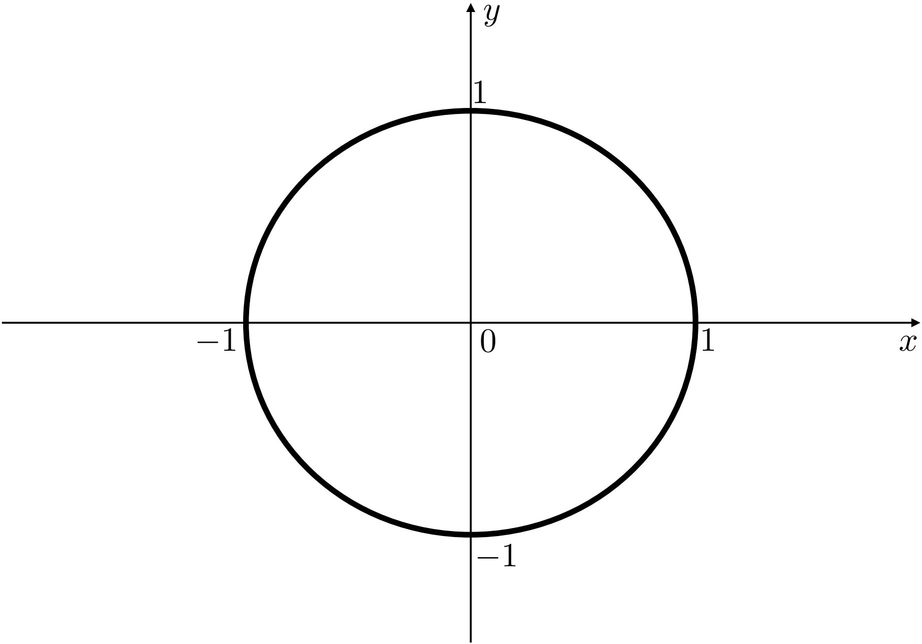
Can we say something on

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = +\infty \times 0 = \text{indeterminate form!}$$

Why indeterminate? It could be anything...

$$\begin{aligned}
\lim_{n\to\infty} n \cdot \frac{1}{n} &= +\infty \times 0 = 1 \\
\lim_{n\to\infty} n^2 \cdot \frac{1}{n} &= +\infty \times 0 = \lim_{n\to\infty} n = +\infty \\
\lim_{n\to\infty} n \cdot \frac{1}{n^2} &= +\infty \times 0 = \lim_{n\to\infty} \frac{1}{n} = 0. \\
\lim_{n\to\infty} n \cdot \frac{7}{n+10} &= +\infty \times 0 = 7
\end{aligned}$$

Pay always attention to the indeterminate forms!!!


y
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−1
0
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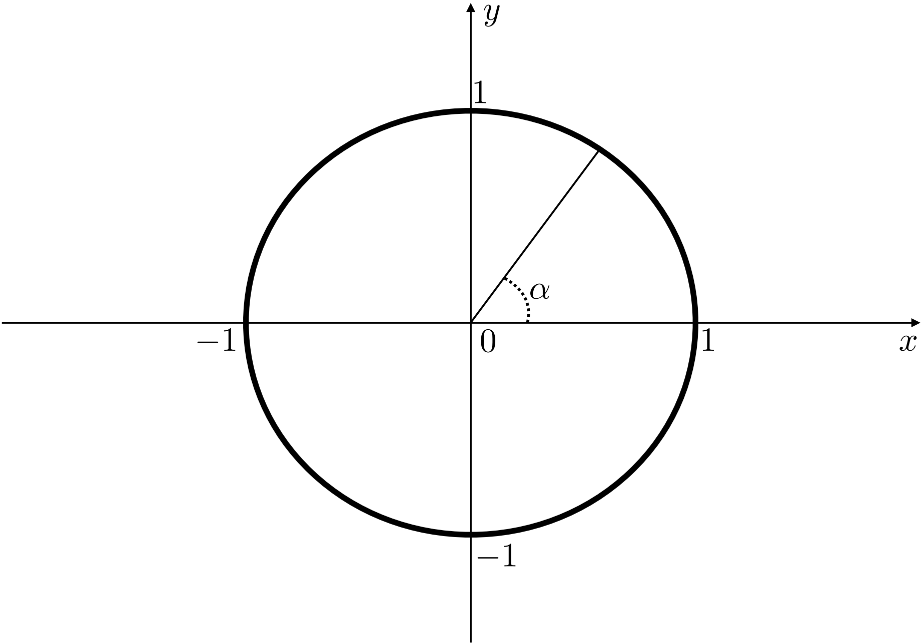
x
−1


y
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α
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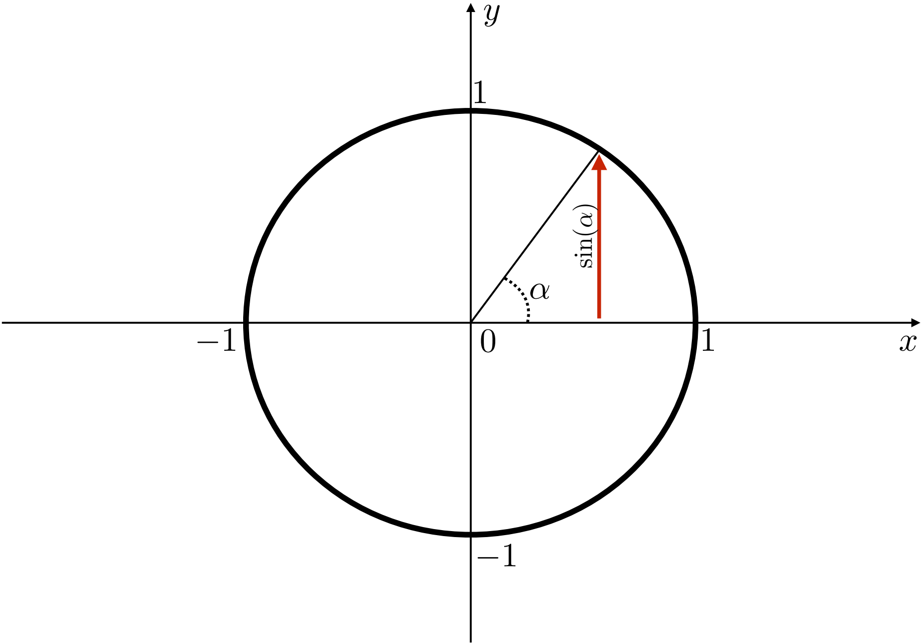

y
1
sin(α)
α
−1
0
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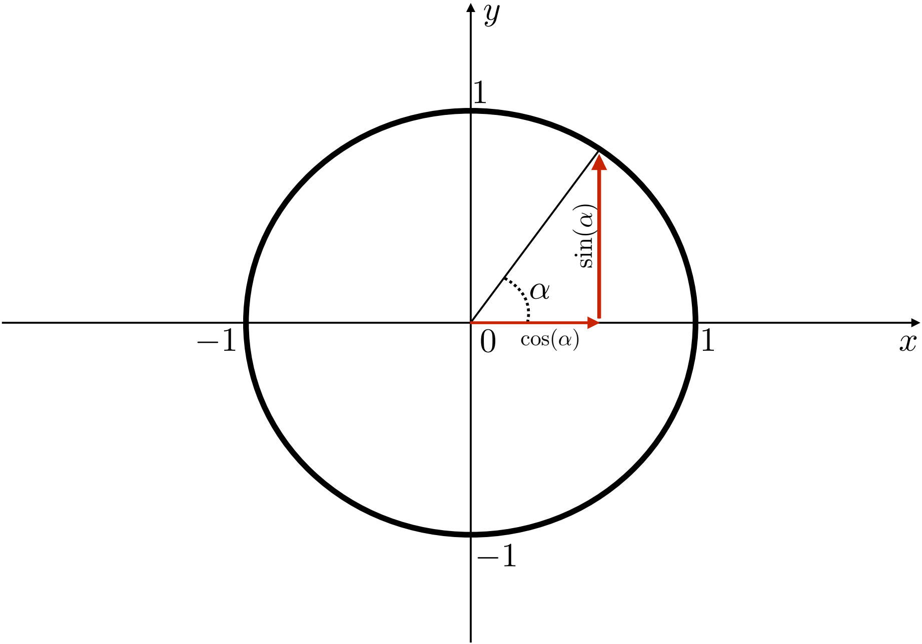
x
−1


y
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α
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x
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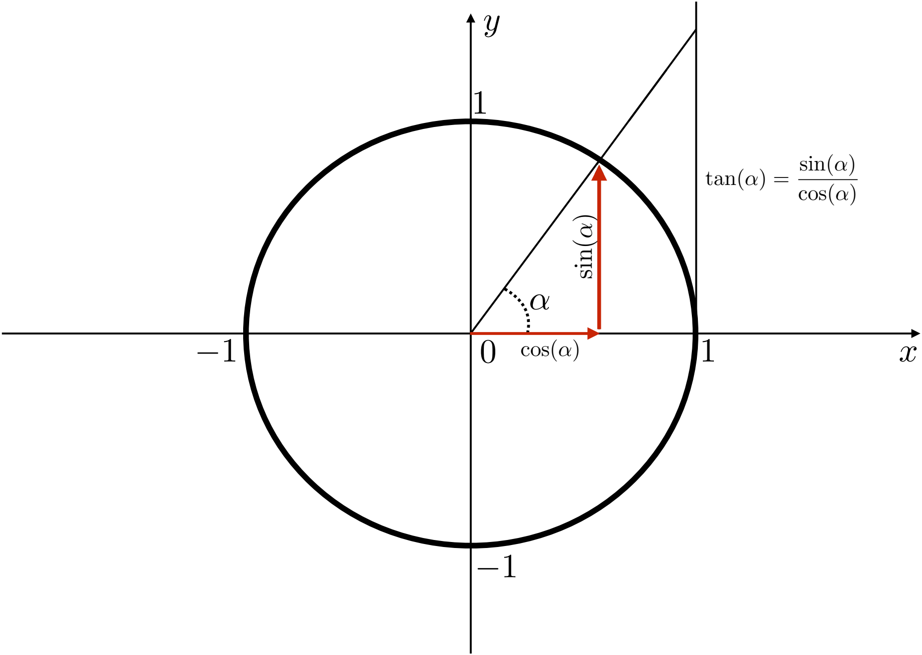
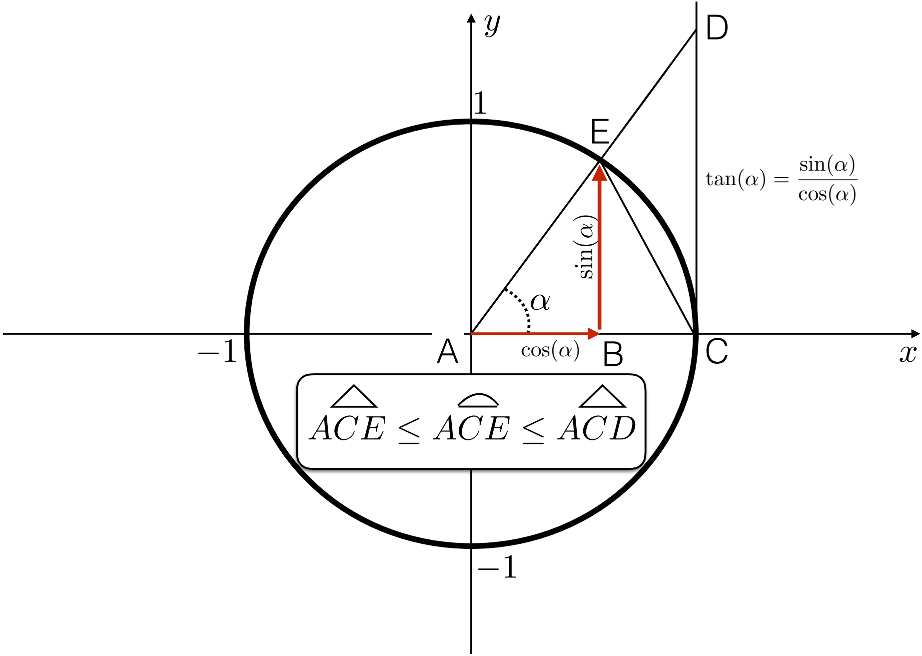
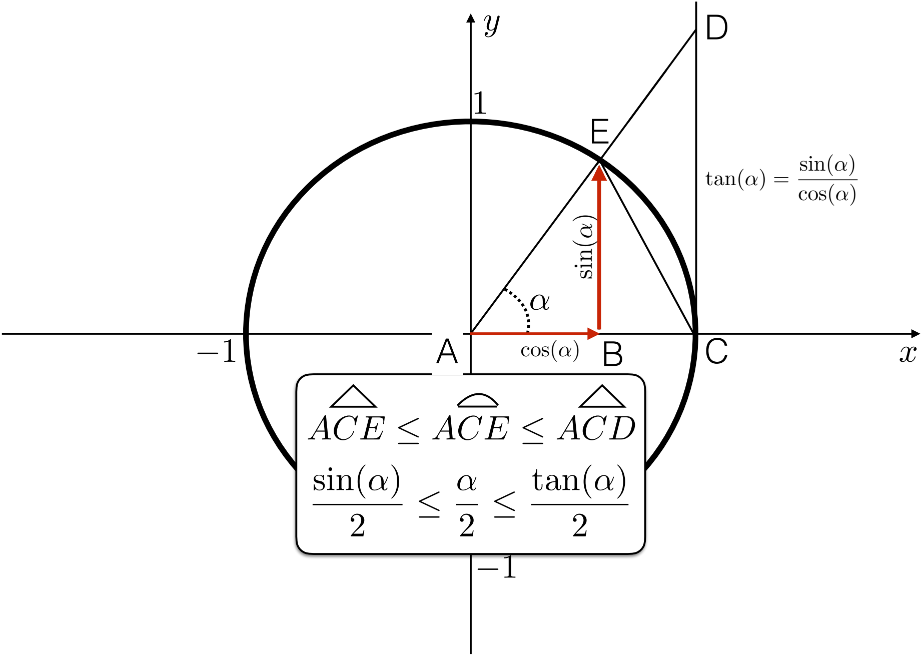

y
1
$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$
$\sin(\alpha)$
α
-1
0
$\cos(\alpha)$
1
x
-1

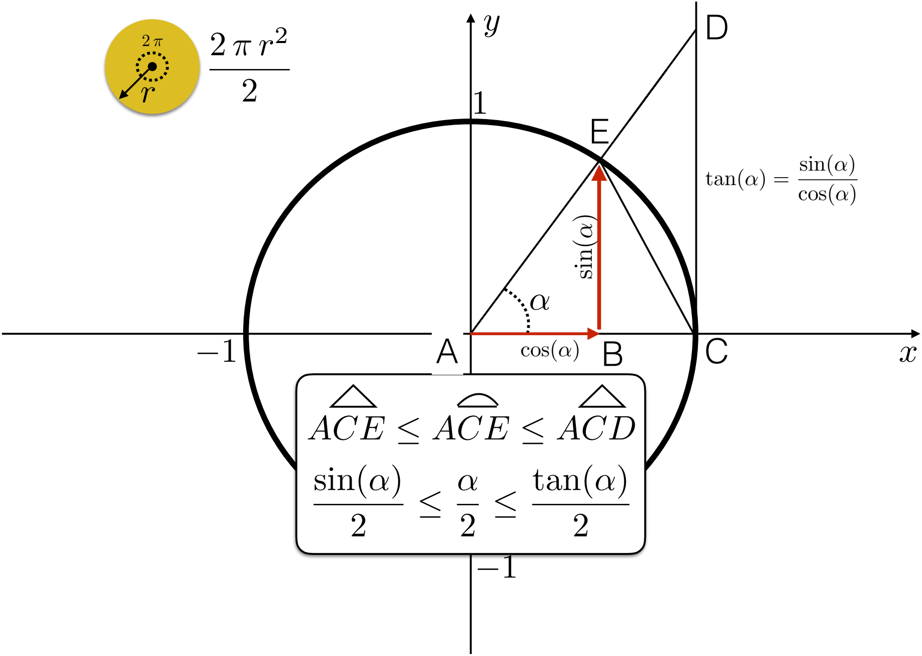


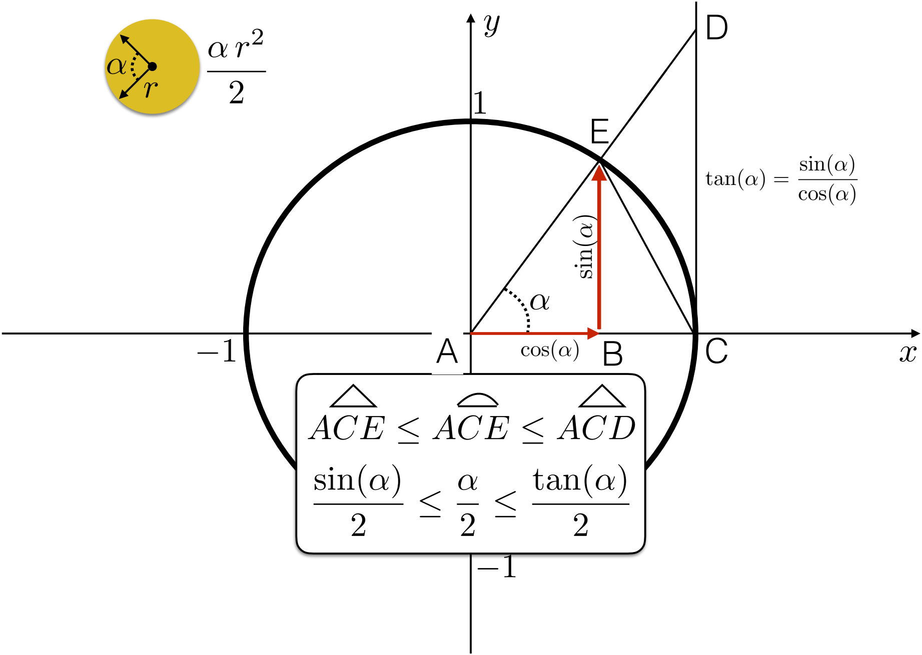
y
D
1
E
tan(α) = sin(α)/cos(α)
sin(α)
α
−1
A
cos(α)
B
C
x
ACE ≤ ACE ≤ ACD
−1



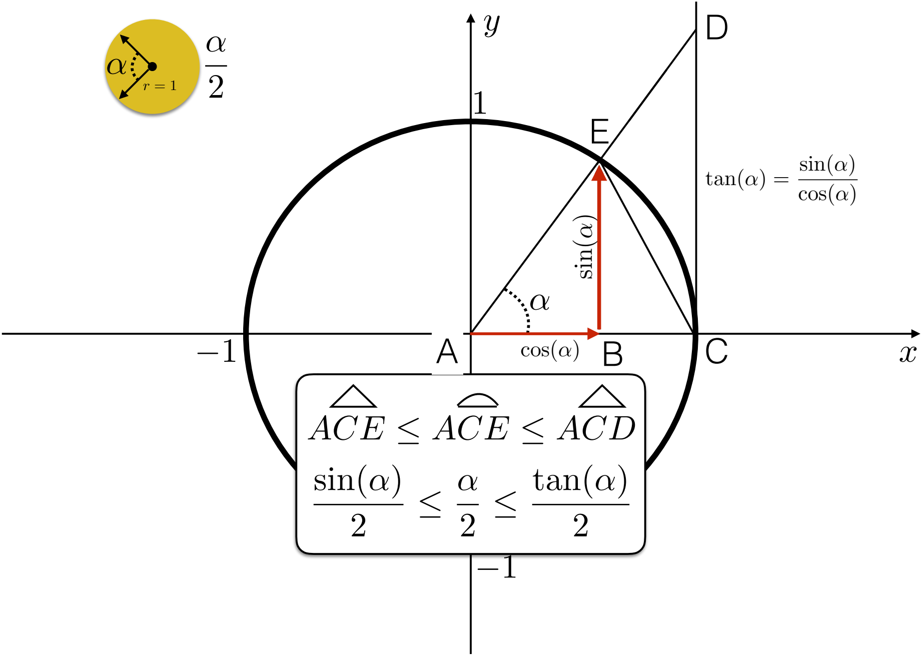
$$\overset{\triangle}{ACE} \leq \overset{\frown}{ACE} \leq \overset{\triangle}{ACD}$$

$$\frac{\sin(\alpha)}{2} \leq \frac{\alpha}{2} \leq \frac{\tan(\alpha)}{2}$$


2π
r
2 π r²/2
y
x
1
−1
−1
D
E
A
B
C
α
cos(α)
sin(α)
tan(α) = sin(α)/cos(α)
ACE ≤ ACE ≤ ACD
sin(α)/2 ≤ α/2 ≤ tan(α)/2



α r^2 / 2
α
r
y
x
1
−1
−1
D
E
A
B
C
α
sin(α)
cos(α)
tan(α) = sin(α) / cos(α)
ACE ≤ ACE ≤ ACD
sin(α)/2 ≤ α/2 ≤ tan(α)/2


α
$r=1$
$\frac{\alpha}{2}$
y
1
D
E
$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$
$\sin(\alpha)$
α
-1
A
$\cos(\alpha)$
B
C
x
-1
$\overset{\triangle}{ACE} \leq \overset{\frown}{ACE} \leq \overset{\triangle}{ACD}$
$\frac{\sin(\alpha)}{2} \leq \frac{\alpha}{2} \leq \frac{\tan(\alpha)}{2}$

Hence, for n sufficiently large,

$$0 \leq \sin\left(\frac{1}{n}\right) \leq \frac{1}{n} \leq \frac{\sin(1/n)}{\cos(1/n)}$$

divide all sides by the positive quantity $\sin\left(\frac{1}{n}\right)$, obtaining

$$0 \leq 1 \leq \frac{1}{n\sin(1/n)} \leq \frac{1}{\cos(1/n)}$$

flip al the fractions

$$1 \geq n\sin(1/n) \geq \cos(1/n)$$

more comfortable expression....

$$\cos(1/n) \leq n\sin(1/n) \leq 1$$

Since $\cos(1/n) \to 1$ then we get

$$\lim_{n\to\infty} n\sin(1/n) = 1.$$

Exercize

Let $a \in \mathbb{R}$, $a \neq 1$. Prove that

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1 \\ +\infty & \text{if } a > 1 \\ \nexists & \text{if } a \leq -1 \end{cases}$$

Solution. Case $|a| < 1$. Define $h = \frac{1}{|a|} - 1$. We get

$$|a| < 1 \implies \frac{1}{|a|} > 1 \implies \frac{1}{|a|} - 1 > 0 \implies h = \frac{1}{|a|} - 1 > 0,$$

and $|a| = \frac{1}{1+h}$. We know that

$$(1+h)^n \geq 1 + n\,h \geq n\,h \implies 0 < \frac{1}{(1+h)^n} \leq \frac{1}{n\,h}$$

So finally:

$$0 < |a|^n = \frac{1}{(1+h)^n} \leq \frac{1}{n\,h} \to 0 \implies a^n \to 0.$$

Exercize

Let $a \in \mathbb{R}$, $a \neq 1$. *Prove that*

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1 \\ +\infty & \text{if } a > 1 \\ \nexists & \text{if } a \leq -1 \end{cases}$$

Solution. Case $a > 1$. Define $h = a - 1$. We get

$$a > 1 \implies a - 1 > 0 \implies h = a - 1 > 0,$$

and $a = 1 + h$. Whence

$$a^n = (1 + h)^n \geq 1 + n\,h \to \infty.$$

Exercize

Let $a \in \mathbb{R}$, $a \neq 1$. Prove that

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1 \\ +\infty & \text{if } a > 1 \\ \nexists & \text{if } a \leq -1 \end{cases}$$

Solution. Case $a > 1$. Define $h = a - 1$. We get

$$a > 1 \implies a - 1 > 0 \implies h = a - 1 > 0,$$

and $a = 1 + h$. Whence

$$a^n = (1 + h)^n \geq 1 + n\,h \to \infty.$$

Exercize

Let $a \in \mathbb{R}$, $a \neq 1$. Prove that

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1 \\ +\infty & \text{if } a > 1 \\ \nexists & \text{if } a \leq -1 \end{cases}$$

Solution. Case $a = -1$.

We already know that $(-1)^n$ has no limit.

Exercize

Let $a \in \mathbb{R}$, $a \neq 1$. Prove that

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1 \\ +\infty & \text{if } a > 1 \\ \nexists & \text{if } a \leq -1 \end{cases}$$

Solution. Case $a < -1$. Then $a = -|a|$ with $|a| > 1$ and

$$a^n = (-1)^n \, |a|^n .$$

But then

$$a^{2n} = |a|^{2n} \to \infty$$

and

$$a^{2n+1} = -|a|^{2n+1} \to -\infty$$

Excursus: the summation symbol

For any sequence $(s_n)_{n\in\mathbb{N}}$ we use the notation

$$\sum_{k=0}^{n} s_k = s_0 + s_1 + s_2 + \ldots + s_n.$$

Example

We already know that

$$\sum_{k=0}^{n} k = 0 + 1 + 2 + 3 + \ldots + n = \frac{n\,(n+1)}{2}.$$

Similarly it could be proved that

$$\sum_{k=0}^{n} k^2 = 0 + 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n\,(n+1)\,(2\,n+1)}{6}$$

$$\sum_{k=0}^{n} k^3 = 0 + 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2\,(n+1)^2}{4}$$

and so on and so forth...

The summation symbol: change of variable

Consider

$$\sum_{k=k_1}^{k_2} s_k$$

change variable $k' = k - k_1 \Rightarrow k = k' + k_1$ then

$$\sum_{k=k_1}^{k_2} s_k = \sum_{k'=0}^{k_2-k_1} s_{k'+k_1}$$

but k' or k are dummy variables! Whence

$$\sum_{k=k_1}^{k_2} s_k = \sum_{k'=0}^{k_2-k_1} s_{k'+k_1} = \sum_{k=0}^{k_2-k_1} s_{k+k_1}$$

Example

$$\sum_{k=1}^{5} k = 1+2+3+4+5 = \sum_{k=0}^{4} (k+1) = 1+2+3+4+5$$

Some WARNINGs!

- A tricky notation ....

$$\sum_{k=0}^{n} = \sum_{k=0}^{n} 1 = \underbrace{1+1+1+\ldots+1}_{n+1- \text{ times}} = n+1$$

- It does not make any sense to do something like

$$\sum_{k=0}^{n} \frac{k}{k+1} = k \sum_{k=0}^{n} \frac{1}{k+1}$$

the dummy variable cannot jump out of the summation!!!

Definition

For any integer n we define define the factorial of n as

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1, \quad 0! = 1.$$

Examples and properties

Some simple examples.

$$\begin{cases} 5! & = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \\ 8! & = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320 \end{cases}$$

A useful property. Consider $k < n$, then

$$\begin{aligned} n! &= n \cdot (n-1) \cdots (n-k+1) \cdot \underbrace{(n-k) \cdot (n-k-1) \cdots 2 \cdot 1}_{(n-k)!} \\ &= n \cdot (n-1) \cdots (n-k+1) \cdot (n-k)! \end{aligned}$$

whence

$$\frac{n!}{(n-k)!} = n \cdot (n-1) \cdots (n-k+1)\,.$$

Definition

For any two integers k and n, with $k \leq n$, the binomial coefficient is defined as

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

and it is read "n choose k".

Theorem

For any couple of real numbers a and b and for all integers n, it holds that

$$(a+b)^n = \binom{n}{0} a^n\, b^0 + \binom{n}{1} a^{n-1}\, b^1 + \binom{n}{2} a^{n-2}\, b^2 + \ldots + \binom{n}{n-1} a^1\, b^{n-1} + \binom{n}{n} a^0\, b^n$$

which is also written as

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\, b^k.$$

Example

$$
\begin{aligned}
(a+b)^2 &= \sum_{k=0}^{2} \binom{2}{k} a^{2-k}\, b^k \\
&= \binom{2}{0} a^2\, b^0 + \binom{2}{1} a^{2-1}\, b^1 + \binom{2}{2} a^{2-2}\, b^2
\end{aligned}
$$

$$
\binom{2}{0} = \frac{2!}{0!\,(2-0)!} = 1, \ \binom{2}{1} = \frac{2!}{1!\,(2-1)!} = 2, \ \binom{2}{2} = \frac{2!}{2!\,(2-2)!} = 1
$$

$$
(a+b)^2 = a^2 + 2\,a\,b + b^2.
$$

Assume $k \in \mathbb{N}$ and $a > 1$. We want to compute the limit

$$\lim_{n\to\infty} \frac{n^k}{a^n}$$

- $a > 1$ implies $a = 1 + h$ with $h > 0$.
- Newton's binomial implies

$$\begin{aligned}
a^n &= (1+h)^n = \sum_{k=0}^{n} \binom{n}{k} h^k \\
&= \underbrace{\binom{n}{0}}_{\geq 0} + \underbrace{\binom{n}{1} h}_{\geq 0} + \ldots + \underbrace{\binom{n}{k} h^k}_{\geq 0} + \underbrace{\binom{n}{k+1} h^{k+1}}_{\geq 0} + \ldots + \underbrace{\binom{n}{n} h^n}_{\geq 0} \\
&\geq \binom{n}{k+1} h^{k+1} = \frac{n\,(n-1)\cdots(n-k)}{(k+1)!} h^{k+1}.
\end{aligned}$$

- Whence

$$0 \leq \frac{n^k}{a^n} \leq \frac{\overbrace{n^k}^{k-\text{factors}}}{h^{k+1} \underbrace{n\,(n-1)\cdots(n-k)}_{k+1-\text{factors}}} (k+1)! = \frac{(k+1)!}{h^{k+1}\, n\,\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k}{n}\right)} \to 0$$

Let $x \in \mathbb{R}$ be a real number. First remember that

$$S_n(x) = 1 + x + x^2 + x^3 + \ldots + x^n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

so

$$\lim_{n \to \infty} S_n(x) = \begin{cases} \frac{1}{1-x} & \text{if } |x| < 1 \\ +\infty & \text{if } x > 1 \\ \nexists & \text{if } x \leq -1 \end{cases}.$$

IMPORTNAT!

This results is very important!! It is also indicated with

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \Leftrightarrow |x| < 1.$$

$$1 + x + x^2 + x^3 + \ldots = \lim_{n\to\infty} \sum_{k=0}^{n} x^k = \begin{cases} \frac{1}{1-x} & \text{if } |x| < 1 \\ +\infty & \text{if } x > 1 \\ \nexists & \text{if } x \leq -1 \end{cases} .$$

Example

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = \lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = 2$$

$$1 + 2 + 2^2 + 2^3 + \ldots = \lim_{n\to\infty} \sum_{k=0}^{n} 2^k = \infty$$

$$1 - \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 + \ldots = \lim_{n\to\infty} \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

$$1 - 2 + 2^2 - 2^3 + \ldots = \lim_{n\to\infty} \sum_{k=0}^{n} (-2)^k = \nexists.$$

Consider

$$\lim_{n\to\infty} n^{\frac{1}{n}} = (+\infty)^0 = \text{ Indeterminate form!}$$

Why indeterminate? Consider for example

$$\lim_{n\to\infty} n^{1/\log_2(n)} = (+\infty)^0$$

But using the fact that $x = 2^{\log_2(x)}$ we get

$$n^{1/\log_2(n)} = 2^{\log_2\left(n^{1/\log_2(n)}\right)} = 2^{\frac{1}{\log_2(n)}\ \log_2(n)} = 2 \to 2$$

and, similarly

$$n^{1/\log_3(n)} = 3^{\log_3\left(n^{1/\log_3(n)}\right)} = 3^{\frac{1}{\log_3(n)}\ \log_3(n)} = 3 \to 3$$

so $(+\infty)^0$ **can be anything!**.

Consider

$$\lim_{n\to\infty} n^{\frac{1}{n}} = (+\infty)^0 = \text{ Indeterminate form!}$$

For n sufficiently large we have $n^{1/n} > 1$, because if, by contradiction, $n^{1/n} < 1$ then

$$n = \underbrace{n^{1/n} \cdot n^{1/n} \cdot n^{1/n} \cdots n^{1/n}}_{n-\text{times}} < 1 \cdot 1 \cdot 1 \cdots 1 = 1$$

which is impossible. Hence I can write

$$n^{1/n} = 1 + a_n$$

with $a_n \geq 0$, then

$$n = (1 + a_n)^n = 1 + \binom{n}{1} a_n + \binom{n}{2} a_n^2 + \ldots + a_n^n \geq 1 + \binom{n}{2} a_n^2 = 1 + \frac{n\,(n-1)}{2}\, a_n^2$$

or

$$n - 1 \geq \frac{n\,(n-1)}{2}\, a_n^2 \quad \Rightarrow \quad 1 \geq \frac{n}{2}\, a_n^2 \quad \Rightarrow \quad \frac{2}{n} \geq a_n^2 \geq 0$$

so $a_n \to 0$ and thus

$$n^{1/n} = 1 + a_n \to 1.$$

Sequences: notable limits

Consider

$$\lim_{n\to\infty}\frac{n^n}{n!}=\frac{+\infty}{+\infty}= \text{ Indeterminate form!}$$

Let's decompose both n^n and $n!$ in all their factors:

$$\frac{n^n}{n!}=\frac{\overbrace{n\cdot n\cdot n\cdots n\cdot n}^{n-\text{factors}}}{\underbrace{n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1}_{n-\text{factors}}}$$

Now associate to each factor on the numerator the corresponding factor to the denominator....

$$\frac{n^n}{n!}=\frac{n}{n}\cdot\frac{n}{n-1}\cdot\frac{n}{n-2}\cdots\frac{n}{2}\cdot\frac{n}{1}$$

and note that

$$\frac{n^n}{n!}=\underbrace{\frac{n}{n}}_{=1}\cdot\underbrace{\frac{n}{n-1}}_{>1}\cdot\underbrace{\frac{n}{n-2}}_{>1}\cdots\underbrace{\frac{n}{2}}_{>1}\cdot\underbrace{\frac{n}{1}}_{=n}>n\to+\infty.$$

hence

$$\lim_{n\to\infty}\frac{n^n}{n!}=+\infty.$$

Consider, for $a > 1$,

$$\lim_{n\to\infty} \frac{\log_a(n)}{n} = \frac{+\infty}{+\infty} = \text{ Indeterminate form!}$$

Nevertheless

$$\frac{\log_a(n)}{n} = \log_a\left(n^{1/n}\right)$$

since $n^{1/n} \to 1$ we have

$$\lim_{n\to\infty} \frac{\log_a(n)}{n} = \log_a(1) = 0.$$

More generally, for any $b > 0$ we have

$$\lim_{n\to\infty} \frac{\log_a(n)}{n^b} = 0.$$

Consider, for $a > 1$,

$$\lim_{n\to\infty} \frac{a^n}{n!} = \frac{+\infty}{+\infty} = \text{ Indeterminate form!}$$

Define $x_n = \frac{a^n}{n!} \geq 0$ and consider that

$$\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{a^n} \frac{n!}{(n+1)!} = \frac{a^{n+1}}{a^n} \frac{n!}{(n+1)\cdot n!} = a\,\frac{1}{n+1} \to 0.$$

This means that

$$\forall \varepsilon > 0, \exists n^* : \forall n \geq n^* \Rightarrow 0 < \frac{x_{n+1}}{x_n} < \varepsilon.$$

Consider then an $\varepsilon < 1$, then I can find an N such that, for all $n \geq N$

$$0 \leq \frac{x_{n+1}}{x_n} < \varepsilon < 1 \Rightarrow 0 \leq x_{n+1} < \varepsilon\, x_n$$

iterating we get

$$0 \leq x_{n+1} < \varepsilon\, x_n < \varepsilon^2\, x_{n-1} < \varepsilon^3\, x_{n-2} < \cdots < \varepsilon^{n-N} x_{N+1}.$$

N **is fixed**! Whence I can take $n \to +\infty$ and I get $\varepsilon^{n-N} \to 0$, obtaining

$$x_{n+1} \to 0 \Rightarrow \lim_{n\to\infty} \frac{a^n}{n!} = 0.$$

Consider any $a > 1$ and $b > 0$ then

Summary

$$\lim_{n\to\infty} \frac{\log_a(n)}{n^b} = 0 \Rightarrow \text{The power diverges faster than the logarithm}$$

$$\lim_{n\to\infty} \frac{n^b}{a^n} = 0 \Rightarrow \text{The exponential diverges faster than the power}$$

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0 \Rightarrow \text{The factorial diverges faster than the exponential}$$

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0 \Rightarrow n^n \text{ diverges faster than the factorial}$$

Example

$$\lim_{n\to\infty} \frac{3^n + \log_2(n)}{n!} = \lim_{n\to\infty} 3^n \frac{1 + \overbrace{\frac{\log_2(n)}{3^n}}^{\to 0}}{n!} = \lim_{n\to\infty} \frac{3^n}{n!} = 0$$

$$\lim_{n\to\infty} \frac{n^{1/100}}{\log_{20}(n^{100})} = \lim_{n\to\infty} \frac{n^{1/100}}{100\,\log_{20}(n)} = +\infty$$

$$\lim_{n\to\infty} \frac{n! + n^{200000}}{n^n} = \lim_{n\to\infty} n! \frac{1 + \overbrace{\frac{n^{200000}}{n!}}^{\to 0}}{n^n} = \lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

Definition

Let a_n and b_n be two sequences. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

we say that a_n and b_n are asymptotically equivalent, and we write $a_n \sim b_n$.

Theorem

Suppose that $a_n \sim b_n$ and that $b_n \to \ell > 0$. Then $a_n \to \ell$.

Proof. Since $a_n \sim b_n$ then for all $\varepsilon > 0$ and for $n \geq n^*$ we get

$$(1-\varepsilon) < \frac{a_n}{b_n} < (1+\varepsilon) \Rightarrow b_n\,(1-\varepsilon) < a_n < (1+\varepsilon)\;b_n$$

Now as $\varepsilon \to 0$ we have $n \to \infty$ so $b_n \to \ell$ and also $(1 \pm \varepsilon) \to 1$, whence for the comparison theorem

$$a_n \to \ell.$$

Sequences: asymptotic equivalence

Example

$$
\begin{aligned}
n^2 - n &\sim n^2 \to \infty \\
n^{3/2} - n^{1/2} + n^8 &\sim n^8 \to \infty \\
\sin\left(\frac{1}{n}\right) &\sim \frac{1}{n} \to 0 \\
n^2 \sin\left(\frac{1}{n}\right) &\sim n^2 \frac{1}{n} \to \infty \\
\frac{n^9 - n^{1/10000} + \sin(n)}{n^{10}} &\sim \frac{n^9}{n^{10}} = \frac{1}{n} \to 0. \\
\frac{3^n - n^{100}}{6^n} &\sim \frac{3^n}{6^n} = \left(\frac{3}{6}\right)^n = \left(\frac{1}{2}\right)^n \to 0 \\
\frac{\log_7(n) - n^2}{3\,n^2} = &\sim \frac{-n^2}{3\,n^2} = -\frac{1}{3} \to -\frac{1}{3}.
\end{aligned}
$$

Definition

A sequence s_n is said to be

- strictly increasing if $s_n < s_{n+1}$ for all n.
- increasing if $s_n \leq s_{n+1}$ for all n.
- strictly decreasing if $s_n > s_{n+1}$ for all n.
- decreasing if $s_n \geq s_{n+1}$ for all n.

Theorem

Let s_n be increasing. Then $s_n \to \ell$ if and only if $s_n \leq M$.

Theorem

Let s_n be decreasing. Then $s_n \to \ell$ if and only if $s_n \geq M$.

Definition

The sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

is called the Euler sequence.

Theorem

The Euler sequence is increasing.

Proof. Next slide $\Rightarrow$.

Sequences: the Euler sequence

$$
\begin{aligned}
a_n &= \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k = 1+\sum_{k=1}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k \\
&= 1+\sum_{k=1}^{n}\frac{n\cdot(n-1)\cdots(n-k+1)}{k!}\,\frac{1}{n^k} = 1+1+\sum_{k=2}^{n}\frac{n\cdot(n-1)\cdots(n-k+1)}{n^k}\,\frac{1}{k!} \\
&= 2+\sum_{k=2}^{n}\frac{\overbrace{n\cdot(n-1)\cdots(n-k+1)}^{k-\text{ factors}}}{\underbrace{n\cdot n\cdots n}_{k-\text{ factors}}}\,\frac{1}{k!} \\
&= 2+\sum_{k=2}^{n}1\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{1}{k!} \\
&< 2+\sum_{k=2}^{n}1\cdot\left(1-\frac{1}{n+1}\right)\cdot\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)\frac{1}{k!} \\
&< 2+\sum_{k=2}^{n+1}1\cdot\left(1-\frac{1}{n+1}\right)\cdot\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)\frac{1}{k!} = a_{n+1}.
\end{aligned}
$$

In summary $a_n < a_{n+1}$.

Definition

The sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

is called the Euler sequence.

Theorem

The Euler sequence is bounded from below and from above.

Proof. **From below**: Use binomial inequality:

$$a_n = \left(1 + \frac{1}{n}\right)^n \geq 1 + n\frac{1}{n} = 2.$$

From above: Next slide ⇒.

First remember that

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{2^k}=\frac{1}{1-\frac{1}{2}}=2.$$

then note that

$$k!=1\cdot 2\cdot 3\cdots k\geq 1\cdot 2\cdot 2\cdots 2=2^{k-1}.$$

and so

$$\begin{aligned}
2<a_n &= 1+\sum_{k=1}^{n}1\cdot\underbrace{\left(1-\frac{1}{n}\right)}_{<1}\cdot\underbrace{\left(1-\frac{2}{n}\right)}_{<1}\cdots\underbrace{\left(1-\frac{k-1}{n}\right)}_{<1}\frac{1}{k!}\\
&< 1+\sum_{k=1}^{n}\frac{1}{k!}\leq 1+\sum_{k=1}^{n}\frac{1}{2^{k-1}}=1+2\sum_{k=1}^{n}\frac{1}{2^k}\\
&< 1+2\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2^k}=1+2\left(\frac{1}{1-\frac{1}{2}}-1\right)=3. \qquad (0.6)
\end{aligned}$$

Summary

- Euler sequence is defined as

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

- $a_n < a_{n+1}$ for all n.
- $2 \leq a_n < 3$ for all n.

In particular

$$\text{Increasing} \;+\; \text{bounded from above} \;\Rightarrow \exists e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Besides

$$2 \leq e < 3.$$

Definition

The number e is called the Euler's number.

Exercize

Compute

$$\lim_{n\to\infty}\left(1+\frac{7}{n^2}\right)^{n^2}.$$

Solution. Define $m = \frac{n}{\sqrt{7}}$, whence $n = m\sqrt{7}$ so that

$$\lim_{n\to\infty}\left(1+\frac{7}{n^2}\right)^{n^2} = \lim_{m\to\infty}\left(1+\frac{1}{m^2}\right)^{7\,m^2}$$

Call $k = m^2$ so that

$$\lim_{m\to\infty}\left(1+\frac{1}{m^2}\right)^{7\,m^2} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{7\,k} = \lim_{k\to\infty}\left(\left(1+\frac{1}{k}\right)^{k}\right)^{7} = e^7$$

The Euler sequence: an economic application.

- $[0, 1] =$ time horizon, for example one year.
- $R =$ interest rate, defined as (Final Value-Initial Value)/Initial Value.
- If I invest 1\$ at time $t = 0 \Longrightarrow$ I get $(1 + R)$ \$ at time $t = 1$.

What happens if I compound in two periods?

$$\left(1 + \frac{R}{2}\right)\left(1 + \frac{R}{2}\right) = \left(1 + \frac{R}{2}\right)^2.$$

What happens if I compound in three periods?

$$\left(1 + \frac{R}{3}\right)\left(1 + \frac{R}{3}\right)\left(1 + \frac{R}{3}\right) = \left(1 + \frac{R}{3}\right)^3.$$

More generally, I divide the interval $[0, 1]$ into n periods....

$$\underbrace{\left(1 + \frac{R}{n}\right)\left(1 + \frac{R}{n}\right)\cdots\left(1 + \frac{R}{n}\right)}_{n-\text{ times}} = \left(1 + \frac{R}{n}\right)^n \to e^R.$$

This is why $e^R - 1$ is called the **continuously compounded interest rate**.

Sequences: an economic application.

Let p be the price of a one kilogram of rice.

The demand curve

WIKIPEDIA *(I am not an economist): demand curve is the graph depicting the relationship between the price of a certain commodity and the amount of it that consumers are willing and able to purchase at any given price.* We assume a linear demand curve:

$$D(p) = -b\,p + a$$

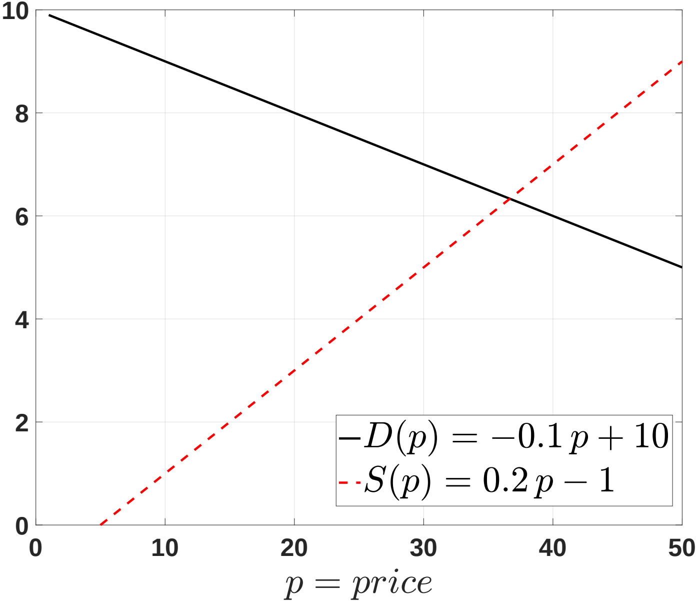
with $b > 0$ and $a > 0$.

The supply curve

WIKIPEDIA*: supply is the amount of something that firms, consumers, labourers, providers of financial assets, or other economic agents are willing to provide to the marketplace.* We assume a linear supply curve:

$$S(p) = s\,p - m$$

with $s > 0$ and $m > 0$.



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50
$p = price$
$D(p) = -0.1\,p + 10$
$S(p) = 0.2\,p - 1$

Sequences: an economic application.

Let p_0 be the initial price of one kilogram of rice. The dynamics of the price is

$$p_{n+1} = p_n + D(p_n) - S(p_n)$$

- if $D(p_n) > S(p_n)$ then $p_{n+1} > p_n$, i.e. price increases.
- if $D(p_n) < S(p_n)$ then $p_{n+1} < p_n$, i.e. price decreases.
- if $D(p_n) = S(p_n)$ then $p_{n+1} = p_n$, i.e. price stays constant.

Using the expressions of the demand and supply curve we obtain:

$$p_{n+1} = p_n - b\,p_n + a - s\,p_n + m = \underbrace{a + m}_{\alpha} + p_n \underbrace{(1 - b - s)}_{\beta} = \alpha + \beta\, p_n$$

whence ...

$$\begin{aligned}
p_1 &= \alpha + \beta\, p_0 \\
p_2 &= \alpha + \beta\, p_1 = \alpha + \beta\,(\alpha + \beta\, p_0) = \alpha + \alpha\,\beta + \beta^2\, p_0 = \alpha\,(1 + \beta) + \beta^2\, p_0 \\
p_3 &= \alpha + \beta\, p_2 = \alpha + \beta\,\left(\alpha + \alpha\,\beta + \beta^2\, p_0\right) = \alpha\,\left(1 + \beta + \beta^2\right) + \beta^3\, p_0 \\
&\vdots \\
p_n &= \alpha\,\left(1 + \beta + \beta^2 + \beta^3 + \ldots + \beta^{n-1}\right) + \beta^n\, p_0 = \alpha\,\frac{1 - \beta^n}{1 - \beta} + \beta^n\, p_0.
\end{aligned}$$

Summary

$$\begin{cases} D(p) = -b\,p + a \\ S(p) = s\,p - m \\ \alpha = a + m \\ \beta = 1 - b - s \end{cases}$$

$$p_{n+1} = p_n + D(p_n) - S(p_n) = \alpha\,\frac{1-\beta^n}{1-\beta} + \beta^n\,p_0,$$

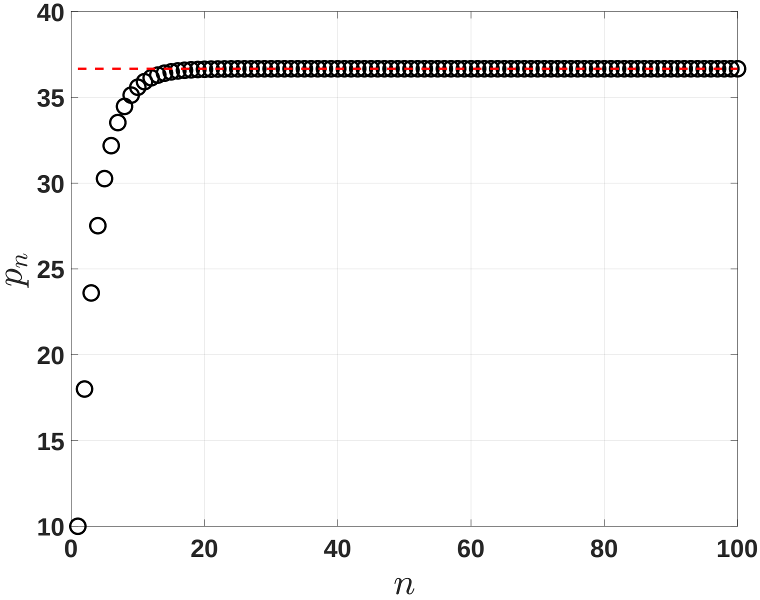
$$|\beta| < 1 \Rightarrow p_{n+1} \to p_e = \alpha\,\frac{1}{1-\beta} = \frac{a+m}{b+s}$$

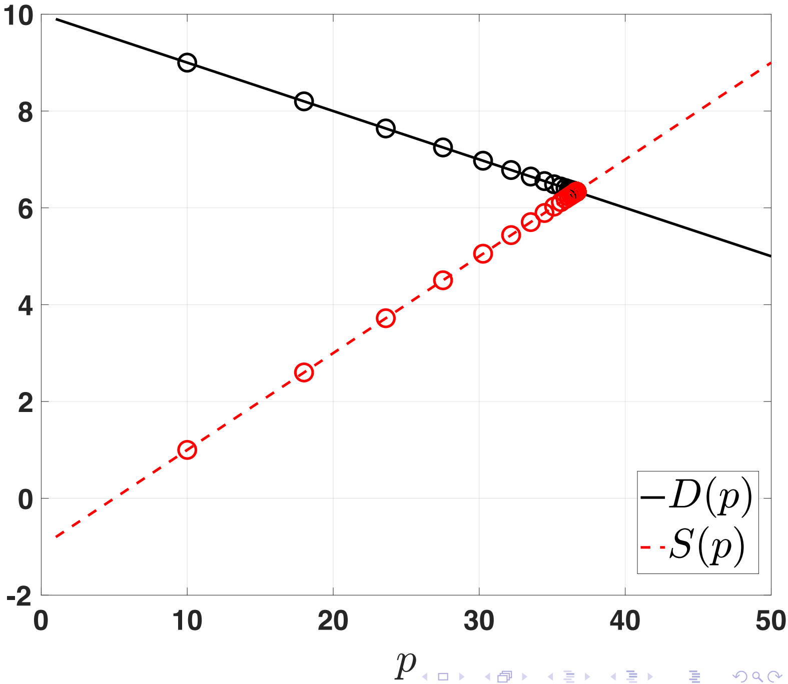
Note that

$$D(p_e) = -b\,\frac{a+m}{b+s} + a = \frac{-b\,a - b\,m + a\,b + a\,s}{b+s} = \frac{a\,s - b\,m}{b+s}$$

$$S(p_e) = s\,\frac{a+m}{b+s} - m = \frac{s\,a + s\,m - m\,b - m\,s}{b+s} = \frac{a\,s - b\,m}{b+s} = D(p_e)$$

$p_0 = 10$, $b = 0.1$, $a = 10$, $s = 0.2$, $m = 1$, $p_e = \frac{a+m}{b+s} = 36.6667$.

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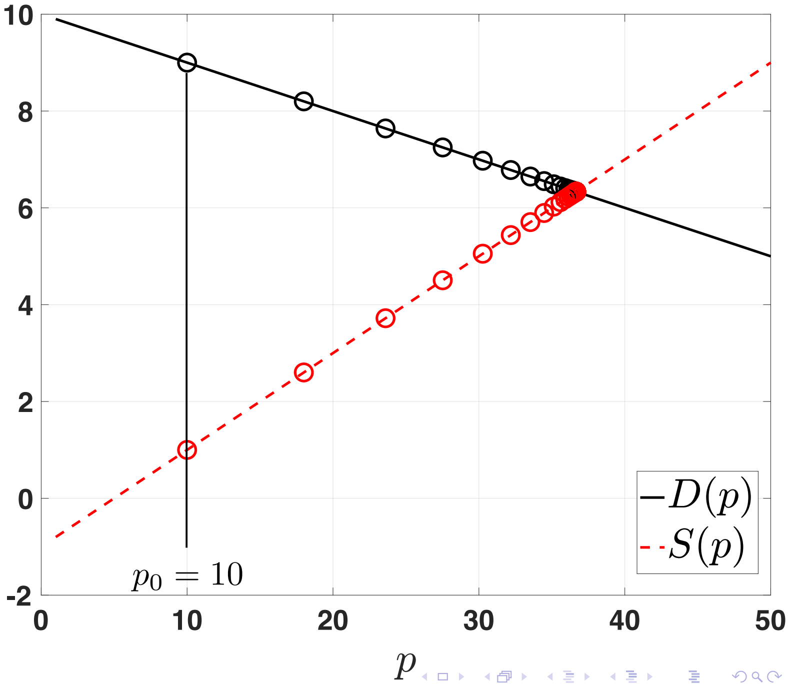
p
$D(p)$
$S(p)$



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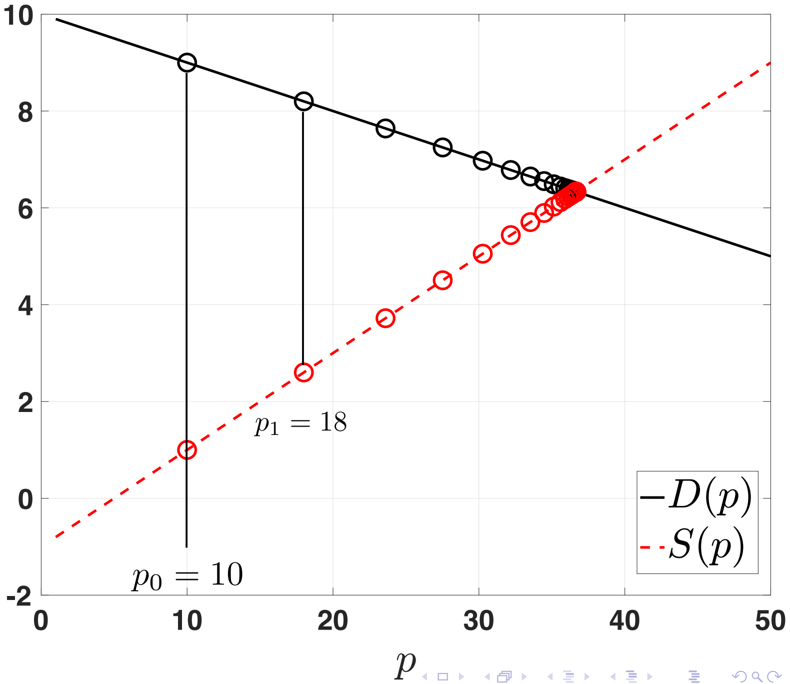
p
$p_0 = 10$
$D(p)$
$S(p)$


10
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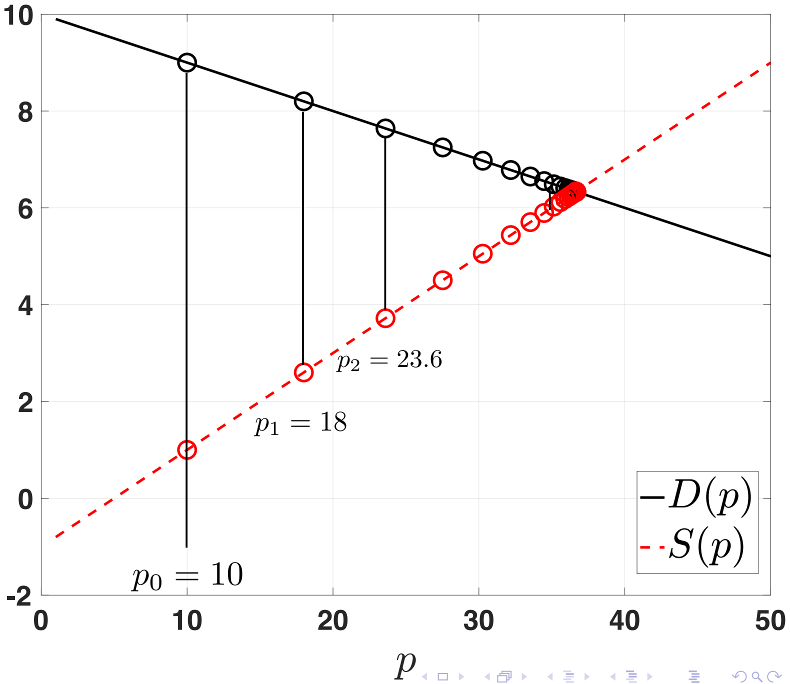
50
p
$p_0 = 10$
$p_1 = 18$
$D(p)$
$S(p)$



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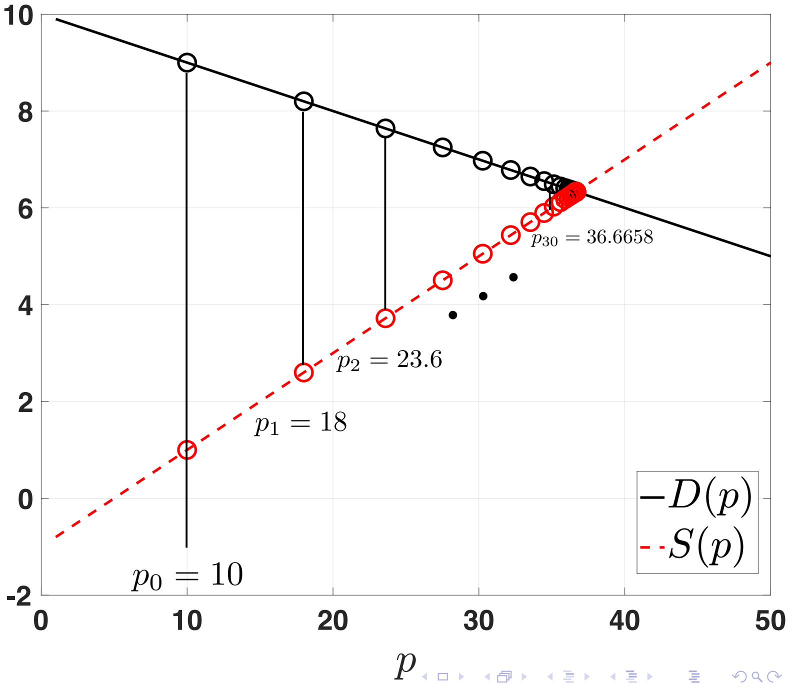
50
p
$p_0 = 10$
$p_1 = 18$
$p_2 = 23.6$
$D(p)$
$S(p)$



10
8
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0
-2
0
10
20
30
40
50
p
$p_0 = 10$
$p_1 = 18$
$p_2 = 23.6$
$p_{30} = 36.6658$
$D(p)$
$S(p)$

Definition

Let $s_n : \mathbb{N} \to \mathbb{R}$ be a sequence. Let $n_k : \mathbb{N} \to \mathbb{N}$ be a **strictly increasing sequence of integers.** The sequence s_{n_k} is called a sub-sequence of s_n.

Example

$$s_n = \frac{1}{n} \quad \Rightarrow \quad s_{n^2} = \frac{1}{n^2}$$

$$s_n = (-1)^n \ n \quad \Rightarrow \quad s_{2\,n} = (-1)^{2\,n} \ (2\,n) = 2\,n.$$

$$s_n = \frac{n}{n+1} \quad \Rightarrow \quad s_{n^2+n} = \frac{n^2+n}{n^2+n+1}. \tag{0.7}$$

Theorem

Let $s_n : \mathbb{N} \to \mathbb{R}$ *be a sequence.*

$$s_n \to \ell \Leftrightarrow \forall s_{n_k} \text{ subsequence of } s_n \text{ it holds that } s_{n_k} \to \ell.$$

Example

The classical example is the alternating sequence $s_n = (-1)^n$. This sequence cannot have limit since the two sub-sequences

$$s_{2\,n} = +1 \to +1$$

and

$$s_{2\,n+1} = -1 \to -1.$$

converge to two different limits.

Exercize

Compute, if exists, the limit of the sequence

$$\begin{cases} s_0 & = 2 \\ s_{n+1} & = \frac{1}{2}\left(s_n + \frac{2}{s_n}\right). \end{cases}$$

Solution. First of all note that $s_0 > 0$ and that if $s_n > 0$ then also $s_{n+1} > 0$. So $s_n > 0$ for all n. We compute explicitly some elements of the sequence:

$$\begin{aligned} s_0 &= 2 \\ s_1 &= \frac{1}{2}\left(s_0 + \frac{2}{s_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{3}{2} < s_0 \\ s_2 &= \frac{1}{2}\left(s_1 + \frac{2}{s_1}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{17}{12} < \frac{3}{2} \end{aligned} \tag{0.8}$$

The guess is that the sequence is decreasing ...

$$\begin{cases} s_0 & = 2 \\ s_{n+1} & = \frac{1}{2}\left(s_n + \frac{2}{s_n}\right). \end{cases}$$

From the recursive equation we get ...

$$s_{n+1} = \frac{1}{2}\,\frac{s_n^2+2}{s_n} \;\Rightarrow\; 2\,s_n\,s_{n+1} = s_n^2+2 \;\Rightarrow\; s_n^2 - 2\,s_{n+1}\,s_n + 2 = 0.$$

the Δ must be positive hence

$$4\,s_{n+1}^2 - 8 \geq 0 \;\Rightarrow\; s_{n+1}^2 \geq 2$$

whence the sequence is **bounded from below**. Is it decreasing? Let's compute

$$s_{n+1} - s_n = \frac{1}{2}\,\frac{s_n^2+2}{s_n} - s_n = \frac{s_n^2+2-2\,s_n^2}{2\,s_n} = \frac{2-s_n^2}{\underbrace{2\,s_n}_{>0}} \leq 0$$

that is $s_{n+1} - s_n \leq 0$ i.e. the sequence is decreasing.

Summary

$$\begin{cases} s_0 & = 2 \\ s_{n+1} & = \frac{1}{2}\left(s_n + \frac{2}{s_n}\right). \end{cases}$$

- $s_n^2 \geq 2$, i.e. the sequence is bounded from below.
- $s_n \geq s_{n+1}$, i.e. the sequence is decreasing.

Whence the sequence has a limit! In other words ... there exists ℓ such that

$$\lim_{n\to\infty} s_n = \ell.$$

How to compute it? If $s_n \to \ell$ then also $s_{n+1} \to \ell$, whence the recursive equation implies

$$\ell = \frac{1}{2}\left(\ell + \frac{2}{\ell}\right) \implies \ell^2 = \frac{\ell^2}{2} + 1 \implies \ell^2 = 2$$

but $s_n > 0$ so $\ell \geq 0$, whence $\ell = +\sqrt{2}$.

Sequences defined by recurrence

Remarks

$$\begin{cases} s_0 &= 2 \\ s_{n+1} &= \frac{1}{2}\left(s_n + \frac{2}{s_n}\right). \end{cases}$$

- $s_n \in \mathbb{Q}$, for all n.
- $s_n \to \sqrt{2} = 1.414213562373095\ldots \notin \mathbb{Q}$.

The sequence is also an algorithm to compute (with arbitrary precision) the number $\sqrt{2}$.

$$\begin{aligned} s_0 &= 2 \\ s_1 &= \frac{1}{2}\left(s_0 + \frac{2}{s_0}\right) = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{3}{2} = 1.5 \\ s_2 &= \frac{1}{2}\left(s_1 + \frac{2}{s_1}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{17}{12} = 1.41\bar{6} \\ s_3 &= \frac{1}{2}\left(s_2 + \frac{2}{s_2}\right) = \frac{1}{2}\left(\frac{17}{12} + \frac{24}{17}\right) = \frac{577}{408} = 1.41421568627451 0\ldots \end{aligned} \tag{0.9}$$

The larger the n the better the approximation...

Sequences defined by recurrence

Exercize

Compute, if exists, the limit of the sequence

$$\begin{cases} a_1 & = 1 \\ a_{n+1} & = 3 - \frac{1}{a_n} \end{cases}$$

Solution. Let's compute explicitly some elements of the sequence

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= 3 - \frac{1}{a_1} = 3 - 1 = 2 < 3 \\
a_3 &= 3 - \frac{1}{a_2} = 3 - \frac{1}{2} = \frac{5}{2} = 2.5 < 3 \\
a_4 &= 3 - \frac{1}{a_3} = 3 - \frac{2}{5} = \frac{13}{5} = 2.6 < 3 \\
a_5 &= 3 - \frac{1}{a_4} = 3 - \frac{5}{13} = \frac{39-5}{13} = \frac{34}{13} = 2.6\overline{15384} < 3 \\
a_6 &= 3 - \frac{1}{a_5} = 3 - \frac{13}{34} = \frac{102-13}{34} = \frac{89}{34} = 2.617647058823529 < 3 \ldots
\end{aligned}$$

It seems increasing and bounded from above...let's prove it.

- **Induction hypothesis:** $a_n \leq 3$. The induction hypothesis implies $\frac{1}{a_n} \geq \frac{1}{3}$ or $-\frac{1}{a_n} \leq -\frac{1}{3}$.

  Then

  $$a_{n+1} = 3 - \frac{1}{a_n} \leq 3 - \frac{1}{3} = \frac{9-1}{3} = \frac{8}{3} = 2.\overline{6} < 3.$$

  and since, for example, $a_1 < 3$, form the principle of induction we have $a_n < 3$ for all n.

- **Induction hypothesis:** $a_n < a_{n+1}$. The induction hypothesis implies $\frac{1}{a_n} > \frac{1}{a_{n+1}}$.

  Then

  $$a_{n+2} = 3 - \frac{1}{a_n + 1} > 3 - \frac{1}{a_n} = a_{n+1}.$$

  Since, for example, $a_1 < a_2$, form the principle of induction we have $a_n < a_{n+1}$ for all n.

Whence

$$\exists \ell = \lim_{n\to\infty} a_n$$

$$a_{n+1} = 3 - \frac{1}{a_n} \Rightarrow \ell = 3 - \frac{1}{\ell} \Rightarrow \ell^2 - 3\,\ell + 1 = 0 \Rightarrow \ell_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

Summary

$$\begin{cases} a_1 & = 1 \\ a_{n+1} & = 3 - \frac{1}{a_n} \end{cases}$$

The limit ℓ exists and it must verifies

$$\ell^2 - 3\,\ell + 1 = 0 \;\Rightarrow\; \ell_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

The limit, when it exists, is unique!! Which is the correct solution?

$$\frac{3-\sqrt{5}}{2} = 0.381966\ldots, \qquad \frac{3+\sqrt{5}}{2} = 2.6180\ldots$$

Since $\frac{3-\sqrt{5}}{2} < 1 = a_1$ and the sequence is increasing the limit is

$$\lim_{n\to\infty} a_n = \frac{3+\sqrt{5}}{2}.$$

Series

The idea

Consider, for example, the sequence $a_k = \frac{1}{k}$.

What can I say about the sequence

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \quad ?$$

Does S_n converge? If yes, which is the limit of S_n ?

Definition

Given a sequence a_n, the sequence of the partial sums of a_n is defined as

$$S_n = \sum_{k=0}^{n} a_k.$$

We call the sum of the series of the a_n the limit, **if it exists**, of the partial sums and we write:

$$S = \lim_{n\to\infty} S_n = \sum_{k=0}^{+\infty} a_k.$$

Example

We already know that

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \Leftrightarrow |x| < 1,$$

so, for example,

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{1-\frac{1}{2}} = 2.$$

Definition

The series

$$\sum_{k=0}^{\infty} x^k$$

is called the geometric series.

The necessary condition

If the series $\sum_{n=0}^{\infty} a_n$ converges then

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Since the series converge, call $L = \sum_{n=0}^{\infty} a_n$. Call $S_n = \sum_{k=0}^{n} a_k$. Then

$$\begin{cases} S_n & \to L \\ S_{n-1} & \to L \end{cases}$$

Now note that

$$a_n = \underbrace{a_n + a_{n-1} + \ldots + a_0}_{S_n} - \underbrace{(a_{n-1} + \ldots + a_0)}_{S_{n-1}} = S_n - S_{n-1} \to L - L = 0$$

The meaning of necessary condition

If a **SEQUENCE** a_n does not verify the necessary condition, i.e. if $\lim_{n\to\infty} a_n \neq 0$ it is **IMPOSSIBLE** that the **CORRESPONDING SERIES**

$$\sum_{n=0}^{\infty} a_n$$

converges!

Example

$$a_k = \frac{k}{k+1} \to 1$$

whence we can say that the series

$$\sum_{k=0}^{\infty} \frac{k}{k+1}$$

does not converge. In particular

$$\sum_{k=0}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots = +\infty.$$

The meaning of necessary condition

The condition

$$\lim_{n\to\infty} a_n = 0$$

is **necessary** but **NOT SUFFICIENT**!!

Consider $a_k = \frac{1}{k} \to 0$. Nevertheless

$$\sum_{k=1}^{2^2} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2}$$

$$\sum_{k=1}^{2^3} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

More generally we can prove, by induction, that

$$\sum_{k=1}^{2^n} \frac{1}{k} > 1 + \frac{n}{2} \to \infty.$$

so the sequence of the partial sum $S_n = \sum_{k=1}^{n} \frac{1}{k}$ cannot have a finite limit.

Theorem

Suppose that a_k and b_k are two sequences such that

$$0 \leq a_k \leq b_k$$

for $k \geq k_0$, with k_0 fixed.
If the series $\sum_{k=0}^{\infty} b_k$ is convergent then also the series $\sum_{k=0}^{\infty} a_k$ is convergent.
If the series $\sum_{k=0}^{\infty} a_k$ is divergent then also the series $\sum_{k=0}^{\infty} b_k$ is divergent.

Proof. The proof is a straightforward application of the comparison theorem two the sequences of partial sums

$$S_n = \sum_{k=0}^{n} a_k \text{ and } S_n' = \sum_{k=0}^{n} b_k .$$

Theorem

Suppose that a_k is a sequence such that

$$a_1 \geq a_2 \geq a_3 \geq \ldots > 0.$$

Then the series

$$\sum_{k=1}^{\infty} a_k$$

converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k}$$

converges.

Proof. Using $a_1 \geq a_2 \geq a_3 \geq \ldots > 0$ we get

$$\begin{aligned}
\sum_{k=1}^{\infty} a_k &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \ldots \\
&\leq a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + \ldots \\
&= a_1 + 2\,a_2 + 4\,a_4 + \ldots \\
&= \sum_{k=0}^{\infty} 2^k a_{2^k}
\end{aligned}$$

Similarly note that

$$\begin{aligned}
\sum_{k=0}^{\infty} 2^k a_{2^k} &= a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + \ldots \\
&= a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8 + a_8 + \ldots \\
&\leq a_1 + a_1 + a_2 + a_2 + a_3 + a_3 + a_4 + a_4 + a_5 + a_5 + a_6 + a_6 + a_7 + a_7 + \ldots \\
&= 2 \sum_{k=1}^{\infty} a_k ,
\end{aligned}$$

In summary

$$0 \leq \sum_{k=1}^{\infty} a_k \leq \sum_{k=0}^{\infty} 2^k a_{2^k} \leq 2 \sum_{k=1}^{\infty} a_k .$$

whence the thesis.

The harmonic series

Theorem

Let $p > 0$. The series

$$\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{p}$$

is called the harmonic series and it converges if and only if

$$p > 1.$$

Proof. By the condensation test the series converges if and only if it is convergent the series

$$\sum_{k=1}^{\infty} 2^k \left(\frac{1}{2^k}\right)^{p} = \sum_{k=1}^{\infty} \frac{2^k}{2^{k\,p}} = \sum_{k=1}^{\infty} \left(\frac{2}{2^p}\right)^{k} = \sum_{k=1}^{\infty} \left(2^{1-p}\right)^{k}$$

and we know that the geometric series

$$\sum_{k=1}^{\infty} \left(2^{1-p}\right)^{k} = \sum_{k=1}^{\infty} x^k \text{ with } x = 2^{1-p}$$

converges if and only if $2^{1-p} < 1$, which means if and only if $p > 1$.

Example

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k} &= +\infty \\
\sum_{k=1}^{\infty} \frac{1}{k^{1.0000001}} &< +\infty \\
\sum_{k=1}^{\infty} \frac{1}{k^{1/2}} &= +\infty \\
\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} &< +\infty \\
\sum_{k=1}^{\infty} \frac{1}{k^{2}} &< +\infty \\
\vdots \quad & \quad \vdots
\end{aligned}
$$

Definition

Let a_k be a sequence. If

$$\sum_{k=0}^{\infty} |a_k|$$

is convergent then the series $\sum_{k=0}^{\infty} a_k$ is said to be absolutely convergent.

Theorem

If a series $\sum_{k=0}^{\infty} a_k$ is absolutely convergent then it is also convergent, that is

$$\sum_{k=0}^{\infty} |a_k| < \infty \Rightarrow \sum_{k=0}^{\infty} a_k < \infty.$$

Absolute convergence

Proof. Remember that

$$-|a_n| \leq a_n \leq |a_n| \qquad (\triangle)$$

adding $|a_n|$ to $(\triangle)$, we get

$$0 \leq a_n + |a_n| \leq 2\,|a_n|$$

whence

$$0 \leq \sum_{k=1}^{n} (a_k + |a_k|) \leq 2 \sum_{k=1}^{n} |a_k| < 2 \sum_{k=1}^{\infty} |a_k|\,. \qquad (\square)$$

From $(\square)$ we deduce that

$$s_n = \sum_{k=1}^{n} (a_k + |a_k|)$$

is a monotonic increasing bounded (from avoe) sequence $\Rightarrow$ it must converge.

The original series

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (a_k + |a_k|) - \sum_{k=1}^{n} |a_k| = s_n - \sum_{k=1}^{n} |a_k|$$

is the difference of two converging sequences $\Rightarrow$ it must converge.

Exercize

Establish if the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

is convergent or not

Solution. Since the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

is convergent then the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

is absolutely convergent and then it converges.

Theorem

Let a_k be a sequence. Assume that the limit

$$\alpha = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| .$$

exists and it is finite.

- *If $\alpha < 1$ then the series $\sum_{k=0}^{\infty} a_k$ converges.*
- *If $\alpha > 1$ then the series $\sum_{k=0}^{\infty} a_k$ does not converge.*
- *If $\alpha = 1$ the test gives no information.*

Proof. Suppose that

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \alpha < 1.$$

Whence, for sufficiently large k (i.e. $k \geq k_0$), we have

$$0 \leq \left|\frac{a_{k+1}}{a_k}\right| \leq r < 1, \quad (\triangle)$$

for $r \in \mathbb{R}$. From $(\triangle)$ we deduce

$$0 \leq |a_{k+1}| \leq r\, |a_k|$$

Iterating we get

$$0 \leq |a_{k+1}| \leq r\, |a_k| \leq r^2\, |a_{k-1}| \leq r^3 \leq |a_{k-2}| \leq \ldots \leq r^{k-k_0+1} a_{k_0}$$

whence, consider a $K > k_0$

$$0 \leq \sum_{k=0}^{K} |a_k| = \sum_{k=0}^{k_0-1} |a_k| + \sum_{k=k_0}^{K} |a_k| \leq \sum_{k=0}^{k_0-1} |a_k| + a_{k_0}\, r^{-k_0} \sum_{k=k_0}^{K} r^k$$

Ratio test: case $0 < \alpha < 1$

$$0 \le \sum_{k=0}^{K} |a_k| = \sum_{k=0}^{k_0-1} |a_k| + \sum_{k=k_0}^{K} |a_k| \le \sum_{k=0}^{k_0-1} |a_k| + a_{k_0}\, r^{-k_0} \sum_{k=k_0}^{K} r^k$$

Now if $K \to \infty$, since $0 < r < 1$, we have

$$0 < \lim_{K\to\infty} \sum_{k=k_0}^{K} r^k < \lim_{K\to\infty} \sum_{k=0}^{K} r^k = \frac{1}{1-r} < \infty$$

hence

$$0 \le \lim_{K\to\infty} \sum_{k=0}^{K} |a_k| < \sum_{k=0}^{k_0-1} |a_k| + a_{k_0}\, r^{-k_0} \frac{1}{1-r} < \infty$$

which means that the series

$$\sum_{k=0}^{\infty} a_k$$

converges absolutely and then converges.

Exercize

Show that the series

$$\sum_{k=0}^{\infty} \frac{1}{k!}$$

converges.

Proof. Apply the ratio test:

$$\frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{1}{(k+1)!}\, k! = \frac{1}{(k+1)\ k!}\, k! = \frac{1}{k+1} \to 0.$$

Whence, by the ratio test, the series converges.

Theorem

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \sum_{k=0}^{\infty}\frac{1}{k!}$$

Proof. Newton's binomial implies that

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} = \sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!\,n^k} \\
&= \sum_{k=0}^{n}\frac{n\,(n-1)\,(n-2)\cdots(n-k+1)\,(n-k)!}{k!\,(n-k)!\,n^k} \\
&= \sum_{k=0}^{n}\frac{n\,(n-1)\,(n-2)\cdots(n-k+1)}{k!\,n^k} \\
&= \sum_{k=0}^{n}\frac{1}{k!}\frac{\overbrace{n\,(n-1)\,(n-2)\cdots(n-k+1)}^{k\text{ factors}}}{\underbrace{n\,n\cdots n}_{k\text{ factors}}} \\
&= \sum_{k=0}^{n}\frac{1}{k!}1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) < \sum_{k=0}^{n}\frac{1}{k!}.
\end{aligned}$$

In summary

$$\left(1+\frac{1}{n}\right)^n < \sum_{k=0}^{n} \frac{1}{k!}.$$

thus

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq \sum_{k=0}^{\infty} \frac{1}{k!}$$

Now take an $m < n$ and note that

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \frac{1}{k!} 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\\ &> \sum_{k=0}^{m} \frac{1}{k!} 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \qquad (0.10)\end{aligned}$$

Take m fixed and let $n \to \infty$, obtaining

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq \sum_{k=0}^{m} \frac{1}{k!}$$

Now take $m \to \infty$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq \sum_{k=0}^{\infty} \frac{1}{k!}$$

In summary

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \le \sum_{k=0}^{\infty}\frac{1}{k!}$$

and

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \ge \sum_{k=0}^{\infty}\frac{1}{k!},$$

which is possible if and only if

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty}\frac{1}{k!}.$$

We know have a simple way to compute (with some approximation) the value of e:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty}\frac{1}{k!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots$$

Theorem

For all $n \in \mathbb{N}$ it holds that

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{n!\, n}$$

that is, we have an upper bound for the error in approximating e with the series $\sum_{k=0}^{\infty} \frac{1}{k!}$.

Proof.

$$\begin{aligned} e - \sum_{k=0}^{n} \frac{1}{k!} &= \sum_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \dots \\ &= \frac{1}{(n+1)!} + \frac{1}{(n+2)\,(n+1)!} + \frac{1}{(n+3)\,(n+2)\,(n+1)!} + \dots \\ &< \frac{1}{(n+1)!} + \frac{1}{(n+1)\,(n+1)!} + \frac{1}{(n+1)\,(n+1)\,(n+1)!} + \dots \\ &= \frac{1}{(n+1)!}\left[1 + \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \dots\right] = \frac{1}{(n+1)!}\,\frac{1}{1 - \frac{1}{(n+1)}} = \frac{1}{n!\, n}. \end{aligned}$$

Therefore:

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{n!\, n}.$$

Theorem

The number e is irrational, that is $\nexists p, q \in \mathbb{N}$ *such that*

$$e = \frac{p}{q}.$$

Proof. Assume, **by contradiction**, that $\exists p, q \in \mathbb{N}$ such that $e = p/q$. Since e is not integer, it must be

$$q > 1.$$

For all n, it holds that

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{n!\, n}$$

in particular ...

$$0 < e - \sum_{k=0}^{q} \frac{1}{k!} < \frac{1}{q!\, q} \implies 0 < q!e - q! \sum_{k=0}^{q} \frac{1}{k!} < \frac{1}{q} < 1.$$

Now

$$q!\, e = q!\, \frac{p}{q} = (q-1)!\, p \in \mathbb{N}$$

and ...

Theorem

The number e is irrational, that is $\nexists p, q \in \mathbb{N}$ *such that*

$$e = \frac{p}{q}.$$

Proof. (CONTINUED)

$$q! \sum_{k=0}^{q} \frac{1}{k!} = q! \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{q!}\right) = q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \ldots + 1$$

whence also $q! \sum_{k=0}^{q} \frac{1}{k!} \in \mathbb{N}$. Remember that

$$0 < q!e - q! \sum_{k=0}^{q} \frac{1}{k!} < \frac{1}{q} < 1$$

nevertheless since $q!e \in \mathbb{N}$ and $q! \sum_{k=0}^{q} \frac{1}{k!} \in \mathbb{N}$ we are arrived at the conclusion that, $\exists z \in \mathbb{Z}$ such that

$$0 < z < 1$$

which is impossible. Whence e is irrational.

Discounting

Let R be the annual interest rate.

$$\underbrace{1\$}_{\text{Today}} \longrightarrow \underbrace{(1+R)\,\$}_{\text{After one Year}}$$

or ...

$$\underbrace{V_0\$}_{\text{Today}} \longrightarrow \underbrace{V_0\;(1+R)\,\$}_{\text{After one Year}}$$

How much money I have to invest TODAY to get 1$ in one year?

$$\mathrm{PV}^{(1)} = \frac{1}{1+R} = \text{ Present value of one dollar in one year }.$$

Similarly ...

$$\mathrm{PV}^{(2)} = \frac{1}{(1+R)^2} = \text{ Present value of one dollar in two years }.$$

Definition

Present value: the value of all future cash flows (positive and negative) over the entire life of an investment discounted to the present.

Perpetuity

An investment that pays one dollar every year from today to the eternity.

Which is the present value of a perpetuity? I have to discount all the cashflows....

$$\text{PV} = \underbrace{\frac{1}{1+R}}_{\text{Discounting the first cashflow}} + \underbrace{\frac{1}{(1+R)^2}}_{\text{Discounting the second cashflow}} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{(1+R)^n} = \sum_{n=0}^{\infty} \frac{1}{(1+R)^n} - 1 = \frac{1}{1-\frac{1}{1+R}} - 1 = \frac{1}{R}.$$

Theorem

Limit Comparison Test. *Suppose that $a_n \geq 0$ and $b_n \geq 0$. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

($a_n \sim b_n$) then either both series ($\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$) converge or they both diverge.

Proof. For all $\varepsilon > 0$ there exists a $n_0 > 0$ such that for all $n \geq n_0$

$$-\varepsilon < \frac{a_n}{b_n} - 1 < \varepsilon \iff \underbrace{1-\varepsilon}_{c_1} < \frac{a_n}{b_n} < \underbrace{1+\varepsilon}_{c_2}.$$

Take $0 < \varepsilon < 1$ hence $c_1 > 0$ and $c_2 > 0$. So that

$$c_1 < \frac{a_n}{b_n} < c_2 \iff c_1\, b_n < a_n < c_2\, b_n.$$

If $\sum_{n=0}^{\infty} b_n = +\infty$ then $\sum_{n=0}^{\infty} a_n = +\infty$ (first inequality) etc..etc...

Theorem

Let a_k be a sequence. Assume that the limit

$$\alpha = \lim_{k\to\infty} |a_k|^{\frac{1}{k}} .$$

exists and it is finite.

- *If $\alpha < 1$ then the series $\sum_{k=0}^{\infty} a_k$ converges.*
- *If $\alpha > 1$ then the series $\sum_{k=0}^{\infty} a_k$ does not converge.*
- *If $\alpha = 1$ the test gives no information.*

Root test: case $0 < \alpha < 1$

Proof. Suppose that

$$\lim_{k\to\infty} |a_k|^{\frac{1}{k}} = \alpha < 1.$$

Whence, for sufficiently large k, we have

$$0 \leq |a_k|^{1/k} \leq r < 1, \quad (\triangle)$$

for $r \in \mathbb{R}$. From $(\triangle)$ we deduce

$$0 \leq |a_k| < r^k.$$

Nevertheless, since $0 < r < 1$, the geometric series $\sum_{k=0}^{\infty} r^k$ converges, whence

$$0 \leq \sum_{k=0}^{\infty} |a_k| < \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

and hence $\sum_{k=0}^{\infty} a_k$ converges.

Root test: case $\alpha > 1$

Proof. Suppose that

$$\lim_{k\to\infty} |a_k|^{\frac{1}{k}} = \alpha > 1.$$

Whence, for sufficiently large k, we have

$$1 < r < |a_k|^{1/k}, \quad (\triangle)$$

for $r \in \mathbb{R}$. From $(\triangle)$ we deduce

$$1 < r^k < |a_k|.$$

Nevertheless, since $r > 1$ then $\lim_{k\to\infty} r^k = +\infty$ and hence

$$\lim_{k\to\infty} |a_k| = +\infty$$

so it is not possible that $\lim_{k\to\infty} a_k = 0$, whence the **necessary** condition is violated.

Exercize

Determine whether the following series is convergent or not:

$$\sum_{n=1}^{\infty} \frac{n^6 + n^3 - n^2 - \ln(n)}{n^7}.$$

Solution. Note that the necessary condition is satisfied

$$\frac{n^6 + n^3 - n^2 - \ln(n)}{n^7} \to 0.$$

Only the asymptotic behaviour of the sequence that is summed matters.

$$\frac{n^6 + n^3 - n^2 - \ln(n)}{n^7} \sim \frac{n^6}{n^7} = \frac{1}{n}.$$

But since

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

Then

$$\sum_{n=1}^{\infty} \frac{n^6 + n^3 - n^2 - \ln(n)}{n^7} = +\infty.$$

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Solution. Note that the necessary condition is satisfied

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Only the asymptotic behaviour of the sequence that is summed matters.

$$\frac{n^6 + n^3 - n^2 - \ln(n)}{n^7} \sim \frac{n^6}{n^7} = \frac{1}{n}.$$

But since

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$$\sum_{n=1}^{\infty} \frac{n^6 + n^3 - n^2 - \ln(n)}{n^7} = +\infty.$$

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But since

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$$

Then

$$\sum_{n=1}^{\infty} \frac{n^6 + n^3 - n^2 - \ln(n)}{n^7} = +\infty.$$

Exercize

Determine whether the following series is convergent or not:

$$\sum_{n=1}^{\infty}\left(\frac{n}{2\,n+1}\right)^{n}.$$

Solution. Note that the necessary condition is satisfied

$$\ln\left(\left(\frac{n}{2\,n+1}\right)^{n}\right)=n\,\underbrace{\ln\left(\frac{n}{2\,n+1}\right)}_{\to\ln(1/2)<0}\to-\infty \quad\Rightarrow\quad \left(\frac{n}{2\,n+1}\right)^{n}\to 0.$$

Since a power to the n is involved ....

$$\left(\left(\frac{n}{2\,n+1}\right)^{n}\right)^{\frac{1}{n}}=\frac{n}{2\,n+1}\longrightarrow\frac{1}{2}<1.$$

For the root test the series

$$\sum_{n=1}^{\infty}\left(\frac{n}{2\,n+1}\right)^{n}$$

converges.

Determine whether the following series are convergent or not

- $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$. Since
$$\frac{1}{n^2+n} \sim \frac{1}{n^2},$$
the series converges.
- $\sum_{n=1}^{\infty} \frac{1+n}{\sqrt{n}}$. Since
$$\frac{1+n}{\sqrt{n}} \to +\infty$$
the series cannot converge.
- $\sum_{n=0}^{\infty} \frac{1}{n+e^n}$. Since
$$\frac{1}{n+e^n} \sim \frac{1}{e^n}$$
and since
$$\sum_{n=0}^{\infty} \frac{1}{e^n} = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}$$
the series $\sum_{n=0}^{\infty} \frac{1}{n+e^n}$ converges.

Determine whether the following series is convergent or not

$$\sum_{n=0}^{\infty} \frac{k^2}{k!}.$$

Solution. The necessary condition is satisfied

$$\lim_{k\to\infty} \frac{k^2}{k!} = 0.$$

Use ratio test

$$\begin{aligned} \frac{\frac{(k+1)^2}{(k+1)!}}{\frac{k^2}{k!}} &= \frac{(k+1)^2}{(k+1)!}\,\frac{k!}{k^2} = \frac{(k+1)^2}{(k+1)\;k!}\,\frac{k!}{k^2} \\ &= \frac{(k+1)^2}{(k+1)}\,\frac{1}{k^2} \sim \frac{k^2}{k^3} = \frac{1}{k} \to 0. \end{aligned}$$

by the ratio test the series converges.

- Determine whether the following series is convergent or not

$$\sum_{n=0}^{\infty} (-1)^k \, \frac{k^2}{k!}.$$

Solution. Since the series converges absolutely, then it converges.

- Determine whether the following series is convergent or not

$$\sum_{k=0}^{\infty} (-1)^{k-1} \, \frac{k}{7^k}.$$

Solution. Ratio test applied to the absolute value:

$$\frac{k+1}{7^{k+1}} \, \frac{7^k}{k} = \frac{k+1}{7\,k} \to \frac{1}{7} < 1$$

by the ratio test the series absolutely converges and then it converges.

Telescoping series.

Exercize

Compute the sum of the series

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right).$$

Solution. Consider the sequence of the partial sum

$$S_n=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right).$$

Note that

$$\begin{aligned} S_1 &= 1-\frac{1}{2} \\ S_2 &= 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}=1-\frac{1}{3} \\ S_3 &= 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}=1-\frac{1}{4}. \end{aligned} \tag{0.11}$$

So the guess is that

$$S_n=1-\frac{1}{n+1}.$$

Telescoping series.

Exercize

Compute the sum of the series

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right).$$

Solution. **Induction hypothesis.**

$$S_n = 1-\frac{1}{n+1} \quad (\triangle).$$

We already know that the guess is true for $n = 1, 2, 3$. Let's assume that $(\triangle)$ holds. Then

$$\begin{aligned} S_{n+1} &= S_n + \frac{1}{n+1} - \frac{1}{n+2} \\ &= 1 - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} \\ &= 1 - \frac{1}{n+2}, \end{aligned}$$

which is the induction hypothesis for $n+1$. Hence

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1 - \frac{1}{n+1} \to 1.$$