All Lectures - Part II

Davide Pirino

October 17, 2023

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0} f(x) = L$$

if and only if $\exists L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \forall x \in D : 0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0} f(x) = L$$

if and only if $\exists L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 : \forall x \in D : 0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0} f(x) = L$$

if and only if $\exists L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \forall x \in D : 0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

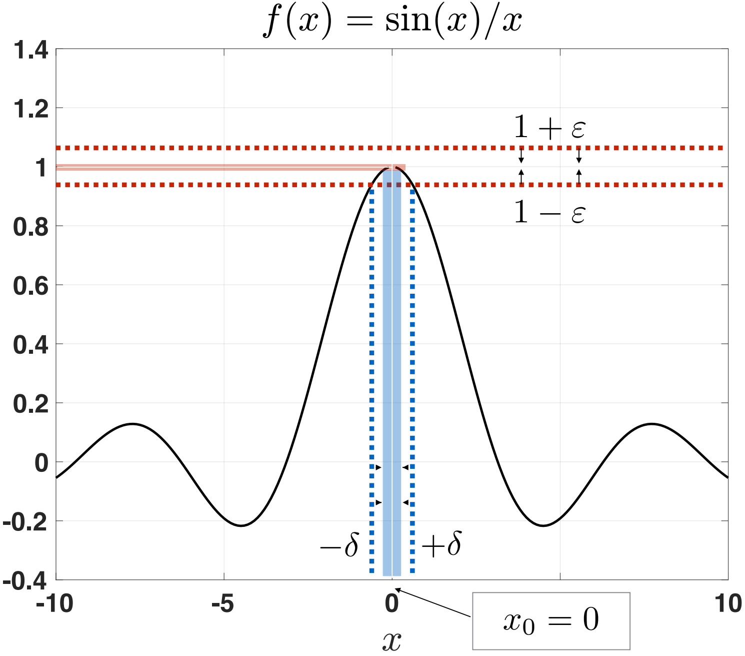
Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0} f(x) = L$$

if and only if $\exists L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 : \forall x \in D : 0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$


$f(x) = \sin(x)/x$
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-10
-5
0
10
x
$1+\varepsilon$
$1-\varepsilon$
$-\delta$
$+\delta$
$x_0 = 0$

Definition

A piecewise-defined function is a function defined by multiple sub-functions.

Example

A standard example is the absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Nevertheless, there is no limit in creating a piecewise-defined function...

$$f(x) = \begin{cases} x^2 + \sin(x) & \text{if } x < -10 \\ 0 & \text{if } -10 \leq x \leq 1 \\ -\ln(x) & \text{if } x > 1 \end{cases}$$

An important remark

Consider the function

$$f(x) = x^2.$$

It is immediate to prove that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0.$$

Consider the **piecewise-defined** function

$$g(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ 9 & \text{if } x = 0. \end{cases}$$

Which is now

$$\lim_{x \to 0} g(x) = \quad ???.$$

The value of the function in x_0 does not matter for the limiting behaviour!

$$\lim_{x \to 0} g(x) = 0.$$

An important remark

Consider the function

$$f(x) = x^2.$$

It is immediate to prove that

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0.$$

Consider the **piecewise-defined** function

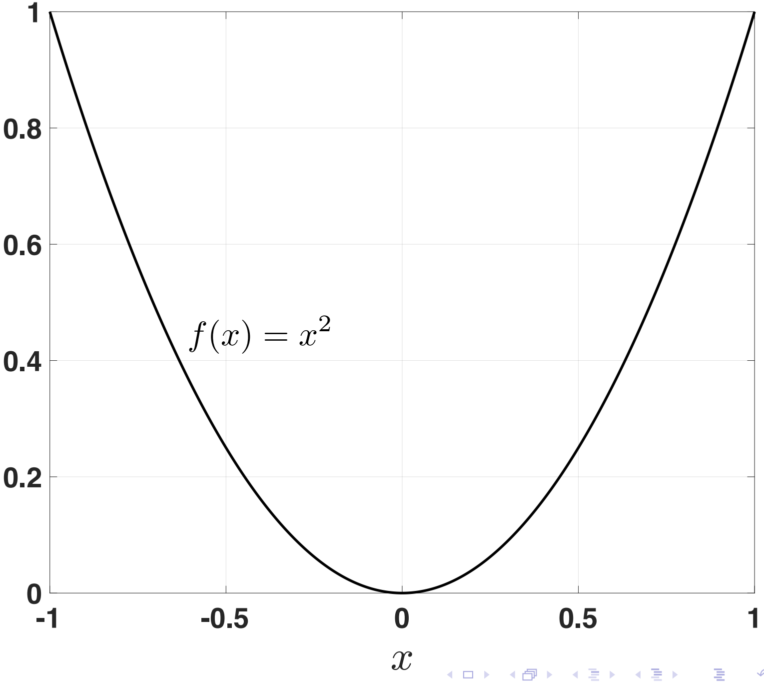
$$g(x) = \begin{cases} f(x) & \text{if } x < -\frac{1}{10} \text{ or } x > \frac{1}{10} \\ 9 & \text{if } -\frac{1}{10} < x < \frac{1}{10}. \end{cases}$$

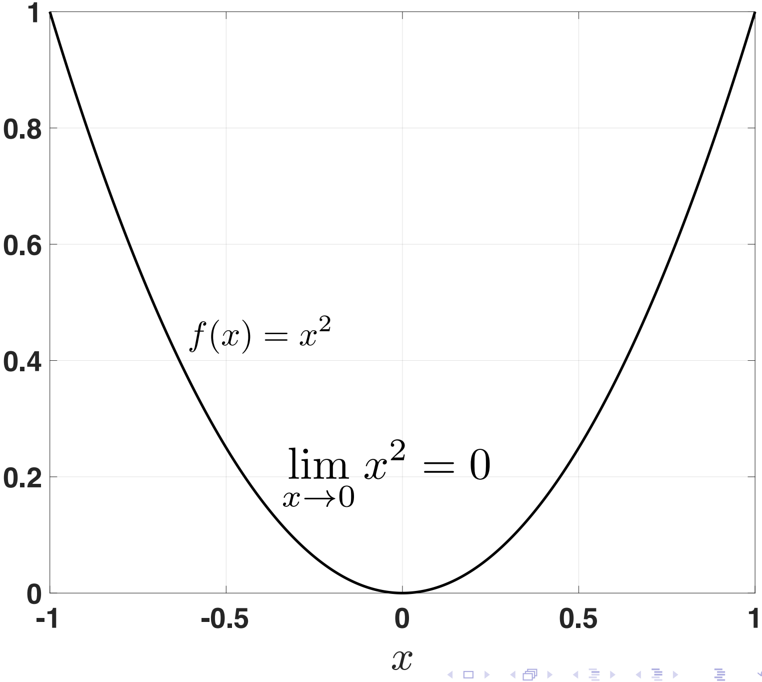
Which is now

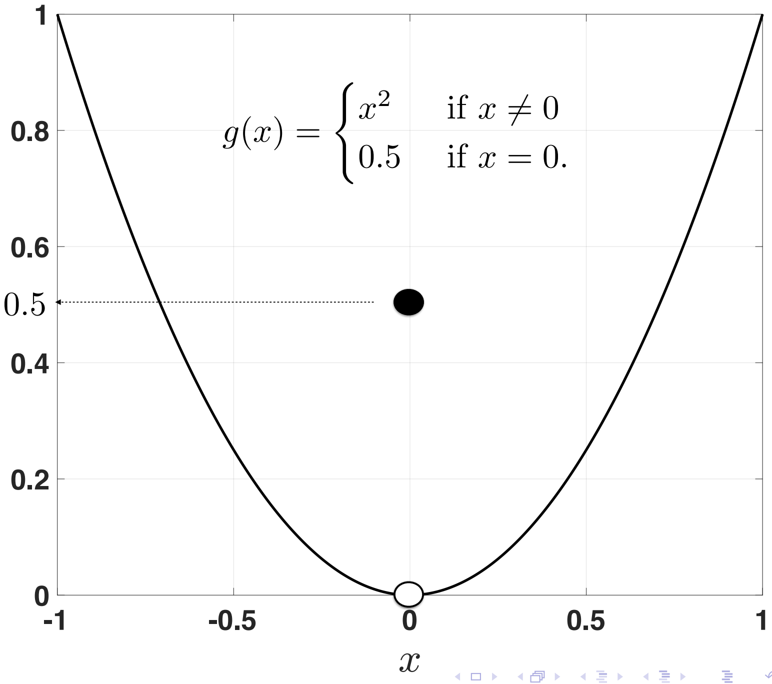
$$\lim_{x \to 0} g(x) = \quad ???.$$

The value of the function in x_0 does not matter for the limiting behaviour! Only the behaviour of the function around x_0 ...

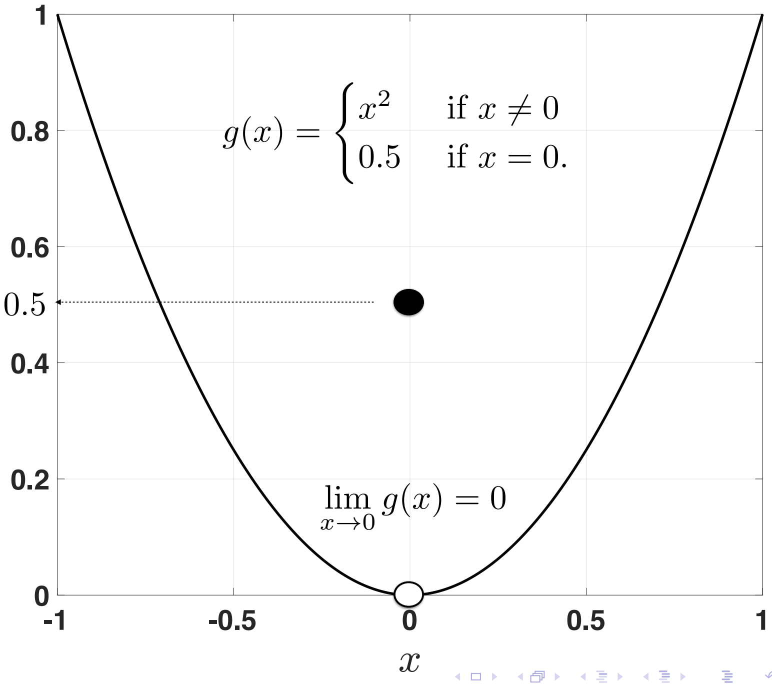
$$\lim_{x \to 0} g(x) = 9.$$


1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
$f(x) = x^2$
x

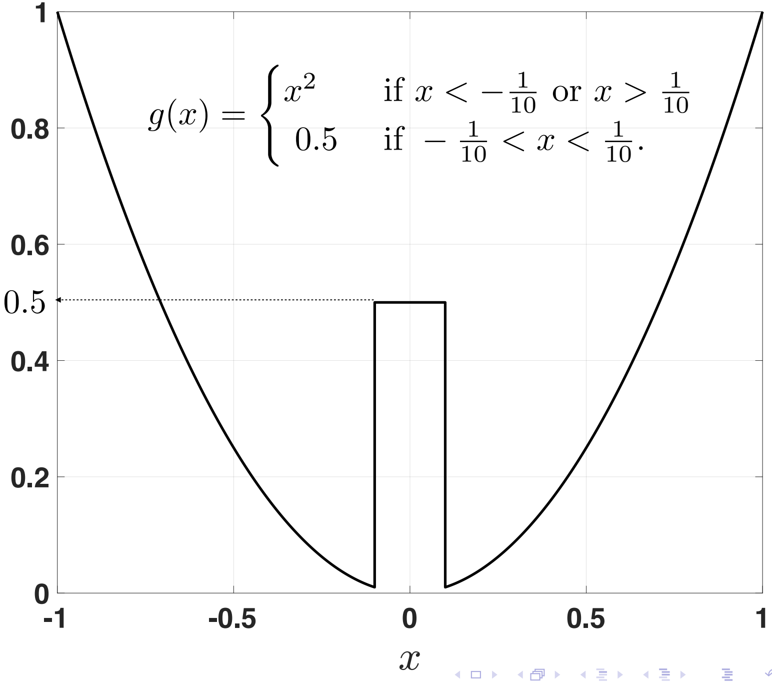

1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
$f(x) = x^2$
$\lim_{x\to 0} x^2 = 0$
x

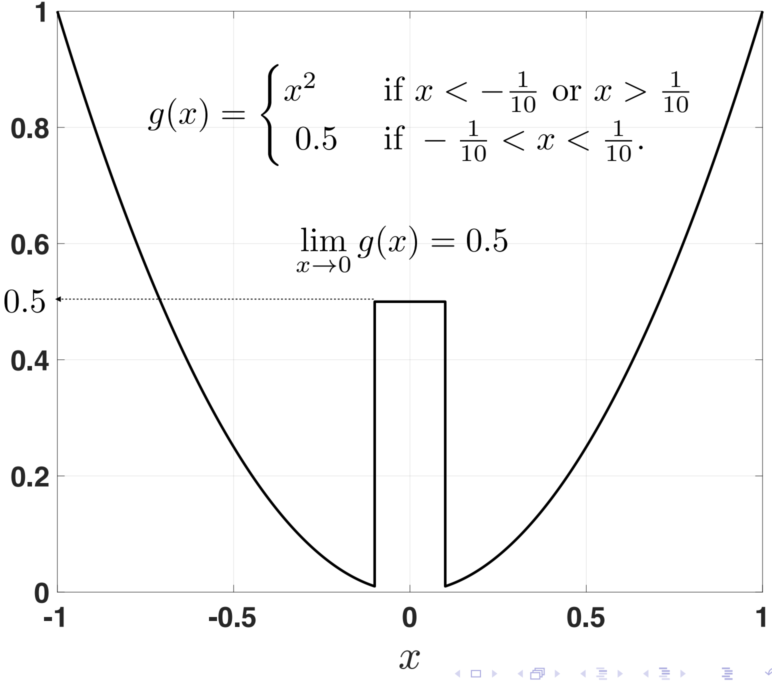


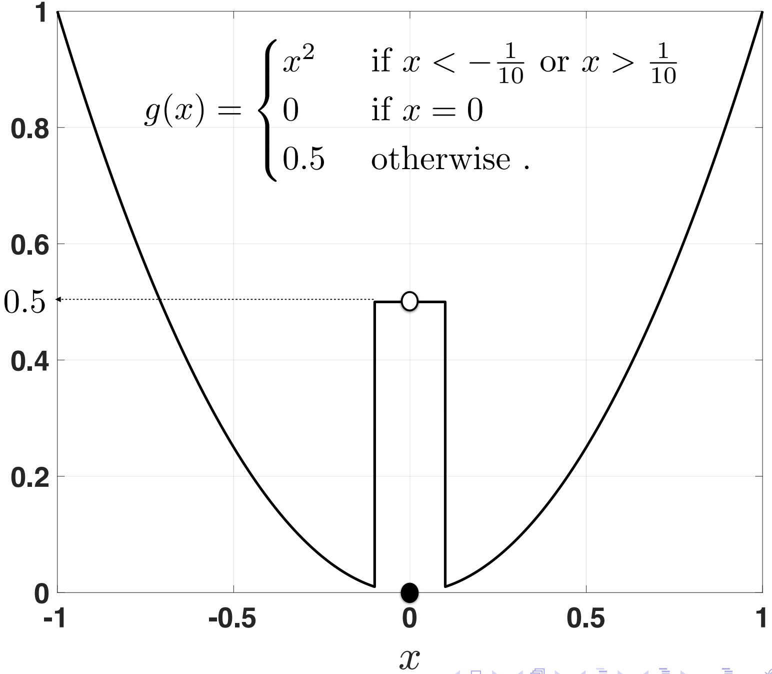
$g(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 0.5 & \text{if } x = 0. \end{cases}$
1
0.8
0.6
0.5
0.4
0.2
0
-1
-0.5
0
0.5
1
x



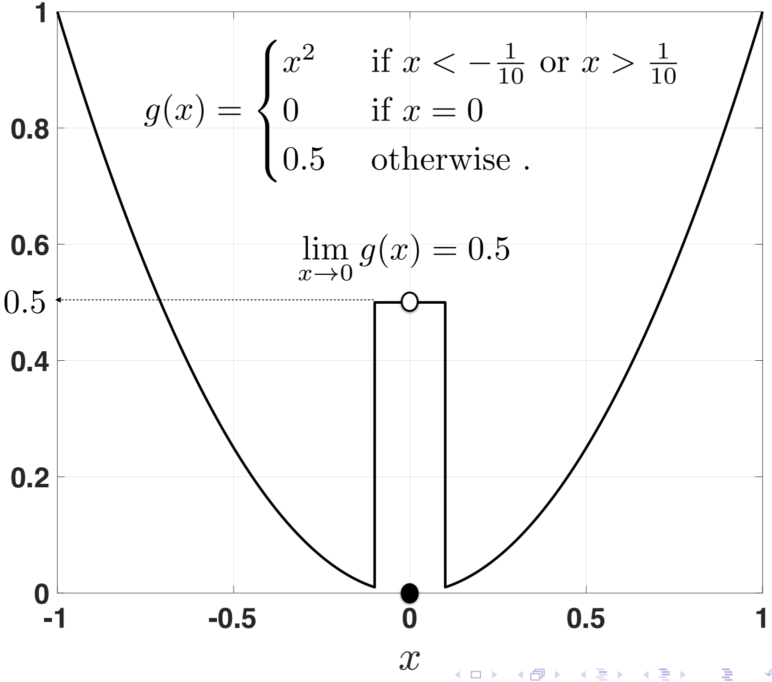
$g(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 0.5 & \text{if } x = 0. \end{cases}$
$\lim_{x \to 0} g(x) = 0$
0.5
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
x


$$g(x) = \begin{cases} x^2 & \text{if } x < -\frac{1}{10} \text{ or } x > \frac{1}{10} \\ 0.5 & \text{if } -\frac{1}{10} < x < \frac{1}{10}. \end{cases}$$
1
0.8
0.6
0.5
0.4
0.2
0
-1
-0.5
0
0.5
1
x


$g(x) = \begin{cases} x^2 & \text{if } x < -\frac{1}{10} \text{ or } x > \frac{1}{10} \\ 0.5 & \text{if } -\frac{1}{10} < x < \frac{1}{10}. \end{cases}$
$\lim_{x \to 0} g(x) = 0.5$
1
0.8
0.6
0.5
0.4
0.2
0
-1
-0.5
0
0.5
1
x



$g(x) = \begin{cases} x^2 & \text{if } x < -\frac{1}{10} \text{ or } x > \frac{1}{10} \\ 0 & \text{if } x = 0 \\ 0.5 & \text{otherwise .} \end{cases}$
1
0.8
0.6
0.5
0.4
0.2
0
-1
-0.5
0
0.5
1
x



$$g(x) = \begin{cases} x^2 & \text{if } x < -\frac{1}{10} \text{ or } x > \frac{1}{10} \\ 0 & \text{if } x = 0 \\ 0.5 & \text{otherwise .} \end{cases}$$
$\lim_{x \to 0} g(x) = 0.5$
1
0.8
0.6
0.5
0.4
0.2
0
-1
-0.5
0
0.5
1
x

Definition

Let $f:(a,+\infty)\subseteq\mathbb{R}\to\mathbb{R}$ be a function. We say that

$$\lim_{x\to+\infty} f(x) = L$$

if and only if $\exists L\in\mathbb{R}$ such that

$$\forall\varepsilon>0 \;\; \exists M>0 \text{ such that if } x>M \quad\Rightarrow\quad |f(x)-L|<\varepsilon.$$

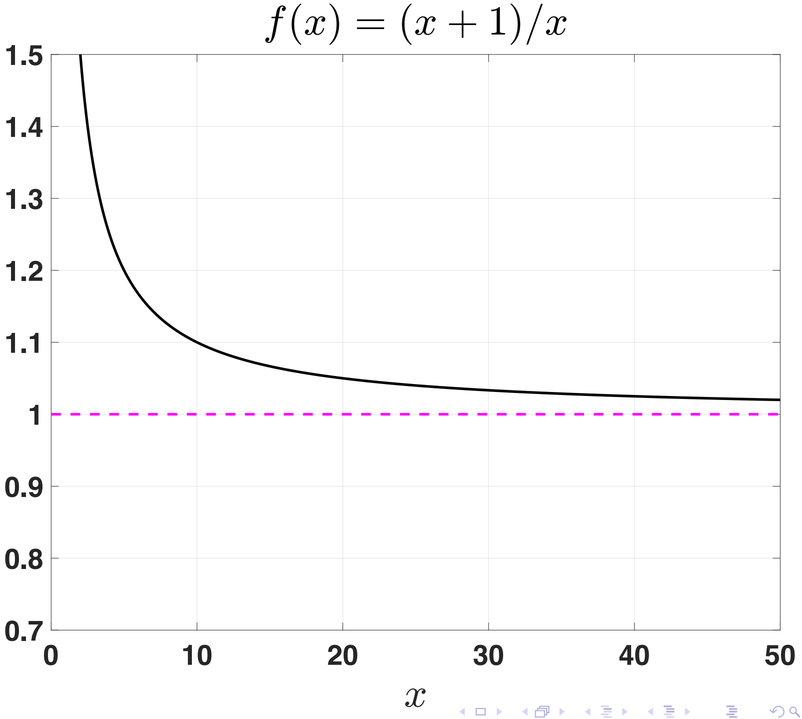
Similarly, Let $f:(-\infty,a)\subseteq\mathbb{R}\to\mathbb{R}$ be a function. We say that

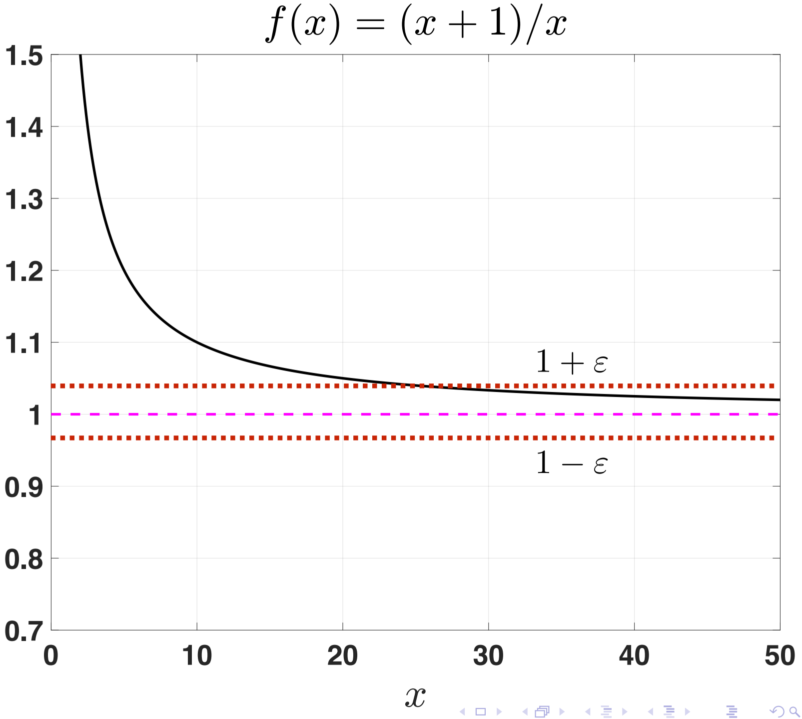
$$\lim_{x\to-\infty} f(x) = L$$

if and only if $\exists L\in\mathbb{R}$ such that

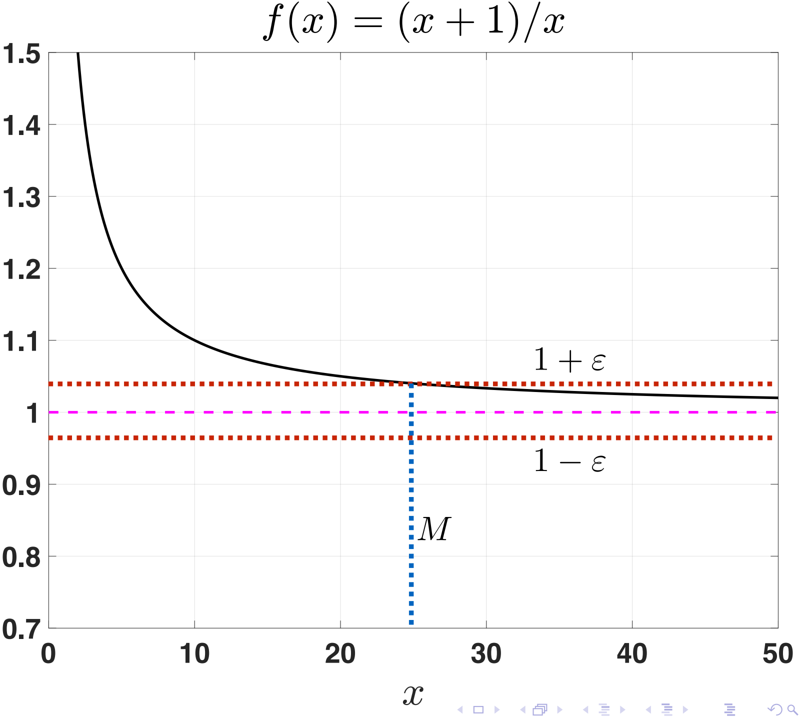
$$\forall\varepsilon>0 \;\; \exists M>0 \text{ such that if } x<-M \quad\Rightarrow\quad |f(x)-L|<\varepsilon.$$

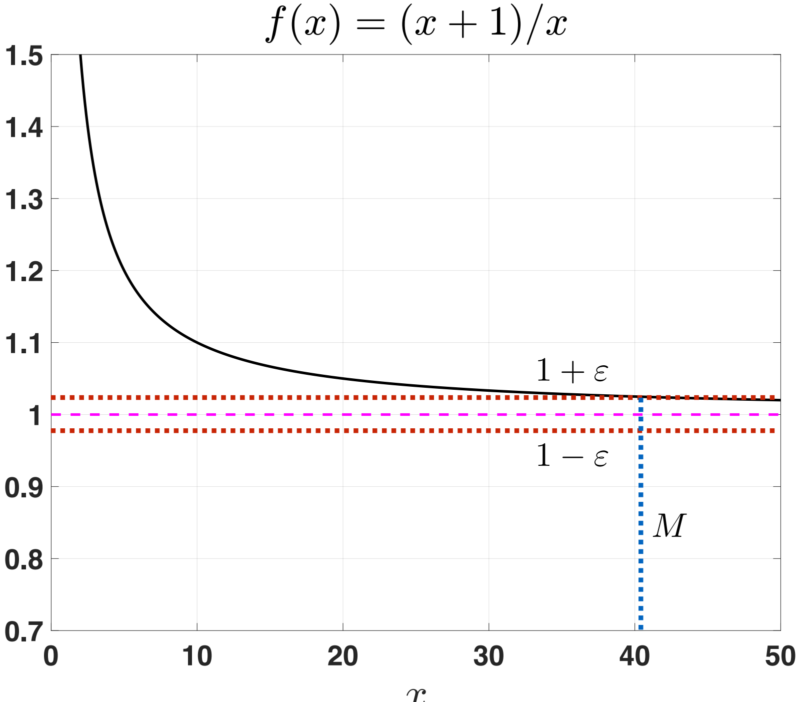
In both cases we say that the function has an horizontal asymptote at L.


$f(x) = (x+1)/x$
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
10
20
30
40
50
x



$f(x) = (x+1)/x$
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
10
20
30
40
50
$1+\varepsilon$
$1-\varepsilon$
x


$f(x) = (x+1)/x$
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
10
20
30
40
50
x
$1+\varepsilon$
$1-\varepsilon$
M


$f(x) = (x+1)/x$
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
10
20
30
40
50
x
$1+\varepsilon$
$1-\varepsilon$
M

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0} f(x) = +\infty$$

if and only if

$$\forall K > 0 \ \ \exists \delta > 0 : \forall x \in D : 0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > K.$$

Equivalently ...

$$\forall K > 0 \ \ \exists \delta > 0 : \forall x \in D : x_0 - \delta < x < x_0 + \delta, x \neq x_0 \quad \Rightarrow \quad f(x) > K.$$

We say that the function has a vertical asymptote at x_0.

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0} f(x) = -\infty$$

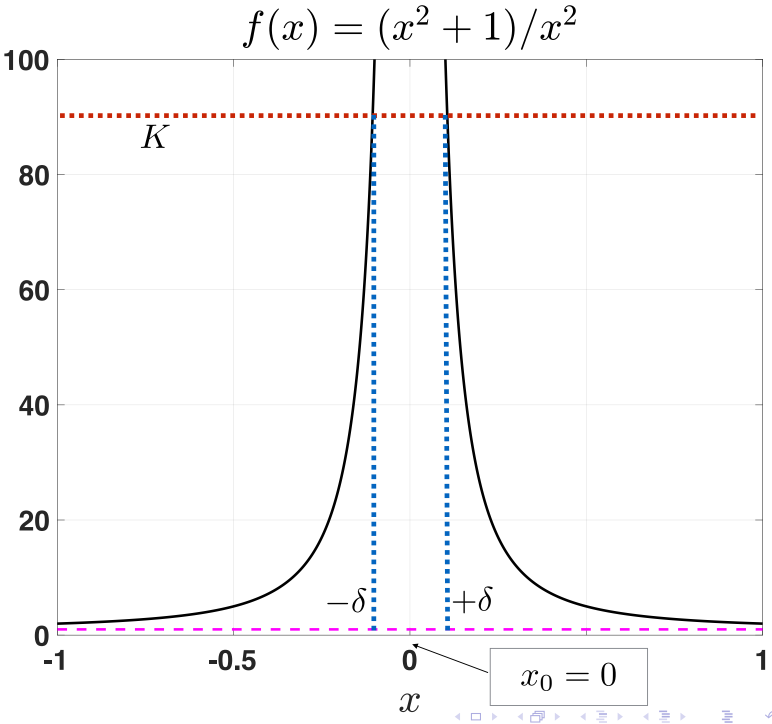
if and only if

$$\forall K > 0 \;\; \exists \delta > 0 : \forall x \in D : \; 0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -K.$$

Equivalently ...

$$\forall K > 0 \;\; \exists \delta > 0 : \forall x \in D : \; x_0 - \delta < x < x_0 + \delta, x \neq x_0 \quad \Rightarrow \quad f(x) < -K.$$

We say that the function has a vertical asymptote at x_0.


$f(x) = (x^2+1)/x^2$
100
80
60
40
20
0
-1
-0.5
0
1
x
K
$-\delta$
$+\delta$
$x_0 = 0$

Definition

Let $f : (a, +\infty) \subseteq \mathbb{R} \to \mathbb{R}$ be a function. We say that

$$\lim_{x \to +\infty} f(x) = +\infty$$

if and only if

$$\forall K > 0 \;\; \exists M > 0 \text{ such that if } x > M \quad \Rightarrow \quad f(x) > K.$$

Similarly ... let $f : (-\infty, a) \subseteq \mathbb{R} \to \mathbb{R}$ be a function. We say that

$$\lim_{x \to -\infty} f(x) = +\infty$$

if and only if

$$\forall K > 0 \;\; \exists M > 0 \text{ such that if } x < -M \quad \Rightarrow \quad f(x) > K.$$

etc ... etc ...

Theorem

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and x_0 be a limit point of D. If $f(x) \to L$ when $x \to x_0$ then the limit is unique.

Proof. Suppose, by contradiction, that there exist L_1 and L_2 with $L_1 \neq L_2$ such that

$$\lim_{x \to x_0} f(x) = L_1, \quad \lim_{x \to x_0} f(x) = L_2.$$

Hence for all $\varepsilon > 0$ I can find $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - x_0| < \delta_1 \text{ and } x \in D \text{ implies } |f(x) - L_1| < \varepsilon/2, \quad (\triangle)$$

$$0 < |x - x_0| < \delta_2 \text{ and } x \in D \text{ implies } |f(x) - L_2| < \varepsilon/2 \quad (\square).$$

Now consider $\delta = \min(\delta_1, \delta_2)$. Since x_0 is a **LIMIT POINT** of D then there exists $x^* \in D$ such that $0 < |x^* - x_0| < \delta$. Since $\delta < \delta_1$ AND $\delta < \delta_2$, for that x^* both conditions $(\triangle)$ and $(\square)$ are satisfied, whence

$$|L_1 - L_2| = |L_1 - f(x^*) + f(x^*) - L_2| \leq \underbrace{|L_1 - f(x^*)|}_{<\varepsilon/2} + \underbrace{|f(x^*) - L_2|}_{<\varepsilon/2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means $L_1 = L_2$.

Theorem

Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a function defined on the set D, subset of $\mathbb{R}$, with value in $\mathbb{R}$. Let x_0 be a limit point of D. Then:

$$\exists \lim_{x \to x_0} f(x) = L \Leftrightarrow \forall x_n \in D \text{ such that } x_n \to x_0 \text{ and } x_n \neq x_0 \Rightarrow f(x_n) \to L.$$

In practice

If we find two sequences $x_n \to x_0$ and $y_n \to x_0$ such that

$$f(x_n) \to \ell_1 \text{ and } f(y_n) \to \ell_2$$

with $\ell_1 \neq \ell_2$, in virtue of the theorem above we can conclude that the limit

$$\lim_{x \to x_0} f(x)$$

does not exist.

Example

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = ?? \quad (\triangle)$$

Remember that

$$\begin{cases} \sin\left(\frac{\pi}{2}\right) & = 1 \\ \sin\left(5\,\frac{\pi}{2}\right) & = 1 \\ \sin\left(9\,\frac{\pi}{2}\right) & = 1 \\ \vdots & \vdots \end{cases} \Rightarrow \quad \sin\left((4\,k+1)\,\frac{\pi}{2}\right) = 1, \quad \forall k \in \mathbb{N}.$$

$$\begin{cases} \sin\left(3\,\frac{\pi}{2}\right) & = -1 \\ \sin\left(7\,\frac{\pi}{2}\right) & = -1 \\ \sin\left(11\,\frac{\pi}{2}\right) & = -1 \\ \vdots & \vdots \end{cases} \Rightarrow \quad \sin\left((4\,k+3)\,\frac{\pi}{2}\right) = -1, \quad \forall k \in \mathbb{N}.$$

Example

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) = ?? \qquad (\triangle)$$

Since

$$\sin\left((4\,k+1)\,\frac{\pi}{2}\right) = 1, \quad \sin\left((4\,k+3)\,\frac{\pi}{2}\right) = -1, \quad \forall k \in \mathbb{N}.$$

consider

$$x_k = \frac{1}{(4\,k+1)\,\frac{\pi}{2}}, \quad y_k = \frac{1}{(4\,k+3)\,\frac{\pi}{2}}, \quad \forall k \in \mathbb{N}.$$

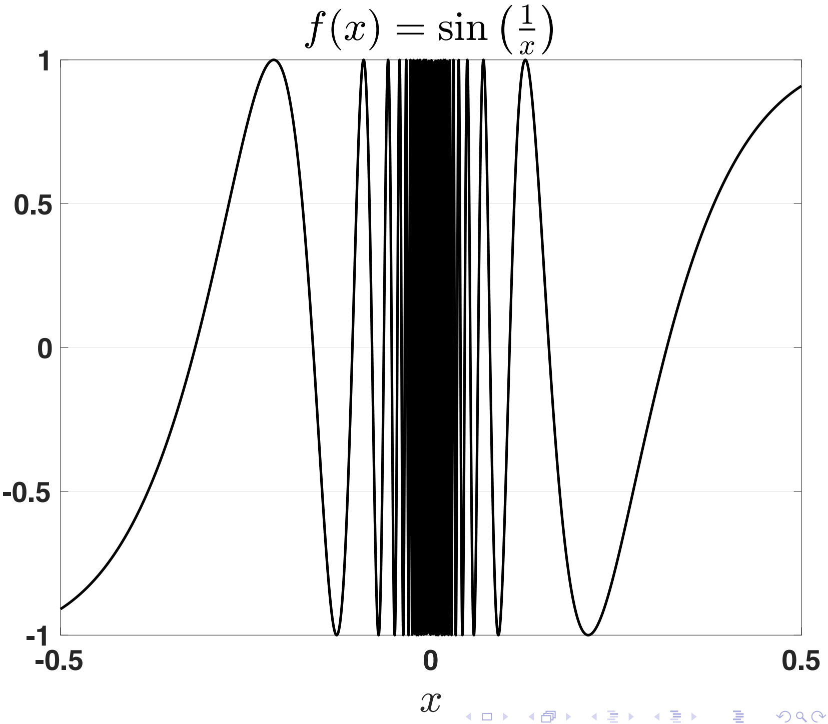
whence $x_k \to 0$ and $y_k \to 0$ nevertheless

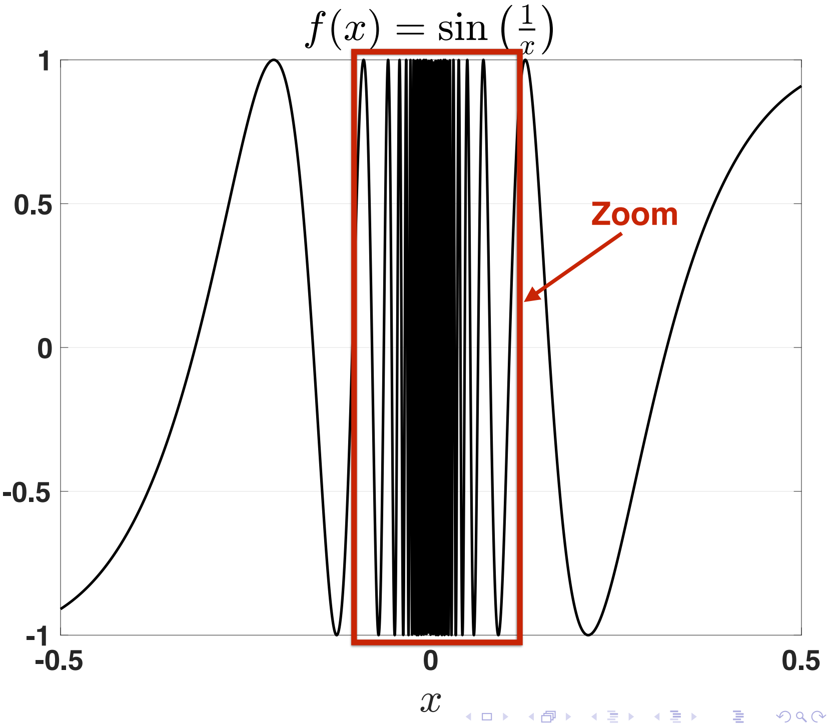
$$f\left(x_k\right) = \sin\left(\frac{1}{x_k}\right) = \sin\left((4\,k+1)\,\frac{\pi}{2}\right) = 1 \to 1$$

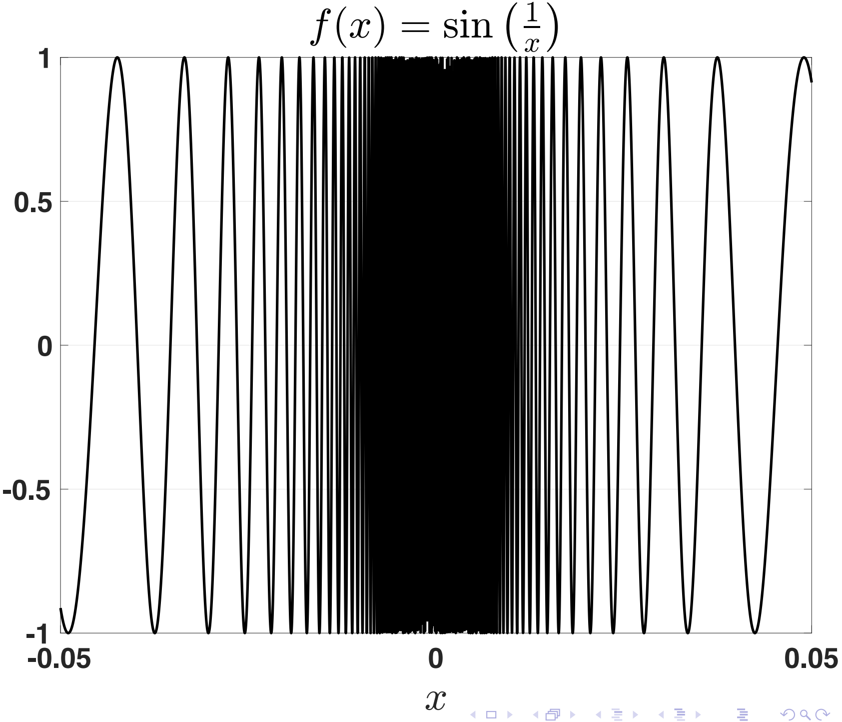
and

$$f\left(y_k\right) = \sin\left(\frac{1}{y_k}\right) = \sin\left((4\,k+3)\,\frac{\pi}{2}\right) = -1 \to -1$$

whence the limit $(\triangle)$ does not exist!


$f(x) = \sin\left(\frac{1}{x}\right)$
1
0.5
0
-0.5
-1
-0.5
0
0.5
x


$f(x) = \sin\left(\frac{1}{x}\right)$
1
0.5
0
-0.5
-1
-0.5
0
0.5
x
Zoom


$f(x) = \sin\left(\frac{1}{x}\right)$
1
0.5
0
-0.5
-1
-0.05
0
0.05
x

Example

The theorem is still valid even if $x_0 = \pm\infty$. Example $f(x) = \sin(x)$ and

$$\lim_{x \to +\infty} \sin(x) = ?? \quad (\triangle)$$

Since

$$\sin\left((4\,k+1)\,\frac{\pi}{2}\right) = 1, \quad \sin\left((4\,k+3)\,\frac{\pi}{2}\right) = -1, \quad \forall k \in \mathbb{N}.$$

consider

$$x_k = (4\,k+1)\,\frac{\pi}{2}, \quad y_k = (4\,k+3)\,\frac{\pi}{2}, \quad \forall k \in \mathbb{N}.$$

whence $x_k \to +\infty$ and $y_k \to +\infty$ nevertheless

$$f(x_k) = \sin(x_k) = \sin\left((4\,k+1)\,\frac{\pi}{2}\right) = 1 \to 1$$

and

$$f(y_k) = \sin(y_k) = \sin\left((4\,k+3)\,\frac{\pi}{2}\right) = -1 \to -1$$

whence the limit $(\triangle)$ does not exist!

Left and right limits

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0^+} f(x) = L$$

if and only if $\exists L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 : \forall x \in D : \; 0 < x - x_0 < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

similarly, we say that

$$\lim_{x \to x_0^-} f(x) = L$$

if and only if $\exists L \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 : \forall x \in D : \; 0 < x_0 - x < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Left and right limits

Theorem

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. Then the limit

$$\lim_{x \to x_0} f(x) = L$$

exists ***IF AND ONLY IF***

$$\lim_{x \to x_0^+} f(x) = L \textit{ and } \lim_{x \to x_0^-} f(x) = L.$$

In practice

We will encounter one among these situations

- $\lim_{x \to x_0^+} f(x) = L$ and $\lim_{x \to x_0^-} f(x) = L$. We can conclude that the limit $\lim_{x \to x_0} f(x)$ exists and it is equal to L.
- At least on between $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ does not exist. We can conclude that the limit $\lim_{x \to x_0} f(x)$ does not exist.
- $\lim_{x \to x_0^+} f(x) = L_1$ and $\lim_{x \to x_0^-} f(x) = L_2$, with $L_1 \neq L_2$. We can conclude that the limit $\lim_{x \to x_0} f(x)$ does not exist.

Example

The $\operatorname{sign}(x)$ function is defined on $(-\infty, 0) \cup (0, +\infty)$ as

$$\operatorname{sign}(x) = \frac{|x|}{x} = \frac{x}{|x|} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

If $x > 0$, by definition, $\operatorname{sign}(x) = +1$ so that

$$\lim_{x \to 0^+} \operatorname{sign}(x) = +1.$$

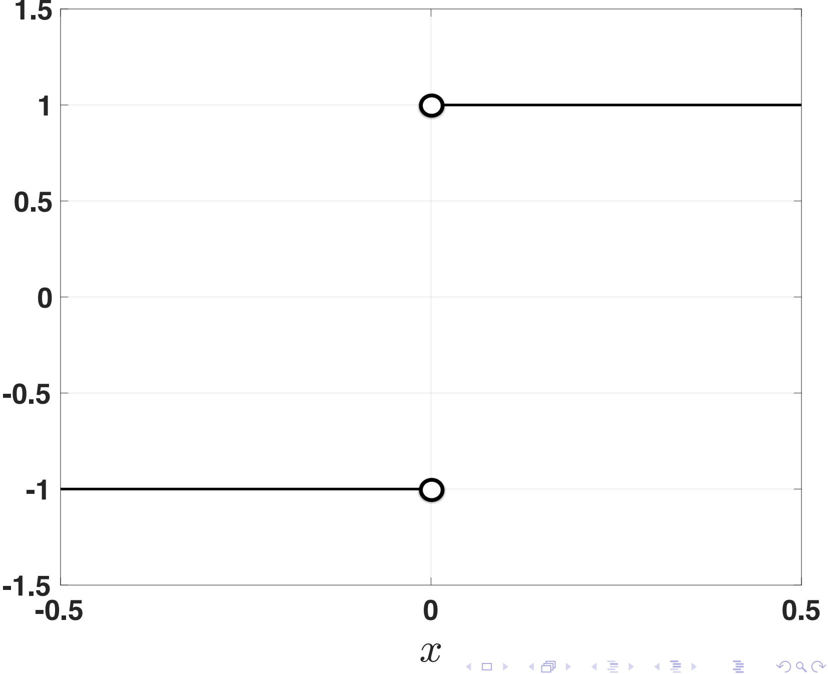
If $x < 0$, by definition, $\operatorname{sign}(x) = -1$ so that

$$\lim_{x \to 0^-} \operatorname{sign}(x) = -1.$$

Whence

$$\nexists \lim_{x \to 0} \operatorname{sign}(x).$$

$f(x) = \text{sign}\,(x)$



Left and right limits

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0^+} f(x) = +\infty$$

if and only if

$$\forall K > 0 \;\; \exists \delta > 0 : \forall x \in D : 0 < x - x_0 < \delta \quad \Rightarrow \quad f(x) > K$$

similarly, we say that

$$\lim_{x \to x_0^-} f(x) = +\infty$$

if and only if

$$\forall K > 0 \;\; \exists \delta > 0 : \forall x \in D : 0 < x_0 - x < \delta \quad \Rightarrow \quad f(x) > K$$

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Let x_0 be a limit point of D. We say that

$$\lim_{x \to x_0^+} f(x) = -\infty$$

if and only if

$$\forall K > 0 \ \ \exists \delta > 0 : \forall x \in D : 0 < x - x_0 < \delta \quad \Rightarrow \quad f(x) < -K$$

similarly, we say that

$$\lim_{x \to x_0^-} f(x) = -\infty$$

if and only if

$$\forall K > 0 \ \ \exists \delta > 0 : \forall x \in D : 0 < x_0 - x < \delta \quad \Rightarrow \quad f(x) < -K$$

Example

Consider the function $f : (-\infty, 0) \cup (0, \infty) \to (-\infty, 0) \cup (0, \infty)$ defined as

$$f(x) = \frac{1}{x}.$$

Take any $K > 0$ then consider an x^* such that $0 < x^* < \frac{1}{K}$. We have

$$f(x^*) = \frac{1}{x^*} > K,$$

hence, by putting $\delta = \frac{1}{K}$ in the definition, we have proved that

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

Using an identical procedure you can try to prove that

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Left and right limits

Definition

Other (intuitive) definitions (in what follows we implicitly assume $x \in D$)

$$\lim_{x \to x_0^+} f(x) = L^+ \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : 0 < x - x_0 < \delta \Rightarrow 0 < f(x) - L < \varepsilon$$

$$\lim_{x \to x_0^+} f(x) = L^- \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : 0 < x - x_0 < \delta \Rightarrow 0 < L - f(x) < \varepsilon$$

$$\lim_{x \to x_0^-} f(x) = L^+ \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : 0 < x_0 - x < \delta \Rightarrow 0 < f(x) - L < \varepsilon$$

$$\lim_{x \to x_0^-} f(x) = L^- \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 : 0 < x_0 - x < \delta \Rightarrow 0 < L - f(x) < \varepsilon$$

$$\lim_{x \to +\infty} f(x) = L^+ \Leftrightarrow \forall \varepsilon > 0, \exists M > 0 : x > M \Rightarrow 0 < f(x) - L < \varepsilon$$

$$\lim_{x \to +\infty} f(x) = L^- \Leftrightarrow \forall \varepsilon > 0, \exists M > 0 : x > M \Rightarrow 0 < L - f(x) < \varepsilon$$

$$\lim_{x \to -\infty} f(x) = L^+ \Leftrightarrow \forall \varepsilon > 0, \exists M > 0 : x < -M \Rightarrow 0 < f(x) - L < \varepsilon$$

$$\lim_{x \to -\infty} f(x) = L^- \Leftrightarrow \forall \varepsilon > 0, \exists M > 0 : x < -M \Rightarrow 0 < L - f(x) < \varepsilon.$$

Example

Using the definitions of limits it is straightforward to prove that

$$\lim_{x\to 0^-} \frac{x}{1+x^2} = 0^-, \quad \lim_{x\to 0^+} \frac{x}{1+x^2} = 0^+$$

$$\lim_{x\to +\infty} \frac{1}{x} = 0^+, \quad \lim_{x\to -\infty} \frac{1}{x} = 0^-$$

$$\lim_{x\to +\infty} \frac{1}{x^2} = 0^+, \quad \lim_{x\to -\infty} \frac{1}{x^2} = 0^+$$

$$\lim_{x\to 1^+} \frac{x-1}{x^2} = 0^+, \quad \lim_{x\to 1^-} \frac{x-1}{x^2} = 0^-$$

Left and right limits

Theorem

Let $f(x)$ and $g(x)$ be two functions defined on a set $D \subseteq \mathbb{R}$. Let x_0 be a limit point of D.

- *If when $x \to x_0$ we have $g(x) \to 0^+$ and $f(x) \to L > 0$ then $f(x)/g(x) \to +\infty$.*
- *If when $x \to x_0$ we have $g(x) \to 0^-$ and $f(x) \to L > 0$ then $f(x)/g(x) \to -\infty$.*
- *If when $x \to x_0$ we have $g(x) \to 0^+$ and $f(x) \to L < 0$ then $f(x)/g(x) \to -\infty$.*
- *If when $x \to x_0$ we have $g(x) \to 0^-$ and $f(x) \to L < 0$ then $f(x)/g(x) \to +\infty$.*

Identical results hold for $x \to x_0^-$ and $x \to x_0^+$.

Theorem

Let $f(x)$ and $g(x)$ be two functions defined on a set $D \subseteq \mathbb{R}$. Let x_0 be a limit point of D.

- *If when $x \to x_0$ we have $g(x) \to L > 0$ and $f(x) \to 0^+$ then $f(x)/g(x) \to 0^+$.*
- *If when $x \to x_0$ we have $g(x) \to L > 0$ and $f(x) \to 0^-$ then $f(x)/g(x) \to 0^-$.*
- *If when $x \to x_0$ we have $g(x) \to L < 0$ and $f(x) \to 0^+$ then $f(x)/g(x) \to 0^-$.*
- *If when $x \to x_0$ we have $g(x) \to L < 0$ and $f(x) \to 0^-$ then $f(x)/g(x) \to 0^+$.*

Identical results hold for $x \to x_0^-$ and $x \to x_0^+$.

Assume $a > 1$.

$$\lim_{x \to 0^+} \log_a(x) = ??$$

For all $K > 0$ consider $\delta = a^{-K}$. So if

$$0 < x < \delta = a^{-K}$$

we have (for $a > 1$ the $\log_a$ is increasing)

$$\log_a(x) < \log_a(\delta) = -K.$$

whence

$$\lim_{x \to 0^+} \log_a(x) = -\infty.$$

Assume $a > 1$.

$$\lim_{x \to +\infty} \log_a(x) = ??$$

For all $K > 0$ consider $M = a^K$. So if

$$x > M = a^K$$

we have (for $a > 1$ the $\log_a$ is increasing)

$$\log_a(x) > \log_a(M) = K.$$

whence

$$\lim_{x \to +\infty} \log_a(x) = +\infty.$$

Similar computations show that, for $0 < a < 1$, we have

$$\lim_{x \to 0^+} \log_a(x) = +\infty, \quad \lim_{x \to +\infty} \log_a(x) = -\infty.$$

Assume $a > 1$.

$$\lim_{x \to -\infty} a^x = ??$$

For all $\varepsilon > 0$ consider $M = \log_a(1/\varepsilon)$. So if

$$x < -M$$

we have (if $a > 1$ then a^x is increasing)

$$0 < a^x < a^{-M} = a^{-\log_a(1/\varepsilon)} = \varepsilon.$$

whence

$$\lim_{x \to -\infty} a^x = 0.$$

Assume $a > 1$.

$$\lim_{x\to+\infty} a^x = ??$$

For all $K > 0$ consider $M = \log_a(K)$. So if

$$x > M = \log_a(K)$$

we have (if $a > 1$ then a^x is increasing)

$$a^x > a^M = a^{\log_a(K)} = K.$$

whence

$$\lim_{x\to+\infty} a^x = +\infty.$$

Similar computations show that, for $0 < a < 1$, we have

$$\lim_{x\to-\infty} a^x = +\infty, \quad \lim_{x\to+\infty} a^x = 0.$$

Example

$$\lim_{x\to 0^+} \log_2(x) = -\infty \qquad \lim_{x\to +\infty} \log_2(x) = +\infty,$$

$$\lim_{x\to 0^+} \log_{1/2}(x) = +\infty \qquad \lim_{x\to +\infty} \log_{1/2}(x) = -\infty,$$

$$\lim_{x\to -\infty} e^x = 0 \qquad \lim_{x\to +\infty} e^x = +\infty,$$

$$\lim_{x\to -\infty} \left(\frac{1}{e}\right)^x = +\infty \qquad \lim_{x\to +\infty} \left(\frac{1}{e}\right)^x = 0.$$

The red indicates that the sign must be established. Let $a > 0$ be a strictly positive number.

$$\frac{0}{\pm\infty} = 0, \quad \frac{\pm\infty}{0} = \pm\infty, \quad \frac{a}{0} = \pm\infty, \quad \frac{a}{\pm\infty} = 0, \quad \frac{\pm\infty}{a} = \pm\infty.$$

$$0^{-\infty} = +\infty, \quad 0^{+\infty} = 0, \quad (-\infty)^{2\,n} = +\infty, \quad (-\infty)^{2\,n+1} = -\infty$$

$$(+\infty)^{+\infty} = +\infty, \quad (+\infty)^{-\infty} = 0, \quad +\infty^{-a} = 0, \quad +\infty^{a} = +\infty$$

$$+\infty + \infty = +\infty, \quad -\infty - \infty = -\infty, \quad (+\infty) \cdot (\pm a) = \pm\infty$$

Indeterminate form: 0/0

$$\lim_{x\to\infty} \frac{1/x}{1/x^2} = \frac{0}{0} = \lim_{x\to\infty} \frac{x^2}{x} = +\infty,$$

$$\lim_{x\to\infty} \frac{1/x^2}{1/x} = \frac{0}{0} = \lim_{x\to\infty} \frac{x}{x^2} = 0.$$

Indeterminate form: ∞/∞

$$\lim_{x\to\infty} \frac{x}{x^2} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1}{x} = 0,$$

$$\lim_{x\to\infty} \frac{x^2}{x} = \frac{\infty}{\infty} = \lim_{x\to\infty} x = \infty$$

Indeterminate form: $\infty - \infty$

$$\lim_{x\to\infty} x^2 - x = \infty - \infty = \lim_{x\to\infty} x^2 \left(1 - \frac{1}{x}\right) = (+\infty)\cdot(+1) = +\infty,$$

$$\lim_{x\to\infty} \sqrt{x^2+x} - x = \infty - \infty = \frac{1}{2}.$$

$$\lim_{x\to 0^+} x^x = 0^0 = 1,$$

$$\lim_{x\to 0^+} \left(e^{-1/x^2}\right)^x = 0^0 = \lim_{x\to 0^+} e^{-1/x} = 0.$$

Indeterminate form: 1^{∞}

$$\lim_{x \to +\infty} \left(1 + \frac{\alpha}{x}\right)^{x} = 1^{\infty} = e^{\alpha},$$

For any $a > 1$

$$\lim_{x \to +\infty} x^{1/\log_a x} = (+\infty)^0 = \lim_{x \to +\infty} a^{\log_a\left(x^{1/\log_a x}\right)} = \lim_{x \to +\infty} a^{\frac{\log_a(x)}{\log_a(x)}} = a.$$

Theorem

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ *and* $g : D \subseteq \mathbb{R} \to \mathbb{R}$ *be two functions,* x_0 *a limit point of* D*. If*

$$\lim_{x \to x_0} f(x) = L_1 \text{ and } \lim_{x \to x_0} g(x) = L_2$$

then

$$\lim_{x \to x_0} (f(x) + g(x)) = L_1 + L_2.$$

The results still holds if $x_0 = +\infty$ *or* $x_0 = -\infty$.

Warning!

The converse is not true! Consider $f(x) = x$ and $g(x) = -x$. Then

$$\lim_{x \to +\infty} (f(x) + g(x)) = \lim_{x \to +\infty} (x - x) = 0$$

Nevertheless

$$\lim_{x \to +\infty} f(x) = +\infty \text{ and } \lim_{x \to +\infty} g(x) = -\infty.$$

Theorem

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and $g : D \subseteq \mathbb{R} \to \mathbb{R}$ be two functions, x_0 a limit point of D. If

$$\lim_{x \to x_0} f(x) = L_1 \text{ and } \lim_{x \to x_0} g(x) = L_2$$

then

$$\lim_{x \to x_0} f(x) \cdot g(x) = L_1 \cdot L_2.$$

The results still holds if $x_0 = +\infty$ or $x_0 = -\infty$.

Warning!

The converse is not true! Consider $f(x) = x$ and $g(x) = \frac{1}{x}$. Then

$$\lim_{x \to +\infty} f(x) \cdot g(x) = \lim_{x \to +\infty} x \cdot \frac{1}{x} = 1$$

Nevertheless

$$\lim_{x \to +\infty} f(x) = +\infty \text{ and } \lim_{x \to +\infty} g(x) = 0.$$

Rules for limits

Theorem

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and $g : D \subseteq \mathbb{R} \to \mathbb{R}$ be two functions, x_0 a limit point of D. If

$$\lim_{x \to x_0} f(x) = L_1 \text{ and } \lim_{x \to x_0} g(x) = L_2 \neq 0$$

then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}.$$

The results still holds if $x_0 = +\infty$ or $x_0 = -\infty$.

Warning!

The converse is not true! Consider $f(x) = x$ and $g(x) = x$. Then

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{x}{x} = 1$$

Nevertheless

$$\lim_{x \to +\infty} f(x) = +\infty \text{ and } \lim_{x \to +\infty} g(x) = +\infty.$$

Heuristically

A function is said to be continuous in a point x_0 of its domain if arbitrarily small variations around x_0 generate small variations around $f(x_0)$.

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $x_0 \in D$. We say that f is continuous in x_0 if and only if

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 : \forall x \in D : \; |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Theorem

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $x_0 \in D$. The function f is continuous in x_0 if and only if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Checklist for continuity

Suppose I give you a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and a point $x_0 \in \mathbb{R}$.

Question: Is f continuous in x_0 ?

1. Verify that x_0 is a point of the domain!! That is $x_0 \in D$.
2. Compute
$$\lim_{x \to x_0} f(x).$$
3. If the limit in point 2 does not exists or is infinite $\Rightarrow$ the function cannot be continuous. If the limit in point 2 exists and it is finite, call it L.
4. Verify if
$$L = f(x_0).$$

Exercize

Is the function

$$\ln(x)$$

continuous in $x_0 = 0$ *?*

Solution. Trivially no. The domain of $\ln(x)$ is

$$D = (0, \infty)$$

and, hence, does not include $x_0 = 0$.

Exercize

Is the piecewise-defined function

$$f(x) = \begin{cases} \ln(x) & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

continuous in $x_0 = 0$ *?*

Solution. The domain of $f(x)$ is the entire real line, so $x_0 \in D$.

Nevertheless

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \ln(x) = -\infty,$$

while

$$\lim_{x\to 0^-} f(x) = 0,$$

whence $\nexists \lim_{x\to 0} f(x) \Rightarrow$ The function is not continuous in $x_0 = 0$.

Theorem

The function

$$f(x) = \sqrt{x}$$

is continuous in all its domain $D = [0, +\infty)$.

Proof. Consider a $x_0 > 0$. Let $x \in D$, then

$$0 \leq |f(x) - f(x_0)| = \left|\sqrt{x} - \sqrt{x_0}\right| = \left|\frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}\right| < \left|\frac{x - x_0}{\sqrt{x_0}}\right|$$

so if $x \to x_0$, by the comparison theorem, $f(x) \to f(x_0)$.

Consider now $x_0 = 0$. For all $\varepsilon > 0$ take any $\delta < \varepsilon^2$ so that

$$0 \leq x < \delta \Rightarrow 0 \leq \sqrt{x} < \sqrt{\delta} < \varepsilon.$$

Remark

It is possible to prove that the following functions are continuous in all the points of the corresponding domain:

$$\begin{array}{rclcrcl} f(x) & = & \sin(x) & & D & = & \mathbb{R} \\ f(x) & = & \cos(x) & & D & = & \mathbb{R} \\ f(x) & = & \log_a(x) & & D & = & (0, +\infty) \\ f(x) & = & a^x & & D & = & \mathbb{R} \\ f(x) & = & x^n & & D & = & \mathbb{R} \\ f(x) & = & x^\alpha & & D & = & [0, +\infty) \end{array}$$

More generally, all the trigonometric functions are continuous in their corresponding domains.

Theorem

Suppose that f and g are continuous.

- *Then $f + g$, $f \cdot g$ and f/g are continuous in their corresponding domains.*
- *If f (or g) is invertible, then $f^{(-1)}$ (or $g^{(-1)}$) is continuous in its domain.*
- *If it is possible to define $(f \circ g)$ then $(f \circ g)$ is continuous in its domain.*

Exercize

Find the domain of the function

$$f(x) = x^x$$

and establish where the function is continuous.

Solution. The function can be written as

$$x^x = e^{\ln(x^x)} = e^{x \ln(x)}.$$

Whence it is defined for $x \in (0, +\infty)$. Since $\ln(x)$ is continuous for all $x \in (0, +\infty)$ and since the exponential function is continuous everywhere then $f(x)$ is continuous for all $x \in (0, +\infty)$.

Exercize

Find the domain of the function

$$f(x) = \sin\left(\ln\left(1 + x^2\right)\right)$$

and establish where the function is continuous.

Solution. Since for all x it holds that

$$1 + x^2 > 0$$

then the function $f(x)$ is defined for all $x \in \mathbb{R}$. The function $f(x)$ is a composition of the functions

$$x \to x^2 \to 1 + x^2 \to \ln\left(1 + x^2\right) \to \sin\left(\ln\left(1 + x^2\right)\right),$$

whence it is continuous for all $x \in \mathbb{R}$.

Theorem

It holds that

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Proof. Remember that

$$a_n = \left(1+\frac{1}{n}\right)^n \to e.$$

Consider, for all $x \geq 0$, the integer part of x, defined as

$$\lfloor x \rfloor = \max\left\{m \in \mathbb{N} \mid m \leq x\right\}.$$

For example

$$\left\lfloor \frac{7}{2} \right\rfloor = \lfloor 3.5 \rfloor = 3.$$

Call, for simplicity $n_x = \lfloor x \rfloor \in \mathbb{N}$ and note that

$$n_x \leq x < n_x + 1,$$

From

$$n_x \leq x < n_x + 1,$$

we derive

$$\frac{1}{n_x+1} < \frac{1}{x} \leq \frac{1}{n_x} \Rightarrow 1 + \frac{1}{n_x+1} < 1 + \frac{1}{x} \leq 1 + \frac{1}{n_x}.$$

Since $n_x \leq x$ we have

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} \leq \left(1 + \frac{1}{n_x+1}\right)^{x}$$

using $1 + \frac{1}{n_x+1} < 1 + \frac{1}{x}$ we can continue ...

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} \leq \left(1 + \frac{1}{n_x+1}\right)^{x} < \left(1 + \frac{1}{x}\right)^{x}$$

using $1 + \frac{1}{x} \leq 1 + \frac{1}{n_x}$ we can continue ...

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} \leq \left(1 + \frac{1}{n_x+1}\right)^{x} < \left(1 + \frac{1}{x}\right)^{x} \leq \left(1 + \frac{1}{n_x}\right)^{x}$$

finally, using $x < n_x + 1$ we get

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} < \left(1 + \frac{1}{x}\right)^{x} < \left(1 + \frac{1}{n_x}\right)^{n_x+1}$$

From

$$n_x \le x < n_x + 1,$$

we derive

$$\frac{1}{n_x+1} < \frac{1}{x} \le \frac{1}{n_x} \quad \Rightarrow \quad 1 + \frac{1}{n_x+1} < 1 + \frac{1}{x} \le 1 + \frac{1}{n_x}.$$

Since $n_x \le x$ we have

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} \le \left(1 + \frac{1}{n_x+1}\right)^{x}$$

using $1 + \frac{1}{n_x+1} < 1 + \frac{1}{x}$ we can continue ...

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} \le \left(1 + \frac{1}{n_x+1}\right)^{x} < \left(1 + \frac{1}{x}\right)^{x}$$

using $1 + \frac{1}{x} \le 1 + \frac{1}{n_x}$ we can continue ...

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} \le \left(1 + \frac{1}{n_x+1}\right)^{x} < \left(1 + \frac{1}{x}\right)^{x} \le \left(1 + \frac{1}{n_x}\right)^{x}$$

finally, using $x < n_x + 1$ we get

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} < \left(1 + \frac{1}{x}\right)^{x} < \left(1 + \frac{1}{n_x}\right)^{n_x+1}$$

Notable limits

From

$$n_x \leq x < n_x + 1,$$

we derive

$$\frac{1}{n_x+1} < \frac{1}{x} \leq \frac{1}{n_x} \Rightarrow 1 + \frac{1}{n_x+1} < 1 + \frac{1}{x} \leq 1 + \frac{1}{n_x}.$$

Since $n_x \leq x$ we have

$$\left(1+\frac{1}{n_x+1}\right)^{n_x} \leq \left(1+\frac{1}{n_x+1}\right)^{x}$$

using $1+\frac{1}{n_x+1} < 1+\frac{1}{x}$ we can continue ...

$$\left(1+\frac{1}{n_x+1}\right)^{n_x} \leq \left(1+\frac{1}{n_x+1}\right)^{x} < \left(1+\frac{1}{x}\right)^{x}$$

using $1+\frac{1}{x} \leq 1+\frac{1}{n_x}$ we can continue ...

$$\left(1+\frac{1}{n_x+1}\right)^{n_x} \leq \left(1+\frac{1}{n_x+1}\right)^{x} < \left(1+\frac{1}{x}\right)^{x} \leq \left(1+\frac{1}{n_x}\right)^{x}$$

finally, using $x < n_x + 1$ we get

$$\left(1+\frac{1}{n_x+1}\right)^{n_x} < \left(1+\frac{1}{x}\right)^{x} < \left(1+\frac{1}{n_x}\right)^{n_x+1}$$

Notable limits

From

$$n_x \le x < n_x + 1,$$

we derive

$$\frac{1}{n_x+1} < \frac{1}{x} \le \frac{1}{n_x} \quad \Rightarrow \quad 1 + \frac{1}{n_x+1} < 1 + \frac{1}{x} \le 1 + \frac{1}{n_x}.$$

Since $n_x \le x$ we have

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} \le \left(1 + \frac{1}{n_x+1}\right)^{x}$$

using $1 + \frac{1}{n_x+1} < 1 + \frac{1}{x}$ we can continue ...

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} \le \left(1 + \frac{1}{n_x+1}\right)^{x} < \left(1 + \frac{1}{x}\right)^{x}$$

using $1 + \frac{1}{x} \le 1 + \frac{1}{n_x}$ we can continue ...

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} \le \left(1 + \frac{1}{n_x+1}\right)^{x} < \left(1 + \frac{1}{x}\right)^{x} \le \left(1 + \frac{1}{n_x}\right)^{x}$$

finally, using $x < n_x + 1$ we get

$$\left(1 + \frac{1}{n_x+1}\right)^{n_x} < \left(1 + \frac{1}{x}\right)^{x} < \left(1 + \frac{1}{n_x}\right)^{n_x+1}$$

From
$$n_x \le x < n_x + 1,$$
we derive
$$\frac{1}{n_x+1} < \frac{1}{x} \le \frac{1}{n_x} \;\Rightarrow\; 1+\frac{1}{n_x+1} < 1+\frac{1}{x} \le 1+\frac{1}{n_x}.$$
Since $n_x \le x$ we have
$$\left(1+\frac{1}{n_x+1}\right)^{n_x} \le \left(1+\frac{1}{n_x+1}\right)^{x}$$
using $1+\frac{1}{n_x+1} < 1+\frac{1}{x}$ we can continue ...
$$\left(1+\frac{1}{n_x+1}\right)^{n_x} \le \left(1+\frac{1}{n_x+1}\right)^{x} < \left(1+\frac{1}{x}\right)^{x}$$
using $1+\frac{1}{x} \le 1+\frac{1}{n_x}$ we can continue ...
$$\left(1+\frac{1}{n_x+1}\right)^{n_x} \le \left(1+\frac{1}{n_x+1}\right)^{x} < \left(1+\frac{1}{x}\right)^{x} \le \left(1+\frac{1}{n_x}\right)^{x}$$
finally, using $x < n_x + 1$ we get
$$\left(1+\frac{1}{n_x+1}\right)^{n_x} \le \left(1+\frac{1}{n_x+1}\right)^{x} < \left(1+\frac{1}{x}\right)^{x} \le \left(1+\frac{1}{n_x}\right)^{x} < \left(1+\frac{1}{n_x}\right)^{n_x+1}$$

From

$$n_x \leq x < n_x + 1,$$

we derive

$$\frac{1}{n_x + 1} < \frac{1}{x} \leq \frac{1}{n_x} \Rightarrow 1 + \frac{1}{n_x + 1} < 1 + \frac{1}{x} \leq 1 + \frac{1}{n_x}.$$

Since $n_x \leq x$ we have

$$\left(1 + \frac{1}{n_x + 1}\right)^{n_x} \leq \left(1 + \frac{1}{n_x + 1}\right)^{x}$$

using $1 + \frac{1}{n_x+1} < 1 + \frac{1}{x}$ we can continue ...

$$\left(1 + \frac{1}{n_x + 1}\right)^{n_x} \leq \left(1 + \frac{1}{n_x + 1}\right)^{x} < \left(1 + \frac{1}{x}\right)^{x}$$

using $1 + \frac{1}{x} \leq 1 + \frac{1}{n_x}$ we can continue ...

$$\left(1 + \frac{1}{n_x + 1}\right)^{n_x} \leq \left(1 + \frac{1}{n_x + 1}\right)^{x} < \left(1 + \frac{1}{x}\right)^{x} \leq \left(1 + \frac{1}{n_x}\right)^{x}$$

finally, using $x < n_x + 1$ we get

$$\left(1 + \frac{1}{n_x + 1}\right)^{n_x} \leq \left(1 + \frac{1}{n_x + 1}\right)^{x} < \left(1 + \frac{1}{x}\right)^{x} \leq \left(1 + \frac{1}{n_x}\right)^{x} < \left(1 + \frac{1}{n_x}\right)^{n_x+1}$$

From

$$n_x \leq x < n_x + 1,$$

we derive

$$\frac{1}{n_x+1} < \frac{1}{x} \leq \frac{1}{n_x} \;\Rightarrow\; 1+\frac{1}{n_x+1} < 1+\frac{1}{x} \leq 1+\frac{1}{n_x}.$$

Since $n_x \leq x$ we have

$$\left(1+\frac{1}{n_x+1}\right)^{n_x} \leq \left(1+\frac{1}{n_x+1}\right)^{x}$$

using $1+\frac{1}{n_x+1} < 1+\frac{1}{x}$ we can continue ...

$$\left(1+\frac{1}{n_x+1}\right)^{n_x} \leq \left(1+\frac{1}{n_x+1}\right)^{x} < \left(1+\frac{1}{x}\right)^{x}$$

using $1+\frac{1}{x} \leq 1+\frac{1}{n_x}$ we can continue ...

$$\left(1+\frac{1}{n_x+1}\right)^{n_x} \leq \left(1+\frac{1}{n_x+1}\right)^{x} < \left(1+\frac{1}{x}\right)^{x} \leq \left(1+\frac{1}{n_x}\right)^{x}$$

finally, using $x < n_x + 1$ we get

$$\left(1+\frac{1}{n_x+1}\right)^{n_x} < \left(1+\frac{1}{x}\right)^{x} < \left(1+\frac{1}{n_x}\right)^{n_x+1}$$

In summary, $n_x = \lfloor x \rfloor = \max\{m \in \mathbb{N} \mid m \leq x\}$ and

$$\left(1+\frac{1}{n_x+1}\right)^{n_x} < \left(1+\frac{1}{x}\right)^{x} < \left(1+\frac{1}{n_x}\right)^{n_x+1} = \left(1+\frac{1}{n_x}\right)^{n_x}\left(1+\frac{1}{n_x}\right).$$

Nevertheless, if $x \to +\infty$ we have $n_x \to \infty$

$$\lim_{n_x\to\infty} \underbrace{\left(1+\frac{1}{n_x}\right)^{n_x}}_{\to e} \underbrace{\left(1+\frac{1}{n_x}\right)}_{\to 1} = e$$

$$\lim_{n_x\to\infty} \left(1+\frac{1}{n_x+1}\right)^{n_x} = \lim_{n_x\to\infty} \underbrace{\left(1+\frac{1}{n_x+1}\right)^{n_x+1}}_{\to e} \underbrace{\left(1+\frac{1}{n_x+1}\right)^{-1}}_{\to 1} = e$$

whence, by the comparison theorem,

$$\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^{x} = e.$$

What about

$$\lim_{x\to -\infty}\left(1+\frac{1}{x}\right)^x = \quad ?$$

If $x<0$ then $x=-|x|$ and so

$$\begin{aligned}\left(1+\frac{1}{x}\right)^x &= \left(1-\frac{1}{|x|}\right)^{-|x|} = \left(\frac{|x|-1}{|x|}\right)^{-|x|} = \left(\frac{|x|}{|x|-1}\right)^{|x|} \\ &= \left(1+\frac{1}{|x|-1}\right)^{|x|} = \left(1+\frac{1}{|x|-1}\right)^{|x|-1}\left(1+\frac{1}{|x|-1}\right).\end{aligned}$$

When $x\to-\infty$ we have $|x|\to+\infty$ so that $\left(1+\frac{1}{|x|-1}\right)^{|x|-1}\to e$ and $\left(1+\frac{1}{|x|-1}\right)\to 1$.

Whence

$$\lim_{x\to -\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Definition

Given $n+1$ real numbers $a_0, a_1, ..., a_n$, the function

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is called a polynomial of degree n. The numbers $a_0, a_1, ..., a_n$ are called the coefficient of the polynomial.

Any function of the form

$$f(x) = \frac{Q_n(x)}{P_m(x)}, \quad (\triangle)$$

where $Q_n(x)$ is a polynomial of degree n and $P_m(x)$ is a polynomial of degree m, is called a rational function.

Remark

Any rational function such as $(\triangle)$ has the domain

$$D = \{x \in \mathbb{R} \mid P_m(x) \neq 0\}.$$

For all $x \in D$ the function $(\triangle)$ is continuous.

Limits of fractions of polynomials

Put in evidence the highest power:

$$\lim_{x\to+\infty}\frac{x^7-3\,x^2+x}{9-12\,x^7-x^4}=\lim_{x\to+\infty}\frac{1-\overbrace{3\,x^{-5}}^{\to 0}+\overbrace{x^{-6}}^{\to 0}}{\underbrace{9\,x^{-7}}_{\to 0}-12-\underbrace{x^{-3}}_{\to 0}}=-\frac{1}{12}.$$

$$\lim_{x\to+\infty}\frac{x^8-3\,x^2+x}{9-12\,x^7-x^4}=\lim_{x\to+\infty}x\,\frac{1-\overbrace{3\,x^{-6}}^{\to 0}+\overbrace{x^{-7}}^{\to 0}}{\underbrace{9\,x^{-7}}_{\to 0}-12-\underbrace{x^{-3}}_{\to 0}}=-\infty.$$

$$\lim_{x\to+\infty}\frac{x^7-3\,x^2+x}{9-12\,x^8-x^4}=\lim_{x\to+\infty}\frac{1}{x}\,\frac{1-\overbrace{3\,x^{-5}}^{\to 0}+\overbrace{x^{-6}}^{\to 0}}{\underbrace{9\,x^{-8}}_{\to 0}-12-\underbrace{x^{-4}}_{\to 0}}=0.$$

- Applying the same geometrical argument used for $n \sin(1/n) \to 1$ we can prove that
$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$
- Consider
$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \frac{0}{0}.$$
Using the previous notable limit ...

$$\begin{aligned} \frac{1-\cos x}{x} &= \frac{1-\cos x}{x}\,\frac{1+\cos x}{x}\,\frac{x}{1+\cos x} = \frac{1-\cos^2 x}{x^2}\,\frac{x}{1+\cos x} \\ &= \left(\frac{\sin x}{x}\right)^2 \frac{x}{1+\cos x} \to 1 \cdot 0 = 0 \qquad (0.1) \end{aligned}$$

- To compute
$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{0}{0},$$
we use the notable limits that we already know.

$$\begin{aligned}\frac{1-\cos x}{x^2} &= \frac{1-\cos x}{x^2}\frac{1+\cos x}{1+\cos x}=\frac{1-\cos^2 x}{x^2}\frac{1}{1+\cos x}\\ &= \frac{\sin^2 x}{x^2}\frac{1}{1+\cos x}\to 1\cdot\frac{1}{2}=\frac{1}{2}.\end{aligned}$$

- Same procedure for
$$\lim_{x\to 0}\frac{\tan x}{x}=\frac{0}{0}.$$
$$\frac{\tan x}{x}=\frac{\sin x}{x}\frac{1}{\cos x}\to 1.$$

- In the case of

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \frac{0}{0}.$$

we use the properties of the logarithm

$$\frac{\ln(1+x)}{x} = \ln(1+x)^{\frac{1}{x}}$$

Call $t = \frac{1}{x} \to \infty$

$$\begin{aligned} \lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} &= \lim_{t\to\infty} \ln\left(1+\frac{1}{t}\right)^t \\ &= \ln\left(\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t\right) = \ln e = 1. \end{aligned}$$

Note that we obtain the same result both if $x \to 0^+$ and if $x \to 0^-$.

- The following limit
$$\lim_{x\to 0}\frac{e^x-1}{x}=\frac{0}{0}.$$
can be computed changing variable $y=e^x-1\Rightarrow x=\ln(1+y)$
$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{y\to 0}\frac{y}{\ln(1+y)}=\lim_{y\to 0}\frac{1}{\frac{\ln(1+y)}{y}}=1.$$

- To compute
$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x.$$
again we change variable $y=\frac{x}{a}$
$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{a\,y}=\lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^a=e^a.$$

- To compute
$$\lim_{x\to 0} \frac{\log_b (1+x)}{x} = \frac{0}{0}.$$
it is necessary to use the rule
$$\log_b (x) = \frac{\log_a (x)}{\log_a (b)},$$
Obtaining
$$\lim_{x\to 0} \frac{\log_b (1+x)}{x} = \lim_{x\to 0} \frac{\ln (1+x)}{x \ln b} = \frac{1}{\ln b}.$$
- Similarly for
$$\lim_{x\to 0} \frac{a^x - 1}{x} = \frac{0}{0}.$$
define $y = a^x - 1 \Rightarrow x = \log_a (y+1)$
$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{y\to 0} \frac{y}{\log_a (y+1)} = \ln a \qquad (0.2)$$

Recall what we have proved for sequences...

Consider any $a > 1$ and $b > 0$ then

Summary

$$\lim_{n\to\infty} \frac{\log_a(n)}{n^b} = 0 \Rightarrow \text{The power diverges faster than the logarithm}$$

$$\lim_{n\to\infty} \frac{n^b}{a^n} = 0 \Rightarrow \text{The exponential diverges faster than the power}$$

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0 \Rightarrow \text{The factorial diverges faster than the exponential}$$

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0 \Rightarrow n^n \text{ diverges faster than the factorial}$$

The same results can be straightforwardly proved for limits on the real line.

Consider any $a > 1$ and $b > 0$ then

$$\begin{aligned}
\lim_{x\to\infty} \frac{\log_a(x)}{x^b} &= 0 \Rightarrow && \text{The power diverges faster than the logarithm} \\
\lim_{x\to\infty} \frac{x^b}{a^x} &= 0 \Rightarrow && \text{The exponential diverges faster than the power} \\
\lim_{x\to\infty} \frac{x^n}{\lfloor x \rfloor !} &= 0 \Rightarrow && \text{The factorial diverges faster than the exponential} \\
\lim_{x\to\infty} \frac{\lfloor x \rfloor !}{x^x} &= 0 \Rightarrow && n^n \text{ diverges faster than the factorial}
\end{aligned}$$

Exercize

Compute the limit

$$\lim_{x\to 0} \frac{\sin(x)}{\ln(1+x)\,|x-7|} = \frac{0}{0}$$

Solution.

$$\lim_{x\to 0} \frac{\sin(x)}{\ln(1+x)\,|x-7|} = \lim_{x\to 0} \frac{\sin(x)}{x} \frac{x}{\ln(1+x)\,|x-7|}.$$

Using

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1, \quad \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1,$$

we get

$$\lim_{x\to 0} \frac{\sin(x)}{\ln(1+x)\,|x-7|} = \frac{1}{7}.$$

Exercize

Compute the limit

$$\lim_{x\to 0^+} x\,\ln(x) = 0 \times (-\infty)$$

Solution. By changing variable $x = e^{-t} \Rightarrow t = -\ln(x)$ we get

$$\lim_{x\to 0^+} x\ln(x) = \lim_{t\to+\infty} -t\,e^{-t} = -\lim_{t\to+\infty} \frac{t}{e^t}.$$

However e^t diverge faster than t, so

$$\lim_{x\to 0^+} x\ln(x) = 0.$$

Exercize

Compute the limit

$$\lim_{x\to 0^+} x^x = 0^0$$

Solution. Write

$$x^x = e^{\ln(x^x)} = e^{x\ln(x)}.$$

Since $x\ln(x) \to 0$ and since the exponential is a continuous function we have

$$\lim_{x\to 0^+} x^x = 1.$$

Exercize

Let $K > 0$ be a positive constant. Compute the limit

$$\lim_{x\to 0^+} x^{\frac{\ln(K)}{\ln(x)}} = 0^0$$

Solution. The limit is a little bit tricky. Note, in fact, that

$$x^{\frac{\ln(K)}{\ln(x)}} = e^{\ln\left(x^{\frac{\ln(K)}{\ln(x)}}\right)} = e^{\frac{\ln(K)}{\ln(x)}\ln(x)} = e^{\ln(K)} = K$$

so

$$\lim_{x\to 0^+} x^{\frac{\ln(K)}{\ln(x)}} = K.$$

This is why 0^0 is an indeterminate form ...

$$\lim_{x\to 0^+} x^{\frac{\ln(2)}{\ln(x)}} = 0^0 = 2, \quad \lim_{x\to 0^+} x^{\frac{\ln(100)}{\ln(x)}} = 0^0 = 100, \quad \text{etc...etc...}$$

Exercize

Let $K > 0$ be a positive constant. Compute the limit

$$\lim_{x\to 0}(1+x)^{\frac{\ln(K)}{x}} = 1^{\infty}$$

Solution. Again write

$$\begin{aligned}(1+x)^{\frac{\ln(K)}{x}} &= \exp\left(\ln\left((1+x)^{\frac{\ln(K)}{x}}\right)\right) = \exp\left(\frac{\ln(K)}{x}\ln(1+x)\right) \\ &= \exp\left(\ln(K)\,\frac{\ln(1+x)}{x}\right) \to \exp(\ln(K)) = K. \qquad (0.3)\end{aligned}$$

so

$$\lim_{x\to 0}(1+x)^{\frac{\ln(K)}{x}} = K$$

This is why 1^{∞} is an indeterminate form ...

$$\lim_{x\to 0}(1+x)^{\frac{\ln(2)}{x}} = 1^{\infty} = 2, \quad \lim_{x\to 0}(1+x)^{\frac{\ln(100)}{x}} = 1^{\infty} = 100 \quad \cdots$$

A summary of indeterminate forms

The seven indeterminate forms

1. $\frac{0}{0}$ is indeterminate because, for example

$$\lim_{x \to 0} \frac{K\,x}{x} = \frac{0}{0} = K, \text{ for any arbitrary } K.$$

2. $0 \times \infty$ is indeterminate because, for example

$$\lim_{x \to 0} (K\,x) \cdot \left(\frac{1}{x}\right) = 0 \times \infty = K, \text{ for any arbitrary } K.$$

3. $\frac{\infty}{\infty}$ is indeterminate because, for example

$$\lim_{x \to 0} \frac{\frac{K}{x}}{\frac{1}{x}} = \frac{\infty}{\infty} = K, \text{ for any arbitrary } K.$$

4. $\infty - \infty$ is indeterminate because, for example

$$\lim_{x \to 0} \left(K + \frac{1}{x}\right) - \frac{1}{x} = \infty - \infty = K, \text{ for any arbitrary } K.$$

A summary of indeterminate forms

The seven indeterminate forms

5. 1^{∞} is indeterminate because, for example

$$\lim_{x\to 0}(1+x)^{\frac{\ln(K)}{x}} = 1^{\infty} = K, \text{ for any arbitrary } K.$$

6. 0^0 is indeterminate because, for example

$$\lim_{x\to 0^+} x^{\frac{\ln(K)}{\ln(x)}} = 0^0 = K, \text{ for any arbitrary } K.$$

7. ∞^0 is indeterminate because, for example

$$\lim_{x\to 0}\left(\frac{1}{x}\right)^{\ln(K)\,\frac{x}{1-x}} = \infty^0 = K, \text{ for any arbitrary } K.$$

Theorem

Let $f(x) > 0$ and $g(x) > 0$ be two functions such that

$$\lim_{x \to x_0} f(x) = 0, \quad \lim_{x \to x_0} g(x) = +\infty.$$

Then

$$\lim_{x \to x_0} f(x)^{g(x)} = 0.$$

In other words, $0^{+\infty}$ is not an indeterminate form.

Proof. Take an $0 < \varepsilon < 1$, then there exists a δ such that

$$0 \leq f(x) < \varepsilon.$$

Hence

$$0 \leq (f(x))^{g(x)} < \varepsilon^{g(x)} \to 0$$

where the limit derives from the fact that $0 < \varepsilon < 1$ and $g(x) \to +\infty$.

Theorem

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and x_0 be a limit point of D. Assume that

$$\lim_{x \to x_0} f(x) = L$$

with $L > 0$. Therefore it exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \bigcap D \setminus \{x_0\}$.

Proof. In the definition of limit take $\varepsilon = L > 0$. So that it exits $\delta > 0$ such that for all $x \in D$ such that $0 < |x - x_0| < \delta$ we have

$$|f(x) - L| < L \Rightarrow -L < f(x) - L < L \Rightarrow 0 < f(x) < 2L.$$

Exercize

Is the function

$$f(x) = e^{-1/x^2}$$

continuous in $x_0 = 0$?

Solution. No because the function is not defined in 0.

Exercize

Is the piecewise-defined function

$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

continuous in $x_0 = 0$?

Solution. The function is defined in 0. So we compute the limit of the function in 0:

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} e^{-1/x^2} = \lim_{x \to 0} \frac{1}{e^{1/x^2}} = \frac{1}{e^{+\infty}} = \frac{1}{+\infty} = 0 = g(0),$$

hence the function is continuous.

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $x_0 \in \mathbb{R}$ be a limit point of D.

The point x_0 may or may not belong to D.

We say that f has a jump discontinuity in x_0 or, simply, that the function jumps in x_0 if

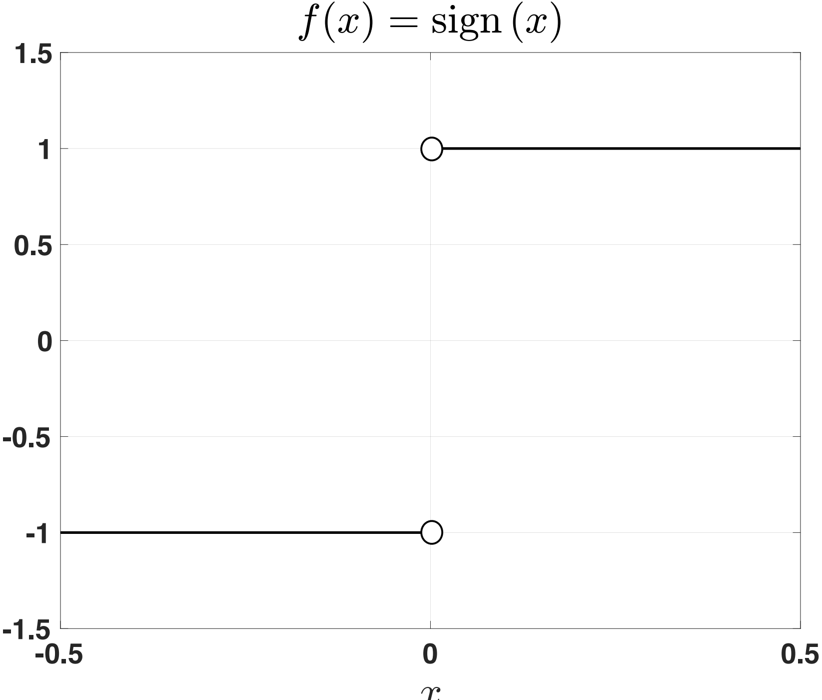
$$\exists L_1 = \lim_{x \to x_0^+} f(x) \quad \text{and} \quad \exists L_2 = \lim_{x \to x_0^-} f(x)$$

but $L_1 \neq L_2$.

Example

The $\operatorname{sign}(x) = |x|/x$ has a jump discontinuity in $x = 0$. In fact

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1, \quad \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$



$f(x) = \mathrm{sign}\,(x)$
1.5
1
0.5
0
-0.5
-1
-1.5
-0.5
0
0.5
x

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $x_0 \in \mathbb{R}$ be a limit point of D. The point x_0 may or may not belong to D.

We say that f has a removable discontinuity in x_0 if

$$\exists L = \lim_{x \to x_0} f(x) \quad \Leftrightarrow \left(\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L \right)$$

but ...

- either $L \neq f(x_0)$ (in case $x_0 \in D$)
- or f is not defined in x_0.

Remark. If f has a removable discontinuity in x_0, then the piecewise-defined function

$$g(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ L & \text{if } x = x_0 \end{cases}$$

is, by definition, continuous in x_0.

Example

Consider the function

$$f(x) = \frac{\sin(x)}{x} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$$

We know that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1,$$

nevertheless $0 \notin D$. This means that the piecewise-defined function

$$g(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is continuous in $x_0 = 0$.

Discontinuities

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $x_0 \in \mathbb{R}$ be a limit point of D. The point x_0 may or may not belong to D.

We say that f has an essential discontinuity in x_0 if either

$$\nexists \lim_{x \to x_0^-} f(x) \text{ or } \lim_{x \to x_0^-} f(x) = \pm\infty \quad (\triangle)$$

or

$$\nexists \lim_{x \to x_0^+} f(x) \text{ or } \lim_{x \to x_0^+} f(x) = \pm\infty \quad (\square)$$

or both condition $(\triangle)$ and $(\square)$ hold simultaneously.

Example

The function

$$f(x) = \sin\left(\frac{1}{x}\right)$$

has an essential discontinuity in $x_0 = 0$.

Exercize

Find the value of the parameter α such that the piecewise-defined function

$$f(x) = \begin{cases} \alpha \frac{\sin x}{x} + x^2 & \textit{if } x \neq 0 \\ \ln(2) & \textit{if } x = 0. \end{cases}$$

is continuous everywhere on $\mathbb{R}$.

Solution.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\alpha \frac{\sin x}{x} + x^2 \right) = \alpha,$$

whence $\alpha = \ln(2)$.

Exercize

Is the function

$$f\left(x\right)=\frac{\ln\left(1+x\right)}{x}$$

continuous in $x_0=0$*? If not classify the type of discontinuity.*

Solution. Since $0\notin D$ the function cannot be continuous in $x_0=0$. Nevertheless

$$\lim_{x\to 0}\frac{\ln\left(1+x\right)}{x}=1,$$

whence the function has a removable discontinuity. This means that the piecewise-defined function

$$g\left(x\right)=\begin{cases}\frac{\ln(1+x)}{x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$$

is defined and continuos everywhere on $\mathbb{R}$.

Exercize

Is the function

$$f(x) = \frac{1}{|x-1|}$$

continuous in $x_0 = 1$? If not classify the type of discontinuity.

Solution. Since $1 \notin D$ the function cannot be continuous in $x_0 = 1$. Besides

$$\lim_{x \to 1^+} \frac{1}{|x-1|} = +\infty, \quad \lim_{x \to 1^-} \frac{1}{|x-1|} = +\infty,$$

whence the function has an essential discontinuity in $x_0 = 1$.

Exercize

Is the function

$$f(x) = \begin{cases} x+7 & \text{if } x > 0 \\ x & \text{if } x \leq 0 \end{cases}$$

continuous on $\mathbb{R}$*? If not classify all the discontinuities.*

Solution. The function is trivially continuous for all $x_0 \neq 0$. Note that

$$\lim_{x \to 0^+} f(x) = 7,$$

while

$$\lim_{x \to 0^-} f(x) = 0,$$

whence the function has a jump discontinuity in $x_0 = 0$.

Theorem

Intermediate value theorem

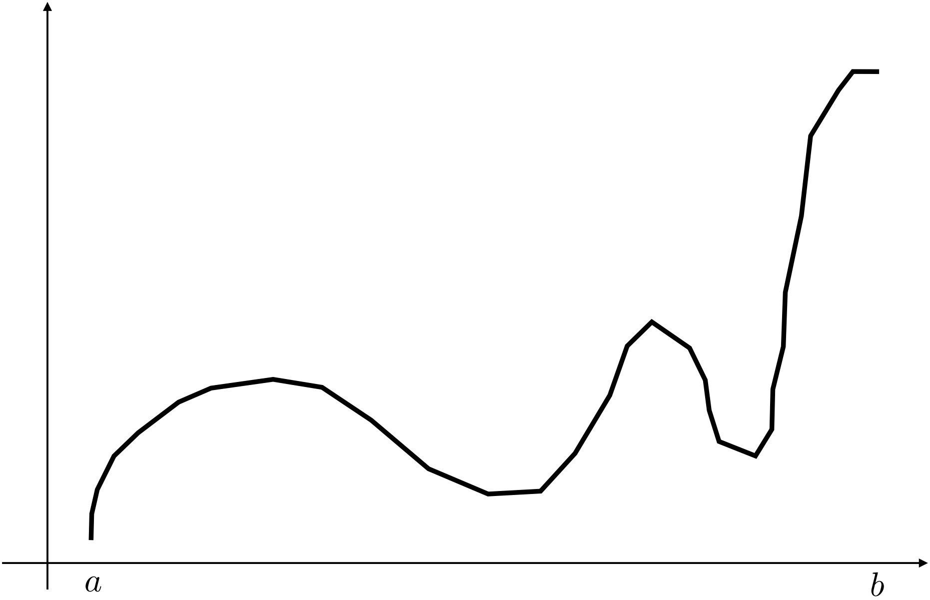
Let $I = [a, b]$ be an interval.

Let $f : I \to \mathbb{R}$ be a ***continuous function*** *defined on I.*

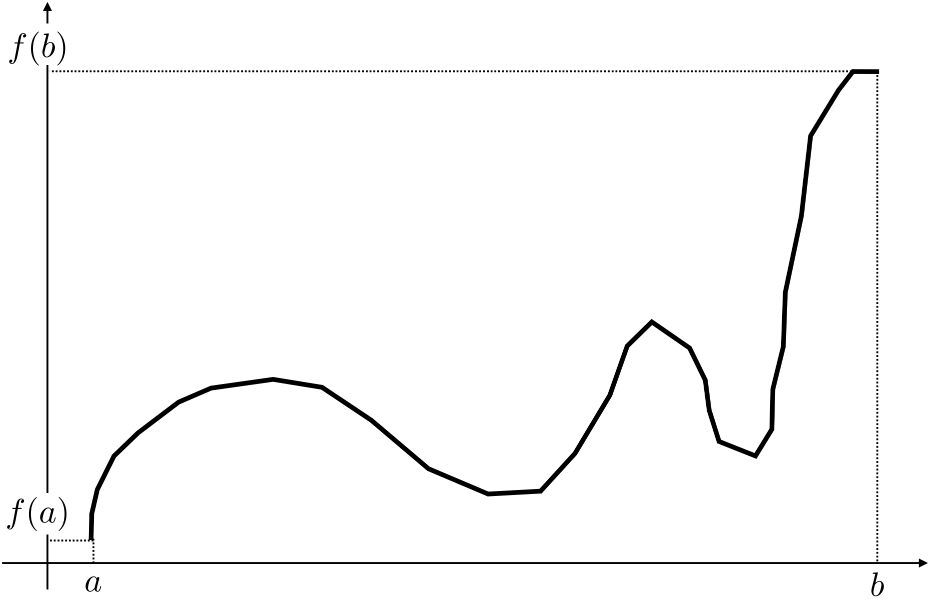
Suppose, without loss of generality, that $f(a) < f(b)$.

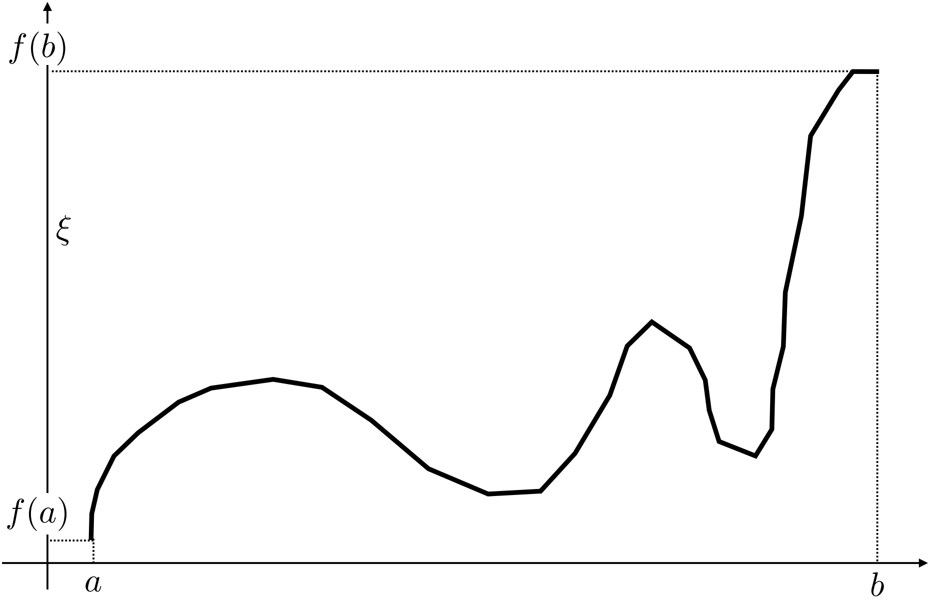
Then

$$\forall \xi \in (f(a), f(b)) \ \exists x_\xi \in (a, b) \text{ such that } f(x_\xi) = \xi.$$

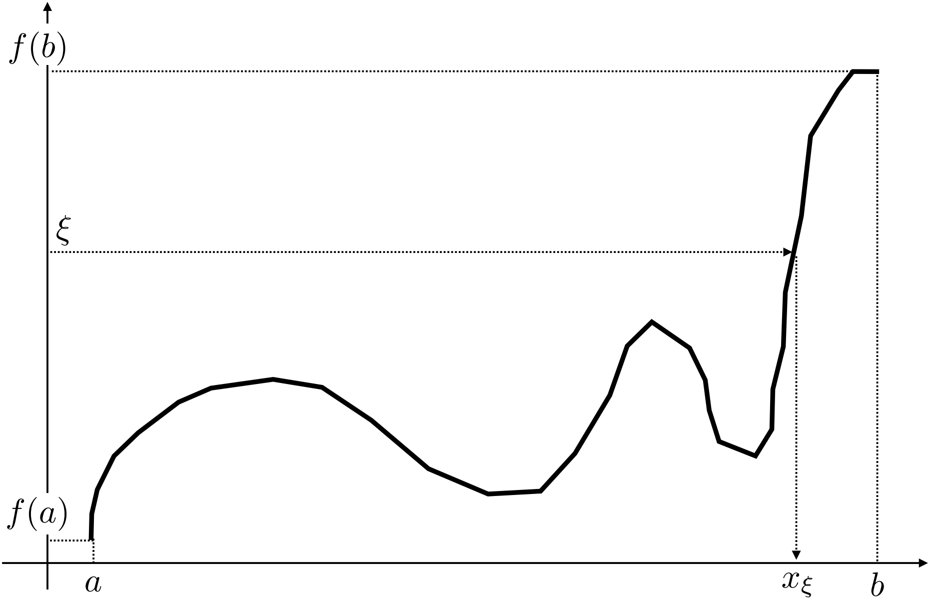


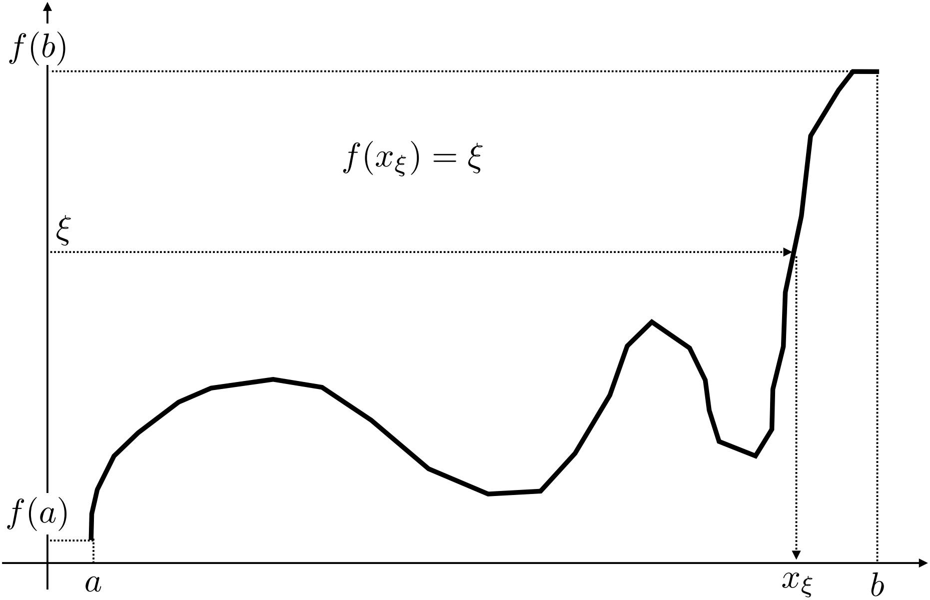
a
b

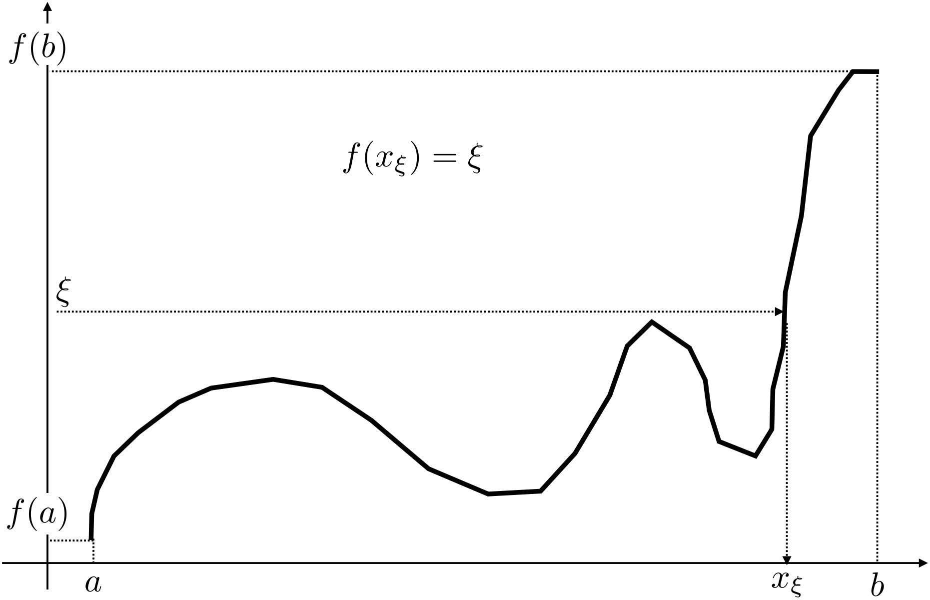

$f(b)$
$f(a)$
a
b



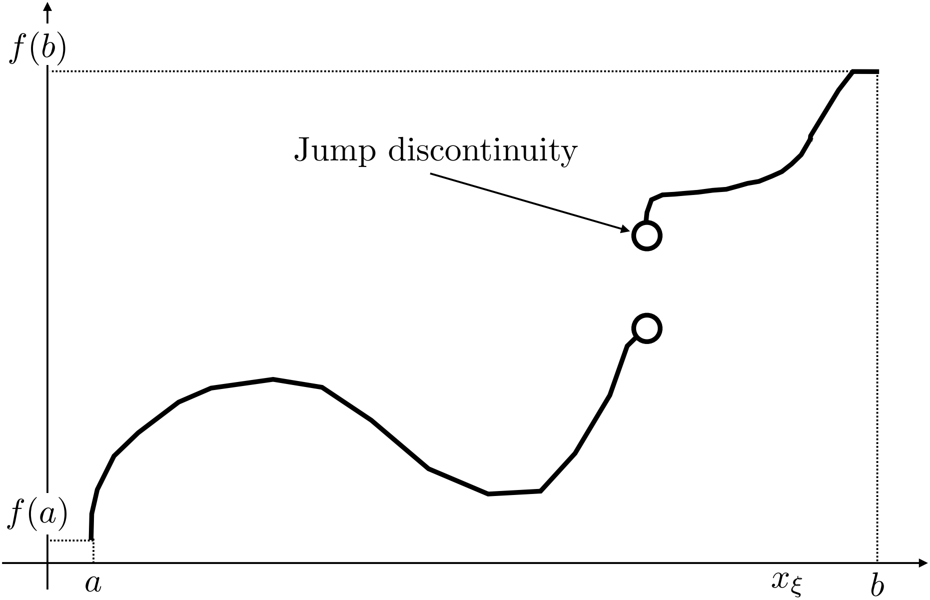
$f(b)$
ξ
$f(a)$
a
b


$f(b)$
ξ
$f(a)$
a
x_ξ
b

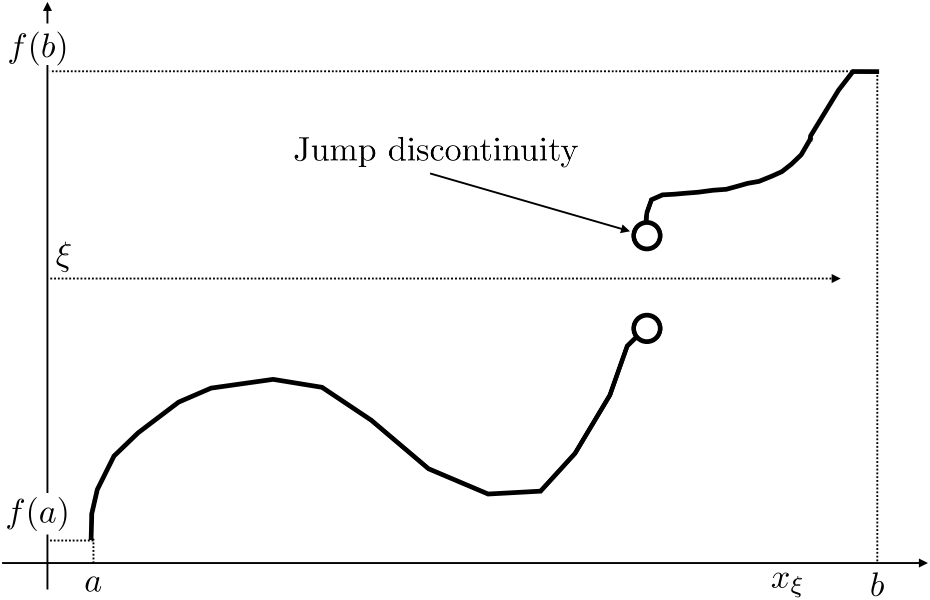

$f(b)$
$f(x_\xi) = \xi$
ξ
$f(a)$
a
x_ξ
b

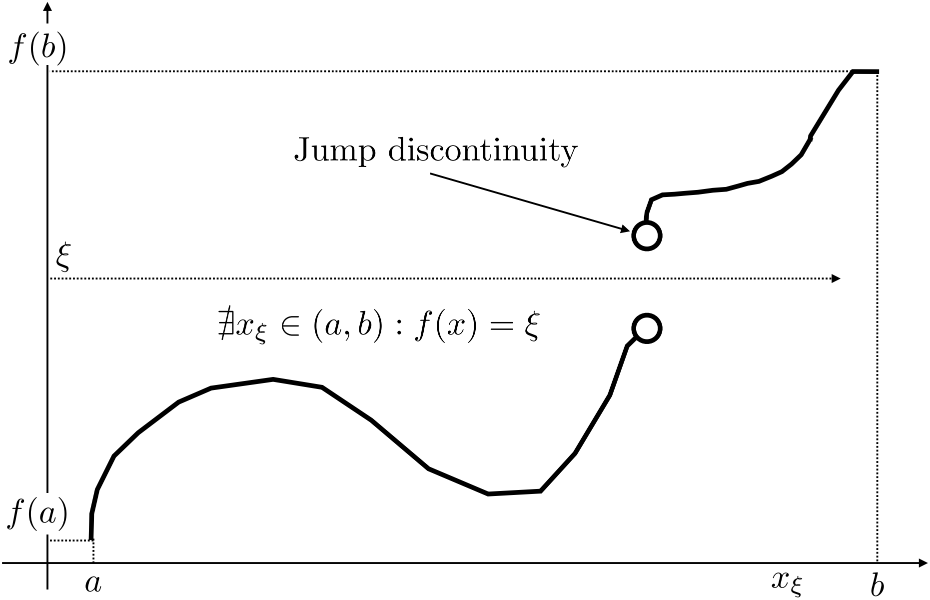


$f(b)$
$f(x_\xi) = \xi$
ξ
$f(a)$
a
x_ξ
b



$f(b)$
Jump discontinuity
$f(a)$
a
x_ξ
b


$f(b)$
Jump discontinuity
ξ
$f(a)$
a
x_ξ
b


$f(b)$
Jump discontinuity
ξ
$\nexists x_\xi \in (a,b) : f(x) = \xi$
$f(a)$
a
x_ξ
b

Let p be the price of a commodity. Assume that $p \in (0,1)$.

Suppose that the demand and supply functions are

$$D(p) = \ln(2-p), \quad S(p) = p \quad (\triangle).$$

Definition

A price p^* is an **equilibrium price** if

$$D(p^*) = S(p^*).$$

Problem

In an economy with demand and supply given by $(\triangle)$ does an equilibrium exist?

Consider the difference

$$f(p) = S(p) - D(p) = p - \ln(2 - p)$$

$$p \in (0,1) \Rightarrow \quad \begin{cases} f(0) & = -\ln(2) < 0, \\ f(1) & = 1 > 0. \end{cases}$$

Since $f(p)$ is continuous in $(0,1)$ then, by the intermediate value theorem,

$$\exists p^* \in (0,1) : f(p^*) = 0,$$

whence

$$S(p^*) = D(p^*).$$

In other words, the economy has at least one equilibrium (we did not prove that is unique).

Definition

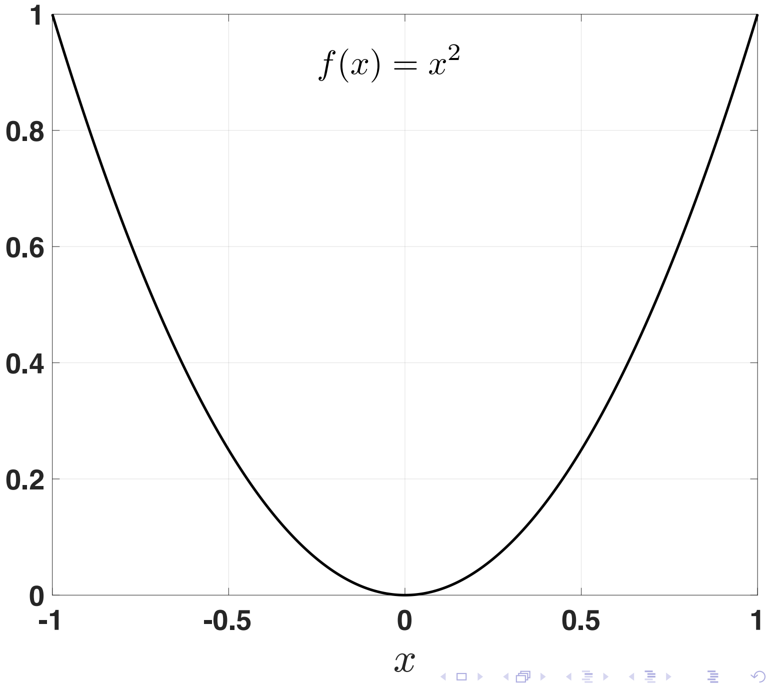
Let $f : I = [a, b] \to \mathbb{R}$ be a function. Suppose that it exists $m \in I$ such that

$$\forall x \in I \Rightarrow f(x) \geq f(m).$$

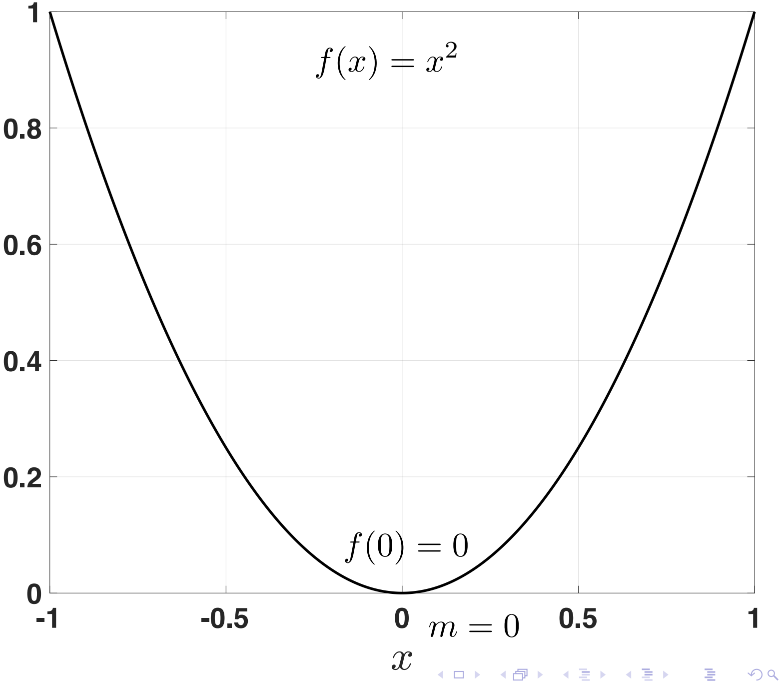
In this case we call $f(m)$ the minimum value of f in I. We also say that f attains a minimum at m.
An identical definition holds for the maximum.

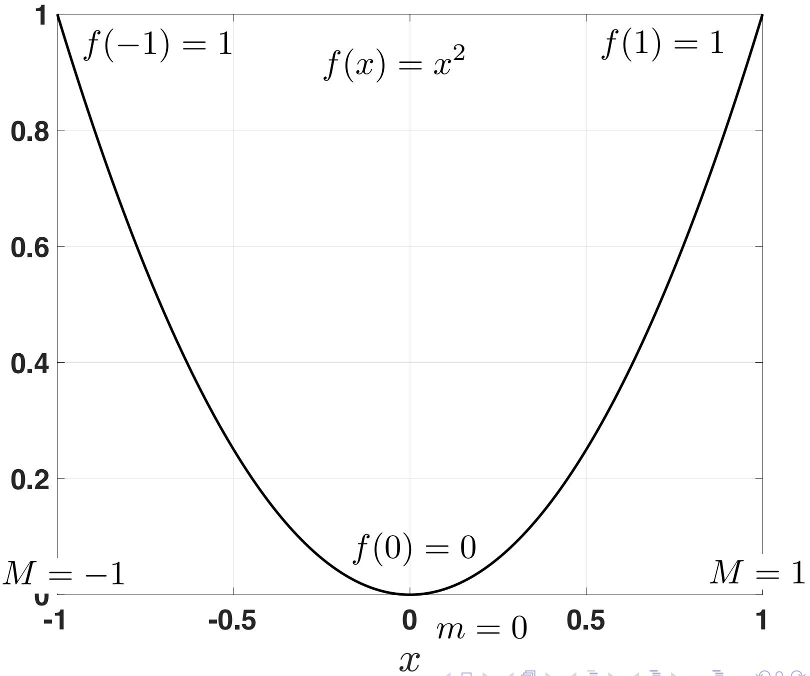
Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function defined on the closed limited interval $[a, b]$. If f is continuous on $[a, b]$ then f attains a minimum and a maximum on $[a, b]$.

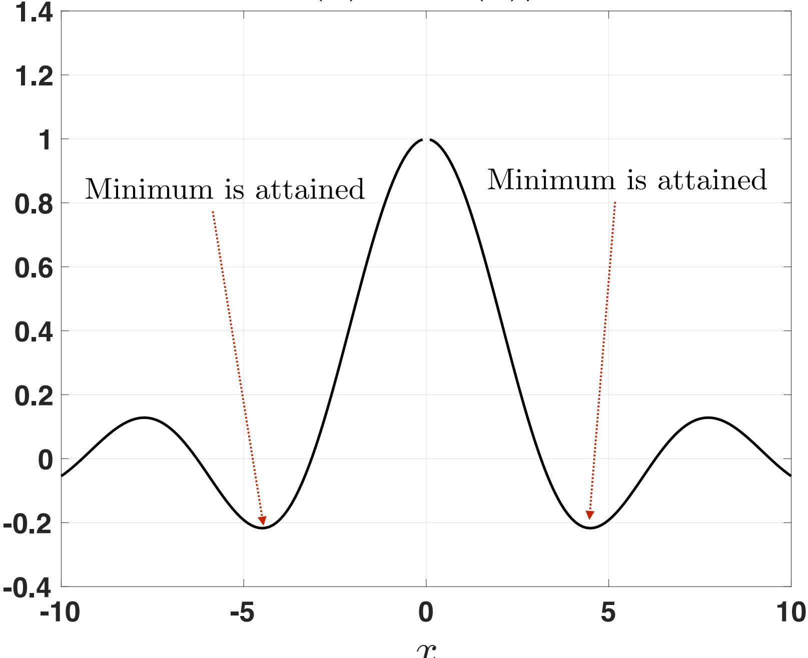


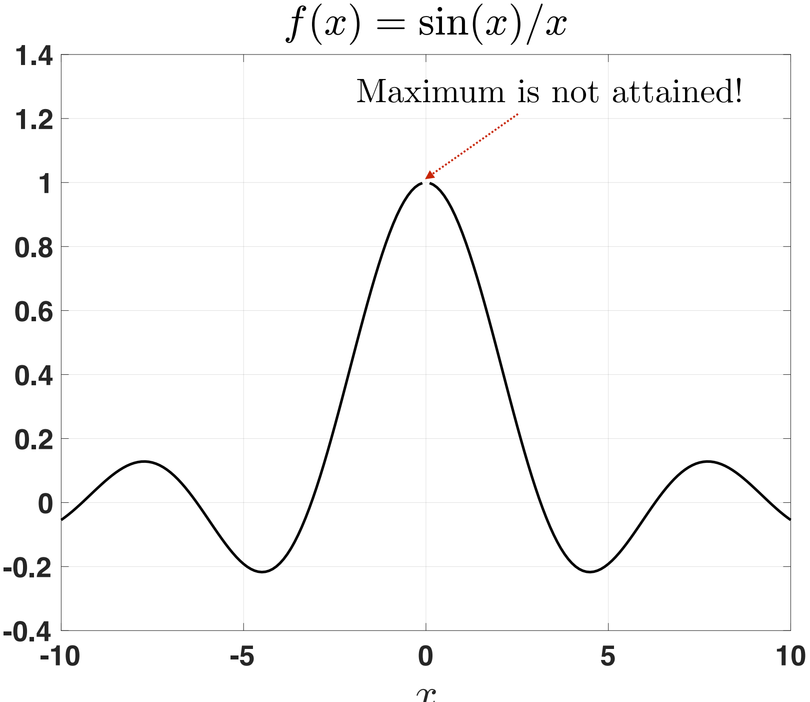
$f(x) = x^2$
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
x


$f(x) = x^2$
$f(0) = 0$
$m = 0$
x
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1

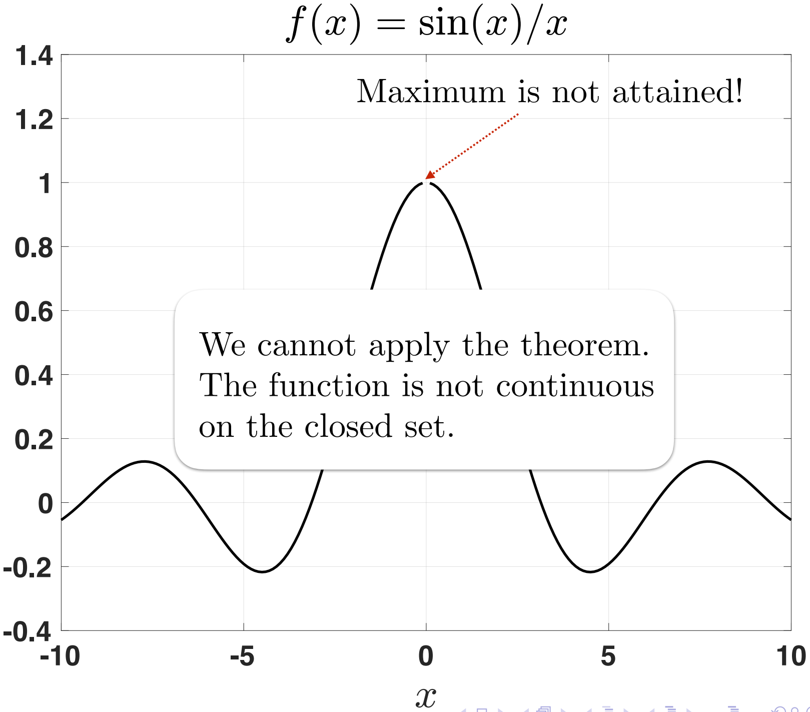

$f(-1) = 1$
$f(x) = x^2$
$f(1) = 1$
$f(0) = 0$
$M = -1$
$M = 1$
$m = 0$
x
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1

$f(x) = \sin(x)/x$

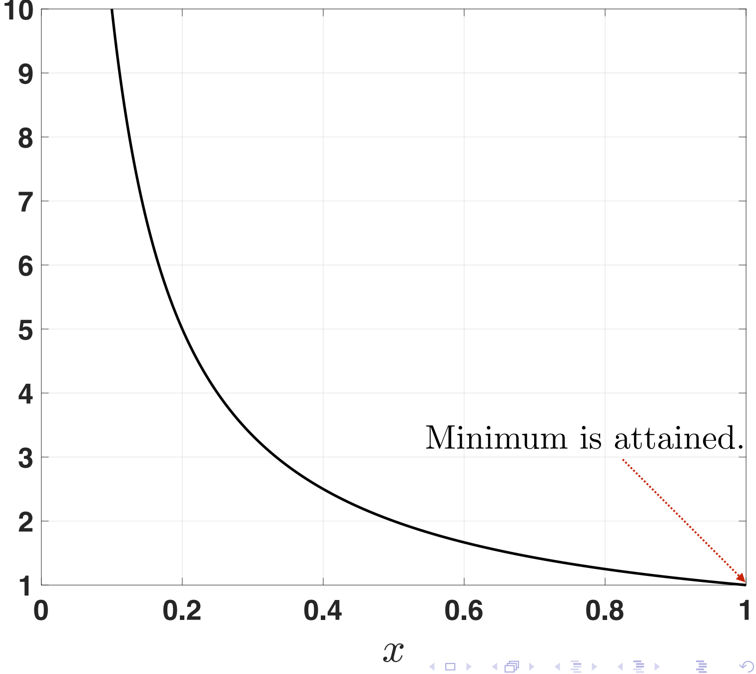

1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-10
-5
0
5
10
x
Minimum is attained
Minimum is attained

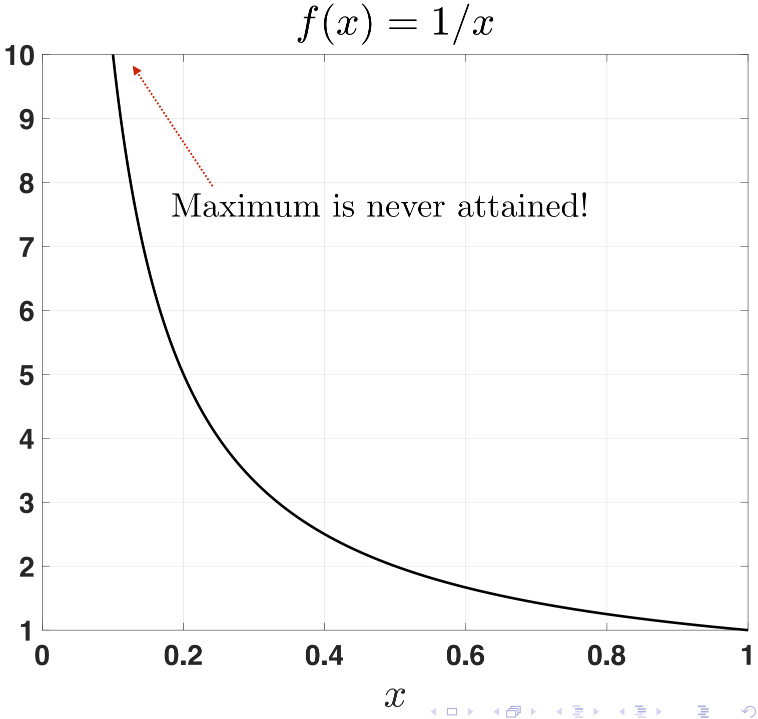


$f(x) = \sin(x)/x$
Maximum is not attained!
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-10
-5
0
5
10
x

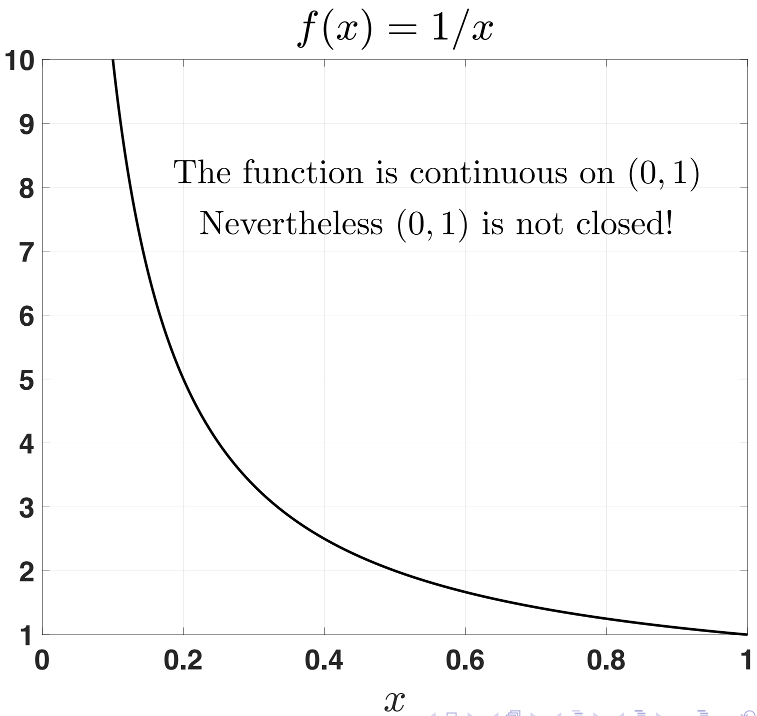

$f(x) = \sin(x)/x$
Maximum is not attained!
We cannot apply the theorem.
The function is not continuous
on the closed set.
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-10
-5
0
5
10
x

$f(x) = 1/x$





$f(x) = 1/x$
Maximum is never attained!
10
9
8
7
6
5
4
3
2
1
0
0.2
0.4
0.6
0.8
1
x


$f(x) = 1/x$
The function is continuous on $(0, 1)$
Nevertheless $(0, 1)$ is not closed!
10
9
8
7
6
5
4
3
2
1
0
0.2
0.4
0.6
0.8
1
x