$\lim_{x \to x_0} f(x) = L$ | $\lim_{x \to x_0} f(x) = \pm\infty$ | $\lim_{x \to \pm\infty} f(x) = L$

$\lim_{x \to \pm\infty} f(x) = \pm\infty$

DEF: $f : [a, +\infty) \to \mathbb{R}$ SI DICE CHE

$$\lim_{x \to +\infty} f(x) = +\infty \iff \forall M > 0 \ \exists N_M : \forall x \geq N_M \Rightarrow f(x) \geq M$$

$$\lim_{x \to -\infty} f(x) = +\infty \iff \forall M > 0 \ \exists N_M : \forall x \leq -N_M \Rightarrow f(x) \geq M$$

$$\lim_{x \to +\infty} f(x) = -\infty \iff \forall M > 0 \ \exists N_M : \forall x \geq N_M \Rightarrow f(x) \leq -M$$

LIMITI IMPORTANTI: USANDO LA DEFINIZIONE DI LIMITE SI PUÒ DIMOSTRARE CHE: $a^x = \frac{1}{a^{-x}}$

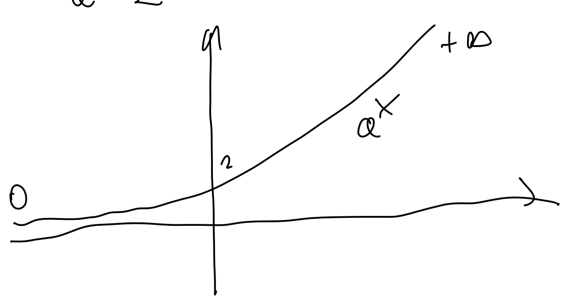
1) SE $a > 1 \Rightarrow$ $\lim_{x \to +\infty} a^x = +\infty$ | $\lim_{x \to +\infty} \log_a(x) = +\infty$

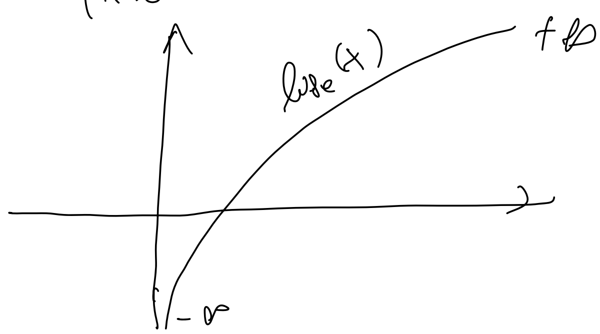
$\lim_{x \to -\infty} a^x = 0$ | $\lim_{x \to 0^+} \log_a(x) = -\infty$

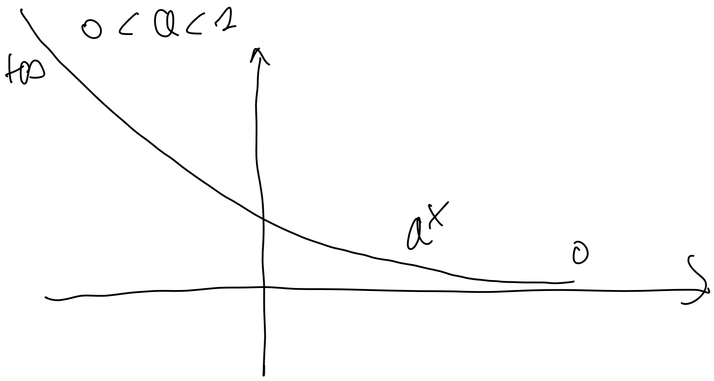
2) SE $0 < a < 1 \Rightarrow$ $\lim_{x \to +\infty} a^x = 0$ | $\lim_{x \to +\infty} \log_a(x) = -\infty$

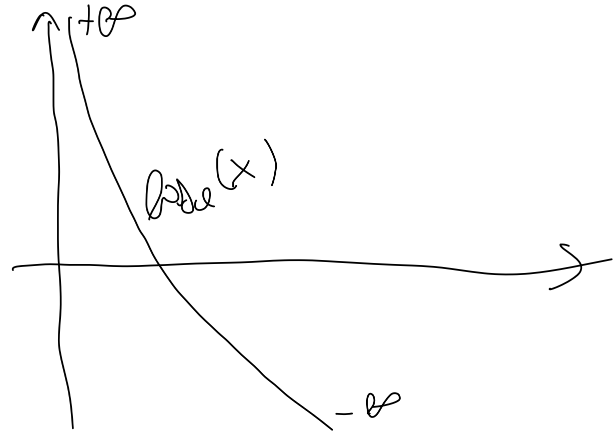
$\lim_{x \to -\infty} a^x = +\infty$ | $\lim_{x \to 0^+} \log_a(x) = +\infty$

$a > 1$









CALCOLARE IL LIMITE IN PRATICA:

VALE LA SEGUENTE "ALGEBRA DEI LIMITI":

SIA $a>0$. SI HA CHE

$$\begin{cases} \dfrac{a}{\pm\infty} = 0 \qquad \dfrac{\pm\infty}{a} = \pm\infty \\[2ex] 0^{-\infty} = +\infty \qquad 0^{+\infty} = 0 \\[2ex] +\infty+\infty = +\infty \qquad -\infty-\infty = -\infty \end{cases}$$

IL PROBLEMA È SE OTTENGO UNA DI QUESTE IDENTITÀ:

$\dfrac{\pm\infty}{\pm\infty} = ?$	$+\infty-\infty = ?$	$-\infty+\infty = ?$
$\dfrac{0}{0} = ?$	$(+\infty)^0 = ?$	$\boxed{1^{+\infty} = ?}$
		$1^{-\infty} = ?$
$0^0 = ?$		

1) $\frac{\infty}{\infty}$

$$\lim_{x\to 0^+} \frac{1/x}{1/x^2} = \frac{+\infty}{+\infty} = \lim_{x\to 0^+} \frac{x^2}{x} = \lim_{x\to 0} x = 0$$

$$\lim_{x\to 0^+} \frac{1/x^2}{1/x} = \frac{+\infty}{+\infty} = \lim_{x\to 0^+} \frac{x}{x^2} = \lim_{x\to 0^+} \frac{1}{x} = +\infty$$

2) $\infty - \infty$

$$\lim_{x\to +\infty} (x^2 - x) = \underline{\infty - \infty} = \lim_{x\to +\infty} x^2\underbrace{\left(1 - \frac{1}{x}\right)}_{\to 1} = \underline{+\infty}$$

$$\lim_{x\to +\infty} \left(\sqrt{x^2+x} - x\right) = \underline{\infty - \infty} = \lim_{x\to +\infty} \frac{\left(\sqrt{x^2+x} - x\right)\left(\sqrt{x^2+x} + x\right)}{\sqrt{x^2+x} + x} =$$

$$= \lim_{x\to +\infty} \frac{\cancel{x^2} + x - \cancel{x^2}}{x + x\sqrt{1+\frac{1}{x}}} = \lim_{x\to +\infty} \frac{\cancel{x}}{\cancel{x} + \cancel{x}\sqrt{1+\frac{1}{x}}} = \underline{\frac{1}{2}}$$

3) 0^0 SI DIMOSTRA CHE

$$\lim_{x\to 0^+} x^x = 1 = 0^0$$

$$\lim_{x\to 0^+} \left(e^{-\frac{1}{x^2}}\right)^x = 0^0 = \lim_{x\to 0^+} e^{-\frac{x}{x^2}}$$

$$= \lim_{x\to 0^+} e^{-\frac{1}{x}} = 0 = 0^0$$

$$\boxed{\begin{aligned} & x\to 0 \Rightarrow \frac{1}{x^2} \to +\infty \\ & \Rightarrow e^{-\frac{1}{x^2}} \to e^{-\infty} = 0 \\ & x\to 0^+ \Rightarrow \frac{1}{x} \to +\infty \\ & \Rightarrow -\frac{1}{x} \to -\infty \\ & \Rightarrow e^{-\frac{1}{x}} \to 0 \end{aligned}}$$

4) 1^∞

ABBIAMO DIMOSTRATO CHE $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e \approx 2{,}7\ldots$

SI PUÓ DIMOSTRARE CHE

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e = \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = 1^{\infty}$$

$$\lim_{x\to+\infty}\left(1+\frac{\alpha}{x}\right)^x = e^{\alpha} = 1^{\infty} \quad \forall\alpha$$

5) $(+\infty)^0$

$$\lim_{x\to+\infty} x^{\frac{1}{\log_a(x)}} = (+\infty)^0 = a$$

$a>1$

$$x^{\frac{1}{\log_a(x)}} = a^{\log_a\left(x^{\frac{1}{\log_a(x)}}\right)} =$$

$$= a^{\frac{1}{\log_a(x)}\cdot\log_a(x)} = a$$

RIASSUNTO: LE INFORME INDETERMINATE SONO

1) $\frac{\infty}{\infty}$ 2) $\infty-\infty$ 3) 1^{∞} 4) 0^0 5) ∞^0 6) $\frac{0}{0}$ 7) $0\times\infty$

DEF: SIA $f: D \subseteq \mathbb{R} \to \mathbb{R}$ E SIA $x_0 \in D$ CHE É ANCHE UN PUNTO LIMITE DI D. DICO CHE LA FUNZIONE f É CONTINUA IN x_0 SE

$$\lim_{x \to x_0} f(x) = f(x_0) = f\left(\lim_{x \to x_0} x\right)$$

ESEMPIO: $f(x) = \sqrt{x}$ É CONTINUA NEL SUO DOMINIO

$$\left| \sqrt{x} - \sqrt{x_0} \right| = \left| \frac{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})}{\sqrt{x} + \sqrt{x_0}} \right| =$$

$$= \left| \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \right| \leq \frac{|x - x_0|}{\sqrt{x_0}} \longrightarrow 0$$

SE $x \to x_0$ ALLORA $\sqrt{x} \to \sqrt{x_0}$

1) a^x $D = \mathbb{R}$ É CONTINUA $\forall x \in D$

2) $\log_a(x)$ $D = (0, \infty)$ É CONTINUA $\forall x \in D = (0, \infty)$

3) x^a $D = [0, +\infty)$ // // $\forall x \in D$

4) TUTTE LE FUNZIONI TRIGONOMETRICHE SONO CONTINUE NEL LORO DOMINIO DI DEFINIZIONE.

ESEMPIO: $\lim_{x \to 0} \sin(x) = \sin(0) = 0$

LIMITI NOTEVOLI

1) $\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{0}{0}$

$\lim_{n \to +\infty} n \cdot \sin\left(\frac{1}{n}\right) = 1$

USANDO ESATTAMENTE LA STESSA DIMOSTRAZIONE SI HA CHE

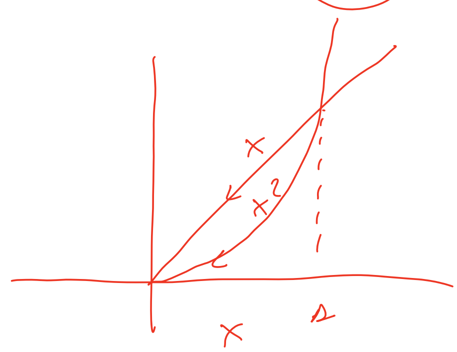
$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

$\sin^2(x) + \cos^2(x) = 1$

2) $\lim_{x \to 0} \frac{1 - \cos(x)}{x} = \frac{0}{0} = \lim_{x \to 0} \frac{1-\cos(x)}{x} \frac{1+\cos(x)}{1} \frac{1}{1+\cos(x)}$

$= \lim_{x \to 0} \frac{1 - \cos^2(x)}{x} \frac{1}{1+\cos(x)} = \lim_{x \to 0} \frac{\sin^2(x)}{x^2} \frac{x}{1+\cos(x)}$

$= \lim_{x \to 0} \underbrace{\left(\frac{\sin(x)}{x}\right)^2}_{1} \cdot \frac{x \to 0}{\underbrace{1+\cos(x)}_{1}} = 1 \cdot 0 = 0$



3) $\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \frac{0}{0}$

$= \lim_{x \to 0} \frac{1-\cos(x)}{x^2} \frac{1+\cos(x)}{1+\cos(x)} = \lim_{x \to 0} \frac{1 - \cos^2(x)}{x^2} \frac{1}{1+\cos(x)} =$

$= \lim_{x \to 0} \left(\frac{\sin(x)}{x}\right)^2 \cdot \frac{1}{1+\cos(x)} = \frac{1}{2}$

1 1

4) $\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{\cos(x)} \frac{1}{x} = \lim_{x\to 0} \frac{\sin(x)}{x} \frac{1}{\cos(x)} = 1$

5) $\lim_{x\to 0} \frac{\log(1+x)}{x}$

NOTA: SE NON SI SPECIFICA LA BASE DEL LOGARITMO È SOTTOINTESO CHE LA BASE È IL NUMERO $e \simeq 2{,}71$

$= \lim_{x\to 0} \frac{\log(1+x)}{x} = \frac{\log(1)}{0} = \frac{0}{0} = 1$

$= \lim_{x\to 0} \frac{1}{x} \cdot \log(1+x) = \lim_{x\to 0} \log(1+x)^{\frac{1}{x}} = \lim_{t\to\infty} \log\left(1+\frac{1}{t}\right)^t$

$t = \frac{1}{x}$ SE $x \to 0 \Rightarrow t \to \infty$ $\quad \left(1+\frac{1}{t}\right)^t \to e$

$= \lim_{t\to\infty} \log\left(1+\frac{1}{t}\right)^t = \log\left(\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^t\right) = \log(e) = 1$

6) $\lim_{x\to 0} \frac{e^x-1}{x} = \frac{e^0-1}{0} = \frac{1-1}{0} = \frac{0}{0}$

$e^x - 1 = t \Rightarrow e^x = 1+t \Rightarrow x = \log(1+t)$

SE $x \to 0 \Rightarrow t \to 0$

$$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{t\to 0} \frac{t}{\log(1+t)} =$$

$$= \lim_{t\to 0} \frac{1}{\frac{1}{t}\log(1+t)} = \lim_{t\to 0} \frac{1}{\underbrace{\log(1+t)^{1/t}}_{\to 1}} = 1$$

7) $\lim_{x\to 0} \frac{\log_a(1+x)}{x}$ REGOLA DEL CAMBIO DI BASE:

$$\log_a(\dots) = \frac{\log_b(\dots)}{\log_b(a)} \qquad b=e$$

$$\log_a(1+x) = \frac{\log(1+x)}{\log(a)}$$

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \lim_{x\to 0} \underbrace{\frac{\log(1+x)}{x}}_{\to 1} \frac{1}{\log(a)} = \frac{1}{\log(a)}$$

8) $\lim_{x\to 0} \frac{a^x-1}{x}$ $\qquad a^x-1=t \Rightarrow x=\log_a(1+t)$

$$= \lim_{t\to 0} \frac{t}{\log_a(1+t)} = \lim_{t\to 0} \frac{1}{\log_a(1+t)^{1/t}} = \log(a)$$

ORDINAMENTO:

$a > 1 \quad b > 0$

$$\lim_{x \to +\infty} \frac{\log_a(x)}{x^b} = 0$$

$$\lim_{x \to +\infty} \frac{x^b}{a^x} = 0$$

$$\lim_{x \to 0^+} x \cdot \log(x) = 0 \times (-\infty) = 0$$

$x = e^{-t}$ se $x \to 0^+ \Rightarrow t \to +\infty$

$$\lim_{x \to 0^+} x \cdot \log(x) = \lim_{t \to +\infty} e^{-t}(-t) = \lim_{t \to +\infty} -\frac{t}{e^t} = 0$$

$$\log(x) = \log(e^{-t}) = -t$$

$$\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{\log(x^x)} = \lim_{x \to 0^+} e^{x \log(x)} = e^0 = 1$$