Part III

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Derivatives

Definition

A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is said to be differentiable in $x_0 \in \operatorname{int}(D)$ ($\operatorname{int}(D)$ denotes the set of the interior points of D) if:

$$\exists L = \lim_{h\to 0} \underbrace{\frac{f(x_0+h) - f(x_0)}{h}}_{\text{difference quotient.}} .$$

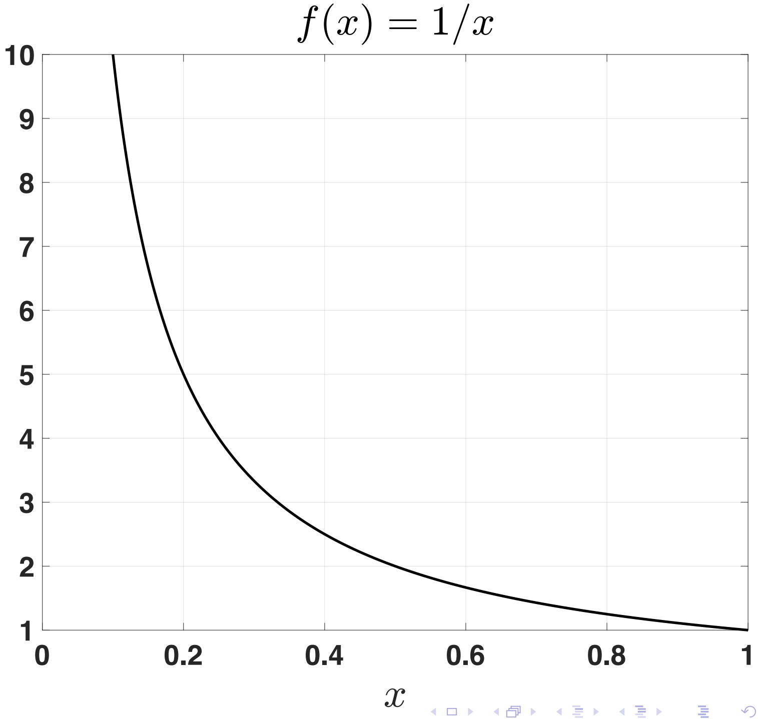
We call $L = f'(x_0)$. We define the left and right derivative of f in x_0 the two limits:

$$\lim_{h\to 0^-} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h},$$

when they exist and we call them $f'\left(x_0^-\right)$ and $f'\left(x_0^+\right)$ respectively.

Remark

Note that the left and right derivatives can be defined even if x_0 belongs to the closure of D.


$f(x) = 1/x$
10
9
8
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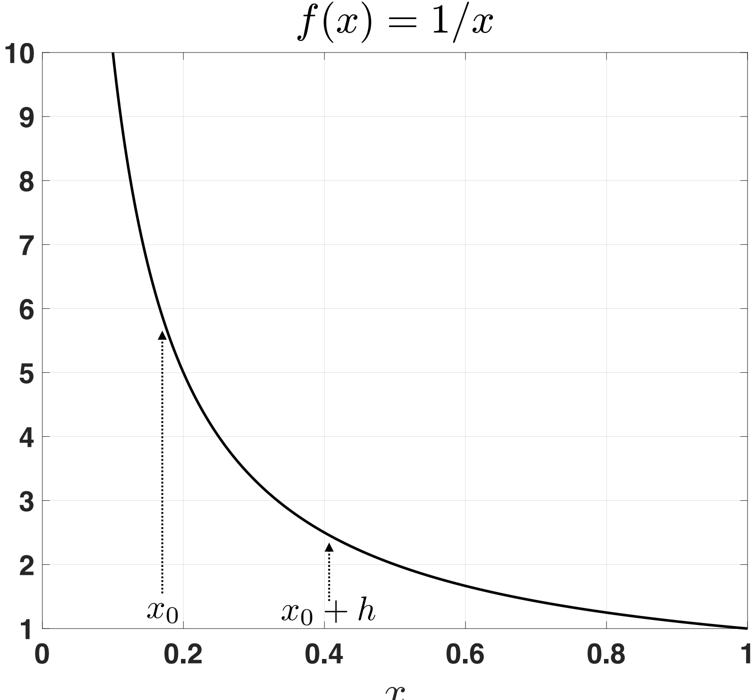
1
x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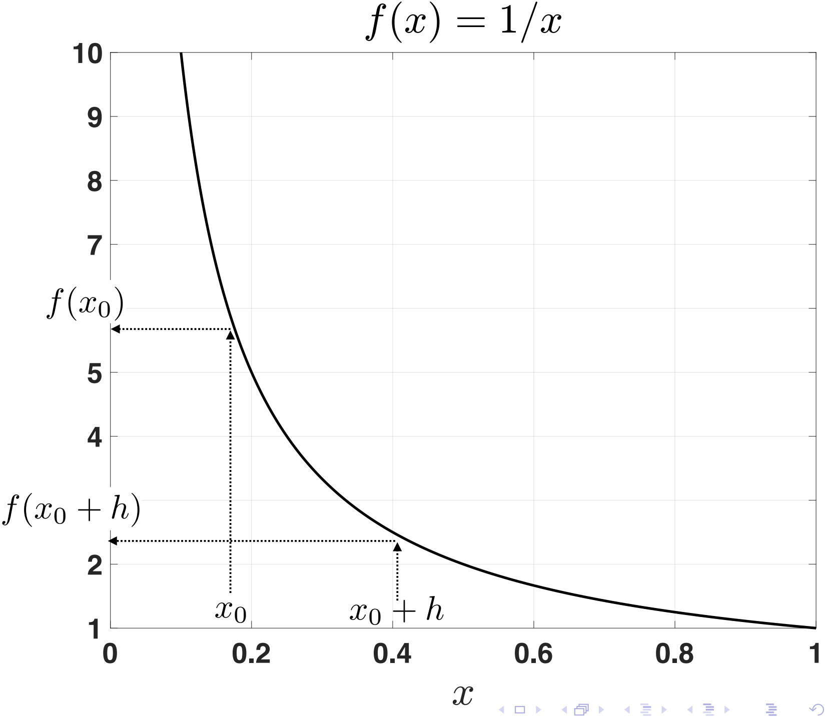
x_0
$x_0 + h$
x


$f(x) = 1/x$
10
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$f(x_0)$
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$f(x_0 + h)$
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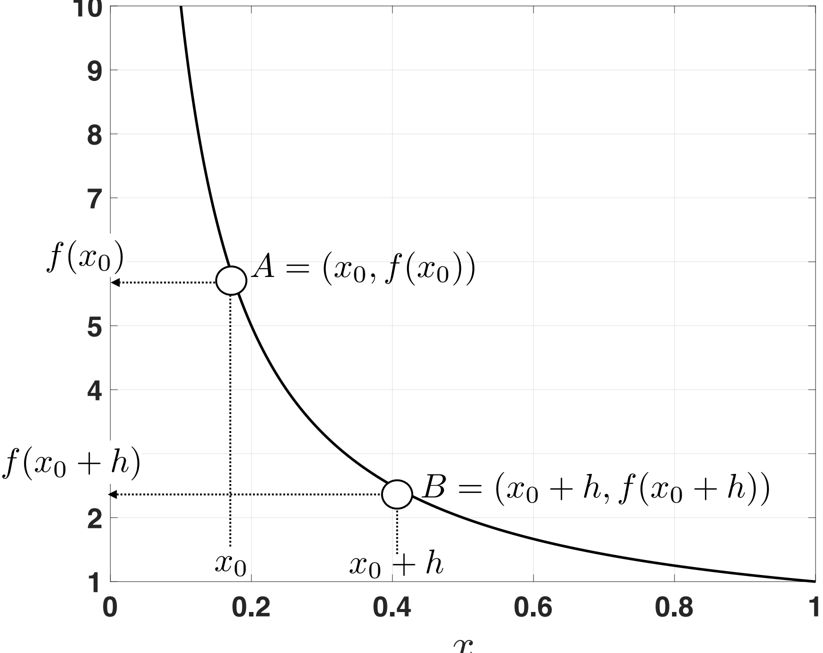
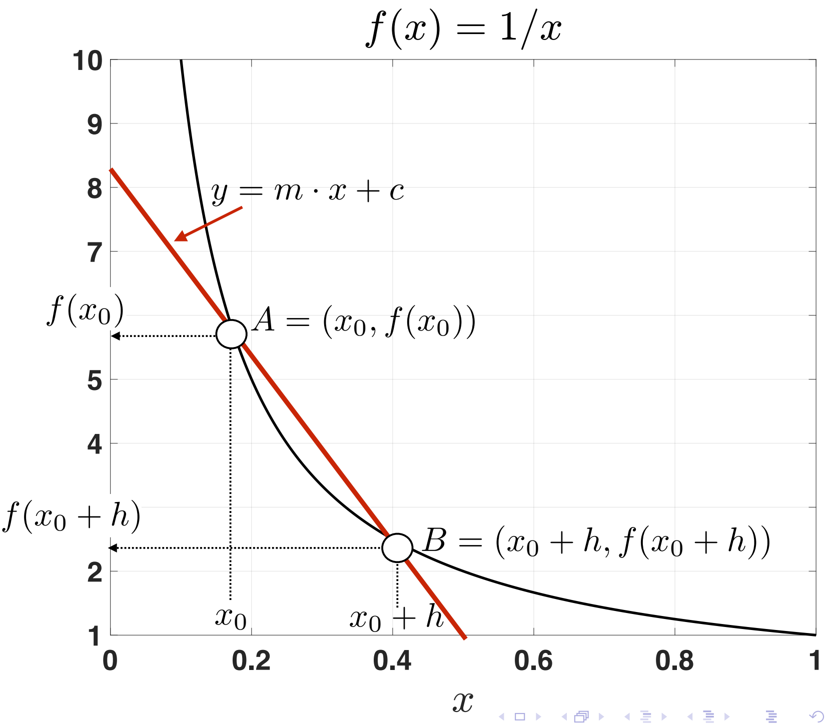
x

$f(x) = 1/x$





$f(x) = 1/x$
$y = m \cdot x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
$f(x_0 + h)$
$B = (x_0 + h, f(x_0 + h))$
x_0
$x_0 + h$
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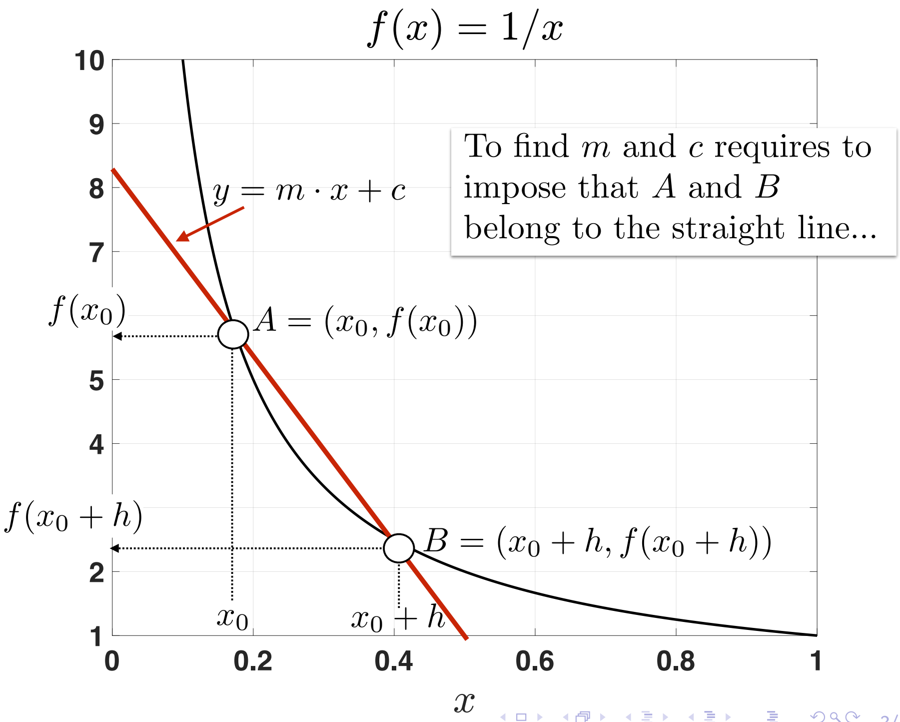

$f(x) = 1/x$
To find m and c requires to impose that A and B belong to the straight line...
$y = m \cdot x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
$f(x_0 + h)$
$B = (x_0 + h, f(x_0 + h))$
x_0
$x_0 + h$
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10
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Geometric interpretation of the derivative

Let f be differentiable in x_0. Find the equation of the straight line passing through $A = (x_0, f(x_0))$ and $B = (x_0 + h, f(x_0 + h))$.

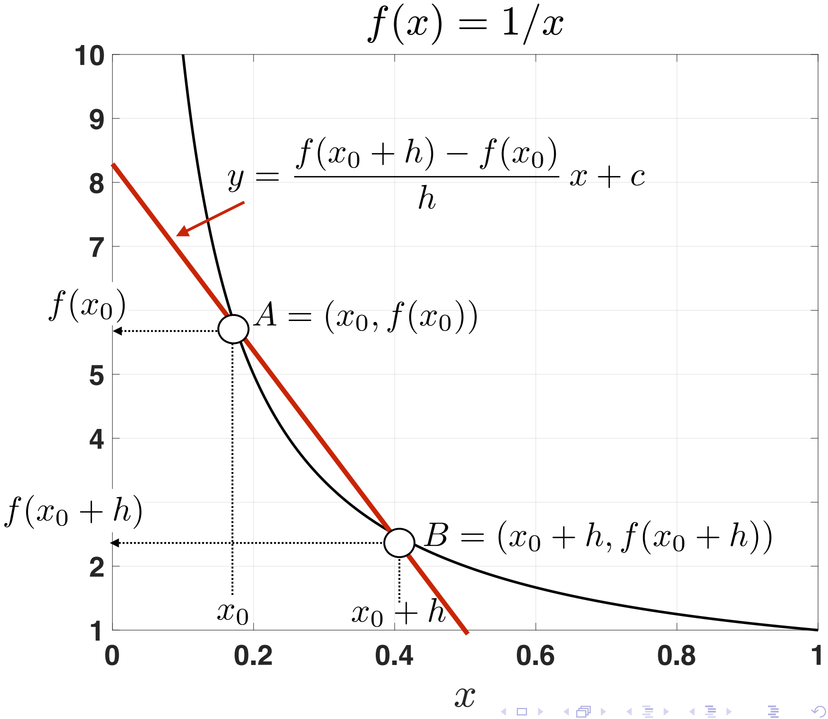
- Generic equation of the straight line $y = m\,x + c$
- A belongs to the line $\Leftrightarrow f(x_0) = m_h\,x_0 + c_h$.
- B belongs to the line $\Leftrightarrow f(x_0 + h) = m_h\,(x_0 + h) + c_h$.

Whence

$$f(x_0 + h) - f(x_0) = m_h\,x_0 + m_h\,h + c_h - m_h\,x_0 - c_h = m_h\,h \Rightarrow$$

$$m_h = \frac{f(x_0 + h) - f(x_0)}{h}$$

If $h \to 0$ then $m_h \to f'(x_0) \Rightarrow$ The derivative is the angular coefficient of the line tangent to the graph.



$f(x) = 1/x$
$y = \frac{f(x_0+h) - f(x_0)}{h} x + c$
$f(x_0)$
$A = (x_0, f(x_0))$
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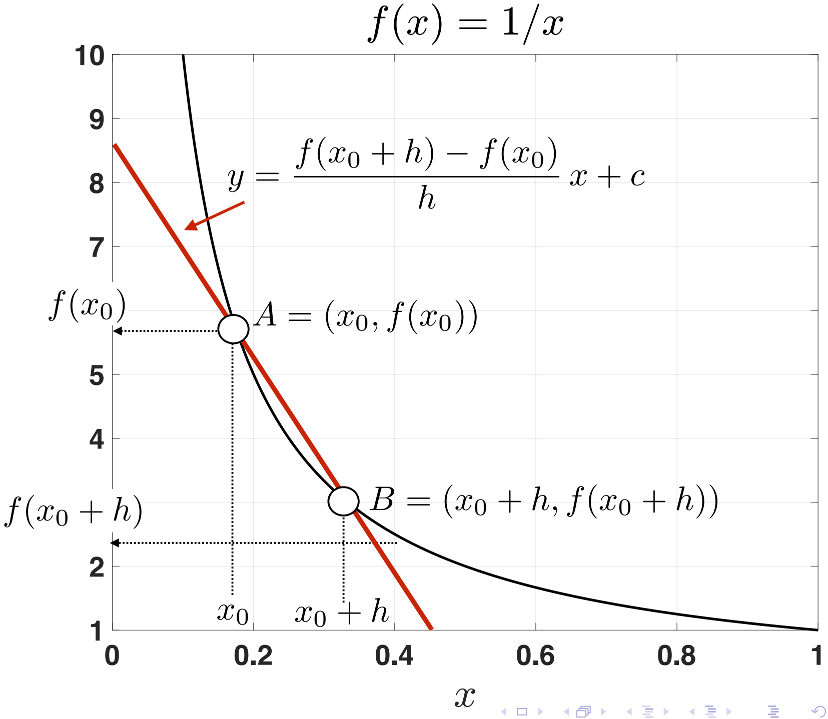
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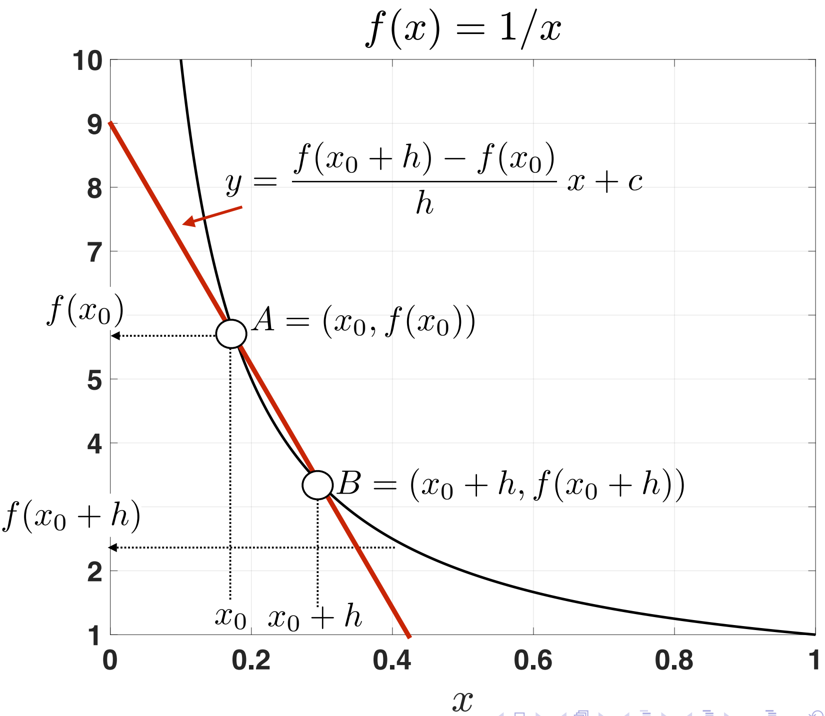
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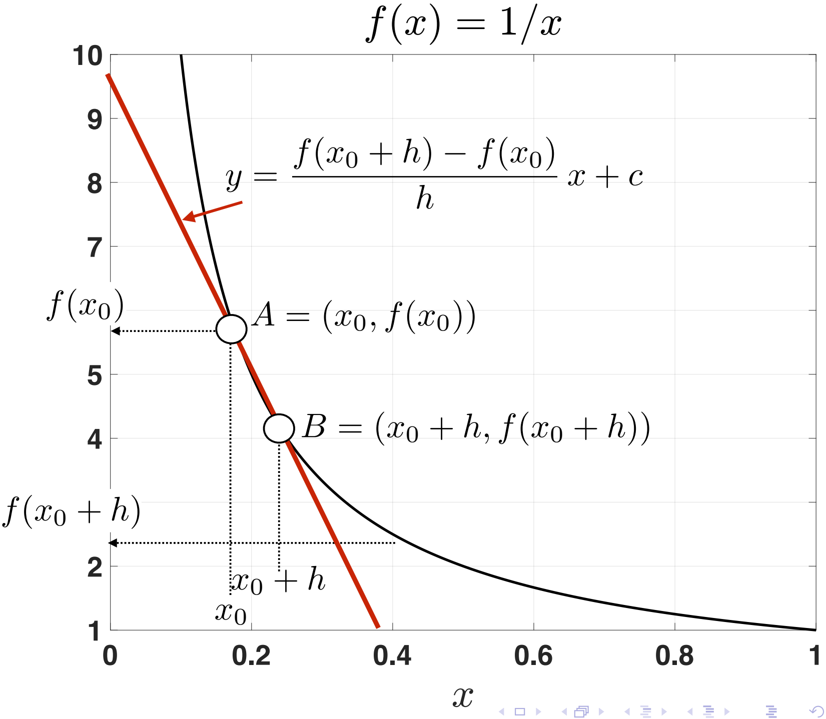

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$A = (x_0, f(x_0))$
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1

Theorem

If f is differentiable in x_0 then it is continuous in x_0.

Proof. We know that the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0),$$

exists and it is finite. Whence

$$\begin{aligned} \lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) \\ &= f'(x_0) \cdot 0 = 0, \end{aligned}$$

which means

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Derivatives

Problem

Is the converse true?

That is, if $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is continuous in x_0, can we say that f is differentiable in x_0?

Answer

No! Differentiability is a condition much stronger than continuity.

As always, to prove our assertion, we need at least one counter-example:

$$f(x) = |x|,$$

is continuous in $x_0 = 0$, nevertheless

$$\lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1, \qquad \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1,$$

that is $f'\left(x_0^+\right) = +1 \neq f'\left(x_0^-\right) = -1$, whence $\nexists f'(0)$.

Definition

The point $x_0 = 0$ it's called an **angle point**. More generally, we say that a function $f(x)$ has an angle point in x_0 whenever the derivative has a jump discontinuity in x_0.

Theorem

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ *and* $g : D \subseteq \mathbb{R} \to \mathbb{R}$ *be two functions differentiable in* x_0.

For all $\alpha \in \mathbb{R}$ *and* $\beta \in \mathbb{R}$, *the function* $\alpha f + \beta g : D \subseteq \mathbb{R} \to \mathbb{R}$ *defined as*

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x), \quad \forall x \in D$$

is differentiable in x_0 *and*

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0).$$

Proof.

We have to proof the the limit of the difference quotient

$$\lim_{h\to 0}\frac{(\alpha f+\beta g)(x_0+h)-(\alpha f+\beta g)(x_0)}{h}$$

exists and it is finite. For this purpose note that

$$\begin{aligned}
&\lim_{h\to 0}\frac{(\alpha f+\beta g)(x_0+h)-(\alpha f+\beta g)(x_0)}{h}\\
=&\lim_{h\to 0}\frac{\alpha f(x_0+h)+\beta g(x_0+h)-\alpha f(x_0)-\beta g(x_0)}{h}\\
=&\lim_{h\to 0}\frac{\alpha f(x_0+h)-\alpha f(x_0)+\beta g(x_0+h)-\beta g(x_0)}{h}\\
=&\lim_{h\to 0}\left[\alpha\frac{f(x_0+h)-f(x_0)}{h}+\beta\frac{g(x_0+h)-g(x_0)}{h}\right]\\
=&\ \alpha f'(x_0)+\beta g'(x_0).
\end{aligned}\tag{0.1}$$

Theorem

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and $g : D \subseteq \mathbb{R} \to \mathbb{R}$ be two functions differentiable in x_0.

The function $f \cdot g : D \subseteq \mathbb{R} \to \mathbb{R}$ defined as

$$(f \cdot g)(x) = f(x) \cdot g(x), \quad \forall x \in D$$

is differentiable in x_0 and

$$(f \cdot g)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0).$$

Proof. We have to proof the the limit of the difference quotient

$$\lim_{h \to 0} \frac{(f \cdot g)(x_0 + h) - (f \cdot g)(x_0)}{h}$$

exists and it is finite. For this purpose note that

$$\begin{aligned}
& \lim_{h \to 0} \frac{(f\,g)(x_0 + h) - (f\,g)(x_0)}{h} \\
= \quad & \lim_{h \to 0} \frac{f(x_0 + h)\, g(x_0 + h) - f(x_0)\, g(x_0)}{h} \\
= \quad & \lim_{h \to 0} \frac{f(x_0 + h)\, g(x_0 + h) - f(x_0)\, g(x_0 + h) + f(x_0)\, g(x_0 + h) - f(x_0)\, g(x_0)}{h} \\
= \quad & \lim_{h \to 0} \frac{g(x_0 + h)\, (f(x_0 + h) - f(x_0)) + f(x_0)\, (g(x_0 + h) - g(x_0))}{h} \\
= \quad & \lim_{h \to 0} \left[g(x_0 + h)\, \frac{(f(x_0 + h) - f(x_0))}{h} + f(x_0)\, \frac{(g(x_0 + h) - g(x_0))}{h} \right].
\end{aligned}$$

$$g \text{ differentiable in } x_0 \Rightarrow g \text{ continuous in } x_0 \Rightarrow g(x_0 + h) \to g(x_0).$$

Whence

$$\lim_{h \to 0} \frac{(f\,g)(x_0 + h) - (f\,g)(x_0)}{h} = g(x_0)\, f'(x_0) + f(x_0)\, g'(x_0).$$

Theorem

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ *and* $g : E \subseteq \mathbb{R} \to \mathbb{R}$ *with* $f(D) \subseteq E$.
Hence it is possible to define $(g \circ f) : D \subseteq \mathbb{R} \to \mathbb{R}$ *in the standard way*

$$(g \circ f)(x) = g(f(x)), \quad \forall x \in D.$$

Assume f *is differentiable in* $x_0 \in D$ *and* g *is differentiable in* $f(x_0) \in E$.
Then $(g \circ f)$ *is differentiable in* x_0 *and*

$$(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0).$$

Proof. The proof is rather technical and we skip it.

- Compute the derivative of a constant function, $f(x) = c$ for all $x \in \mathbb{R}$.

$$\lim_{h\to 0} \frac{f(x+h) - f(x+h)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0 \Rightarrow f'(x) = 0.$$

- Compute the derivative of $f(x) = x$ for all $x \in \mathbb{R}$.

$$\lim_{h\to 0} \frac{f(x+h) - f(x+h)}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = 1 \Rightarrow f'(x) = 1.$$

- For $n \in \mathbb{N}$, $n > 1$, compute the derivative of $f(x) = x^n$ for all $x \in \mathbb{R}$.

$$
\begin{aligned}
& \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\
= \quad & \lim_{h \to 0} \frac{\left(\begin{array}{c} n \\ 0 \end{array}\right) x^n + \left(\begin{array}{c} n \\ 1 \end{array}\right) x^{n-1} h + \left(\begin{array}{c} n \\ 2 \end{array}\right) x^{n-2} h^2 + \ldots + \left(\begin{array}{c} n \\ n \end{array}\right) h^n - x^n}{h} \\
= \quad & \lim_{h \to 0} \frac{x^n + \left(\begin{array}{c} n \\ 1 \end{array}\right) x^{n-1} h + \left(\begin{array}{c} n \\ 2 \end{array}\right) x^{n-2} h^2 + \ldots + h^n - x^n}{h} \\
= \quad & \lim_{h \to 0} \frac{\left(\begin{array}{c} n \\ 1 \end{array}\right) x^{n-1} h + \left(\begin{array}{c} n \\ 2 \end{array}\right) x^{n-2} h^2 + \ldots + h^n}{h} \\
= \quad & \lim_{h \to 0} \left(\begin{array}{c} n \\ 1 \end{array}\right) x^{n-1} + \left(\begin{array}{c} n \\ 2 \end{array}\right) x^{n-2} h + \ldots + h^{n-1}.
\end{aligned}
$$

Since $\lim_{h \to 0} h = \lim_{h \to 0} h^2 = \lim_{h \to 0} h^3 = \ldots = \lim_{h \to 0} h^{n-1} = 0$ we have

$$
f'(x) = \left(\begin{array}{c} n \\ 1 \end{array}\right) x^{n-1} = \frac{n!}{1!\,(n-1)!} x^{n-1} = n\, x^{n-1}.
$$

- For $a > 0, a \neq 1$, compute the derivative of $f(x) = a^x$ for all $x \in \mathbb{R}$.

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \to} \frac{a^h - 1}{h} = a^x \ln(a).$$

Since $\ln(e) = 1$ we have $(e^x)' = e^x$.

- Compute the derivative of $f(x) = \sin x$ for all $x \in \mathbb{R}$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \to 0} \left[\sin x \underbrace{\frac{\cos h - 1}{h}}_{\substack{\downarrow \\ 0}} + \cos x \underbrace{\frac{\sin h}{h}}_{\substack{\downarrow \\ 1}} \right] = \cos x. \end{aligned}$$

- Compute the derivative of $f(x) = \cos x$ for all $x \in \mathbb{R}$. Remember that $\cos x = \sin\left(x + \frac{\pi}{2}\right)$, hence by the rule of derivation of composite function

$$(\cos x)' = \left(\sin\left(x + \frac{\pi}{2}\right)\right)' = \cos\left(x + \frac{\pi}{2}\right)\left(x + \frac{\pi}{2}\right)' = \cos\left(x + \frac{\pi}{2}\right) = -\sin(x).$$

- For $a > 0, a \neq 1$, compute the derivative of $f(x) = \log_a(x)$ for all $x > 0$.

$$f'(x) = \lim_{h\to 0} \frac{\log_a(x+h) - \log_a x}{h} = \lim_{h\to 0} \frac{\log_a\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \log_a\left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$
$$= \lim_{h\to 0} \log_a\left[\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]^{\frac{1}{x}} = \frac{1}{x} \lim_{h\to 0} \log_a\left(1+\frac{h}{x}\right)^{\frac{x}{h}}.$$

Nevertheless, by changing variable, we get

$$\lim_{h\to 0} \log_a\left(1+\frac{h}{x}\right)^{\frac{x}{h}} = \lim_{t\to 0} \log_a(1+t)^{\frac{1}{t}} = \lim_{q\to\infty} \log_a\left(1+\frac{1}{q}\right)^q$$
$$= \log_a\left(\lim_{q\to\infty}\left(1+\frac{1}{q}\right)^q\right) = \log_a e.$$

Whence

$$(\log_a(x))' = \frac{\log_a e}{x} = \frac{1}{x \ln a}.$$

In particular $(\ln(x))' = \frac{1}{x}$.

- For $\alpha \in \mathbb{R}, \alpha \neq 1$, compute the derivative of $f(x) = x^\alpha$ for all $x > 0$.

$$(x^\alpha)' = \left(e^{\ln(x^\alpha)}\right)' = \left(e^{\alpha \ln(x)}\right)' = e^{\alpha \ln(x)} (\alpha \ln(x))' = e^{\alpha \ln(x)} \frac{\alpha}{x} = x^\alpha \frac{\alpha}{x} = \alpha x^{\alpha - 1}.$$

Definition

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let x_0 be a point such that

$$\lim_{x \to x_0^+} f'(x) = +\infty \text{ and } \lim_{x \to x_0^-} f'(x) = -\infty.$$

In this case the point x_0 is called a cusp point.

Example

$$f(x) = \sqrt{|x|} : \mathbb{R} \to \mathbb{R},$$

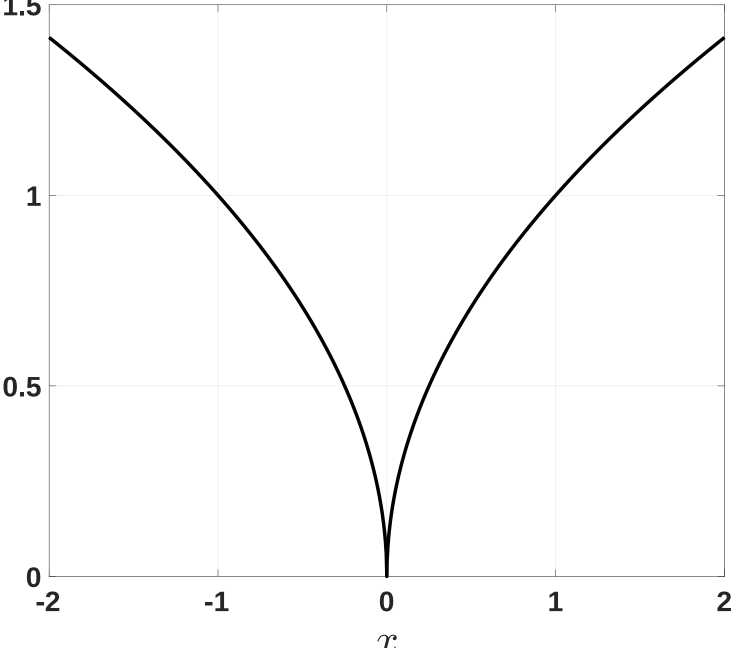
If $x > 0$ then $f(x) = \sqrt{x} = x^{1/2}$, which implies

$$f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \Rightarrow \lim_{x \to 0^+} f'(x) = +\infty.$$

If $x < 0$ then $f(x) = \sqrt{-x} = (-x)^{1/2}$, which implies

$$f'(x) = \frac{1}{2} (-x)^{\frac{1}{2}-1} (-x)' = -\frac{1}{2} (-x)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{-x}} \Rightarrow \lim_{x \to 0^-} f'(x) = -\infty.$$

$$f(x) = \sqrt{|x|}$$



Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then $\exists x_0 \in (a, b)$ such that $f'(x_0) = 0$.

Proof. Continuity on $[a, b]$ and Weiestrass $\Rightarrow \exists m, M \in [a, b]$:

$$f(m) \leq f(x) \leq f(M), \quad \forall x \in \mathbb{R}.$$

If both $m \notin (a, b)$ and $M \notin (a, b)$ then

$$f(a) = f(b) \Rightarrow f(m) = f(M) \Rightarrow f(x) = f(m) = f(M) \Rightarrow f'(x_0) = 0 \forall x_0.$$

Assume, without loss of generality, that at least $M \in (a, b)$.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then $\exists x_0 \in (a, b)$ such that $f'(x_0) = 0$.

Proof. $M \in (a, b)$ hence for $h > 0$ small $M + h \in (a, b)$. $f(M)$ is a maximum, so $f(M + h) - f(M) \leq 0$ and since $h > 0$ we get

$$\frac{f(M+h) - f(M)}{h} \leq 0 \Rightarrow f'\left(M^+\right) \leq 0 \quad (\triangle).$$

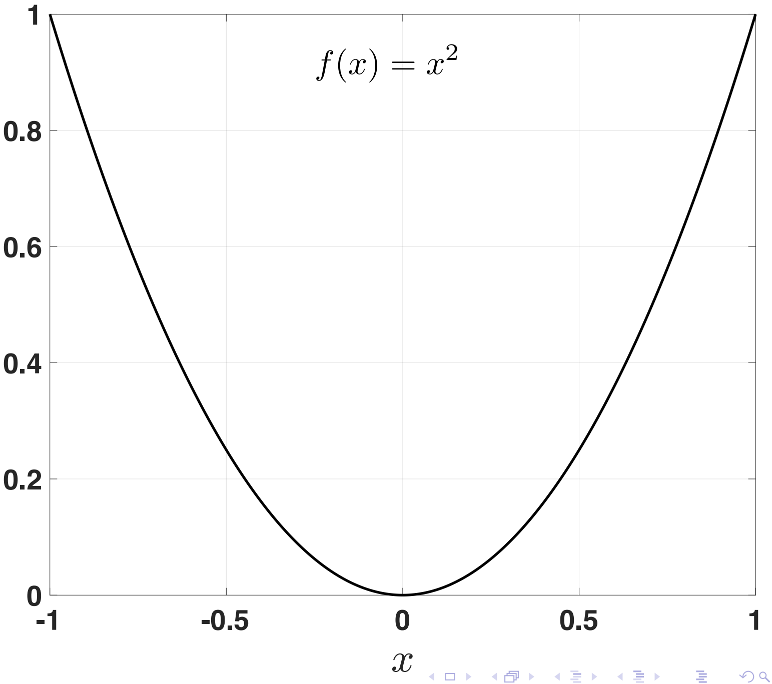
For $h < 0$ (small) the sign changes accordingly

$$\frac{f(M+h) - f(M)}{h} \geq 0 \Rightarrow f'\left(M^-\right) \geq 0 \quad (\square).$$

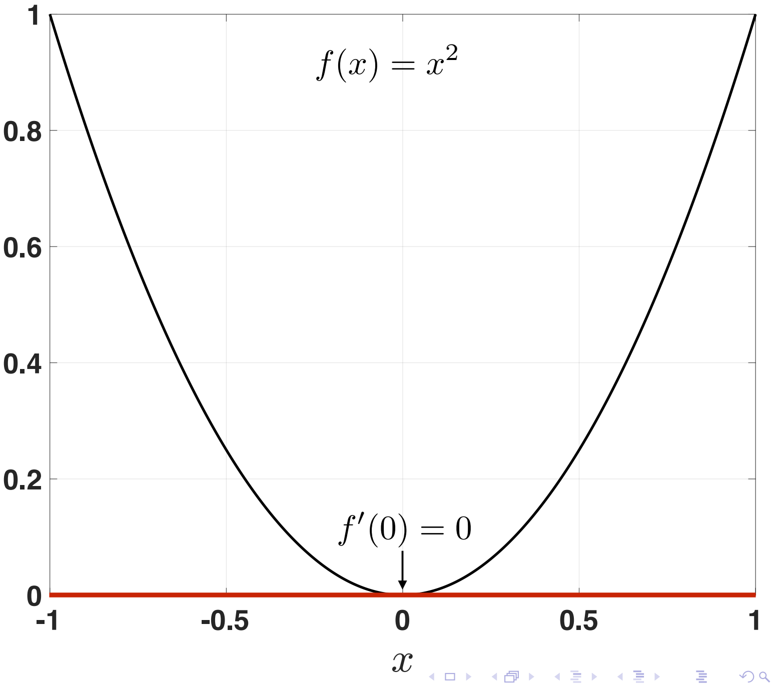
Since f is differentiable we have that

$$f'\left(M^-\right) = f'\left(M^+\right) = f'(M),$$

which combined with $(\triangle)$ and $(\square)$ gives $f'(M) = 0$. $\square$



$f(x) = x^2$
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
x


$f(x) = x^2$
$f'(0) = 0$
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
x

Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). There exists a point $x_0 \in (a, b)$ such that

$$f(b) - f(a) = (b - a)\, f'(x_0).$$

Proof. Consider the function $g(x) = f(x) - \alpha x$. Find α such that

$$\begin{aligned} g(a) = g(b) \quad &\Leftrightarrow \quad f(a) - \alpha a = f(b) - \alpha b \Leftrightarrow \alpha (b - a) = f(b) - f(a) \\ &\Leftrightarrow \quad \alpha = \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

g, as f, is continuous in $[a, b]$ and differentiable in (a, b) and $g(a) = g(b)$ $\Rightarrow$ Rolle's Theorem on g.

$$\exists x_0 \in (a, b) : g'(x_0) = 0 = f'(x_0) - \alpha \Rightarrow f'(x_0) = \frac{f(b) - f(a)}{b - a}. \qquad \square$$

Exercize

Use Lagrange's theorem to prove that

$$\ln(x) \leq x, \quad \forall x \in (0, \infty).$$

Solution.

- If $x \in (0,1)$ we have $\ln(x) < 0 < x$, so the inequality is obvious.
- If $x = 1$ the inequality is $0 \leq 1$, which is true.
- If $x > 1$ Lagrange's theorem on the interval $[1, x]$ gives

$$\frac{\ln(x) - \ln(1)}{x - 1} = \frac{1}{c}$$

with $c \in (1, x)$ and where we have used $(\ln(x))' = 1/x$. In particular $c > 1$. Hence we get

$$\ln(x) = \frac{x-1}{c} = \frac{x}{c} - \frac{1}{c} < \frac{x}{c} < x$$

where the last inequality follows exactly from $c > 1$.

Exercize

Use Lagrange's theorem to prove that

$$\sin(x) < x, \quad \forall x > 0.$$

Solution.

- If $x > \frac{\pi}{2}$ Then $x > 1$ and so $\sin(x) \leq 1 < \frac{\pi}{2} < x$.
- If $x \in \left(0, \frac{\pi}{2}\right)$ apply Lagrange's theorem on the interval $[0, x]$

$$\frac{\sin(x) - \sin(0)}{x - 0} = \cos(c)$$

with $c \in (0, x)$. Since $0 < c < x$ we also have that $0 < c < \frac{\pi}{2}$ and hence $0 < \cos(c) < 1$, whence

$$0 < \frac{\sin(x)}{x} = \cos(c) < 1 \Rightarrow \sin(x) < x$$

Exercize

Let $f(t)$ be the GDP of a country. Assume also that $f(0) = 0$ and that f verifies the hypotheses of the Lagrange's theorem.

Assume that

$$f'(t) \leq 7\% \quad \forall t \geq 0.$$

Which is the maximum GDP at time t ?

Solution. Using the Lagrange's theorem on $[0, t]$ applied to the function f we know that

$$\frac{f(t) - f(0)}{t - 0} = f'(c) \leq 7\%$$

whence $f(t) \leq 7\% \, t$.

Exercize

Does there exist a continuous and differentiable function $f(x)$ such that $f(0) = -1$ and $f(2) = 4$ and $f'(x) \leq 2$ for all x?

Solution. Since such a function would also verify the hypotheses of the Lagrange's theorem we would have

$$\frac{f(2) - f(0)}{2 - 0} = f'(c)$$

with $c \in (0, 2)$. This would imply

$$\frac{4 + 1}{2} = f'(c) \leq 2,$$

which is impossible.

Exercize

Let $f : [0,1] \to \mathbb{R}$ *be a function continuous on* $[0,1]$ *and differentiable in* $(0,1)$ *such that* $f(0) = 0$ *and* $|f'(x)| \le |f(x)|$ *for all* $x \in (0,1)$.
Prove that $f(x) = 0$ *for all* $x \in [0,1]$.

Solution. Fix $x \in (0,1)$. By Lagrange $\exists x_1$ such that $0 < x_1 < x$ and $f(x) = f'(x_1)\, x$, whence

$$|f(x)| = x\,|f'(x_1)| \le x\,|f(x_1)|.$$

By Lagrange $\exists x_2$ such that $0 < x_2 < x_1$ and $f(x_1) = f'(x_2)\, x_1$, whence (using $x_1 < x$ and $|f'(x_2)| \le f(x_2)$)

$$|f(x)| \le x\,|f(x_1)| = x\,x_1\,|f'(x_2)| \le x\,x\,|f(x_2)| = x^2 f(x_2).$$

By iterating we get a sequence $x_n \in (0,1)$ such that

$$|f(x)| \le x^n\,|f(x_n)|.$$

By the Weiestrass theorem $|f(x)|$ is bounded in $[0,1]$, whence $x^n\,|f(x_n)| \to 0$.

Theorem

Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be cont. in $[a, b]$ and diff. in (a, b). Then there exists a $x_0 \in (a, b)$ such that

$$(f(b) - f(a))\, g'(x_0) = (g(b) - g(a))\, f'(x_0).$$

Proof. Suppose first that $g(b) \neq g(a)$. Define $h(x) = f(x) - \alpha\, g(x)$ and find the α such that

$$\begin{aligned} h(a) = h(b) \quad &\Leftrightarrow \quad f(a) - \alpha\, g(a) = f(b) - \alpha\, g(b) \\ &\Leftrightarrow \quad \alpha\,(g(b) - g(a)) = f(b) - f(a) \Leftrightarrow \alpha = \frac{f(b) - f(a)}{g(b) - g(a)}. \end{aligned}$$

h, as f and g, is continuous in $[a, b]$ and differentiable in (a, b). Apply Rolle's Theorem to h obtaining

$$\exists x_0 \in (a, b) : h'(\xi) = 0 = f'(x_0) - \frac{f(b) - f(a)}{g(b) - g(a)}\, g'(x_0)$$

whence the thesis.

Theorem

Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be cont. in $[a, b]$ and diff. in (a, b). Then there exists a $x_0 \in (a, b)$ such that

$$(f(b) - f(a))\, g'(x_0) = (g(b) - g(a))\, f'(x_0).$$

Proof. Suppose now that $g(b) = g(a)$. Apply Rolle's Theorem to g obtaining

$$\exists x_0 \in (a, b) : g'(x_0) = 0.$$

whence the claimed identity $(0 = 0)$ is verified.

Exercize

Use Cauchy's Theorem to prove that

$$0 < 1 - \cos(x) < \frac{x^2}{2}, \quad \forall x > 0.$$

Solution. Apply Cauchy's theorem to

$$f(x) = 1 - \cos(x), \quad g(x) = \frac{x^2}{2}.$$

on the interval $[0, x]$.

$$\exists x_0 \in (0, x) : \frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f'(x_0)}{g'(x_0)} \Rightarrow \frac{1 - \cos(x)}{\frac{x^2}{2}} = \frac{\sin(x_0)}{x_0}.$$

Nevertheless we have proved that $\frac{\sin(x)}{x} < 1$ for $x > 0$, whence

$$0 < \frac{1 - \cos(x)}{\frac{x^2}{2}} < 1 \Rightarrow 1 - \cos(x) < \frac{x^2}{2}.$$

Theorem

Let $f : [a, b] \to \mathbb{R}$ be differentiable in (a, b), then f' cannot have any jump discontinuity on (a, b).

Proof. Let $x_0 \in (a, b)$. We know that the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

exists and is finite. Suppose that the limits $\lim_{x \to x_0^+} f'(x) = A$ and $\lim_{x \to x_0^-} f'(x) = B$ exist and are finite. If $x > x_0$ we have

$$\exists x_1 \in (x_0, x) : \frac{f(x) - f(x_0)}{x - x_0} = f'(x_1).$$

As $x \to x_0^+$ also $x_1 \to x_0^+$ then

$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x_1 \to x_0^+} f'(x_1) = A$$

Similarly by considering $x \to x_0^-$ we can show that $B = f'(x_0)$ and then $A = B$.

Derivatives

Remark

The derivative of a function could, however, have other types of discontinuities.

Example

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

f is continuous everywhere since $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$. If $x \neq 0$ the derivative is

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Nevertheless since

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0,$$

the function is differentiable in $x_0 = 0$ and $f'(0) = 0$. Nevertheless

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \left[2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right] = \nexists$$

whence f' has an essential discontinuity in $x_0 = 0$.

Definition

Let $D \subseteq \mathbb{R} \to \mathbb{R}$ be a function and let $I \subset D$ be an open interval $I = (a, b)$, subset of the domain. We say that the function f is strictly increasing in I if

$$\forall x_1, x_2 \in I : x_1 < x_2 \Rightarrow f(x_1) < f(x_2),$$

we say that the function f is increasing in I if

$$\forall x_1, x_2 \in I : x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2).$$

Theorem

Let $f : [a, b] \to \mathbb{R}$ be differentiable in (a, b).
If $f'(x) \geq 0$ (resp. $f'(x) \leq 0$) for all x in (a, b) then f is increasing (resp. decreasing) in (a, b).

Proof. Consider $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Mean value theorem implies

$$\exists x_0 \in (x_1, x_2) : f(x_2) - f(x_1) = f'(x_0)(x_2 - x_1).$$

If $f'(x_0) \geq 0$ it follows that $f(x_1) \leq f(x_2)$, that is f is increasing.

If $f'(x_0) \leq 0$ it follows that $f(x_2) \leq f(x_1)$, that is f is decreasing.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be differentiable in (a, b).
If f is increasing (resp. decreasing) in (a, b) then $f'(x) \geq 0$ (resp. $f'(x) \leq 0$) for all x in (a, b).

Proof. Assume that f is increasing and let x_0 be in (a, b). For $h > 0$ we have $x_0 \leq x_0 + h$, whence

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0 \Rightarrow f'\left(x_0^+\right) \geq 0.$$

Similarly, for $h < 0$ we have $h = -|h|$ and hence

$$\frac{f(x_0 - |h|) - f(x_0)}{-|h|} = \frac{f(x_0) - f(x_0 - |h|)}{|h|} \geq 0 \Rightarrow f'\left(x_0^-\right) \geq 0$$

whence $f'(x_0) \geq 0$ (the case f decreasing is identical).

Definition

Let $f : [a, b] \to \mathbb{R}$ be a function.

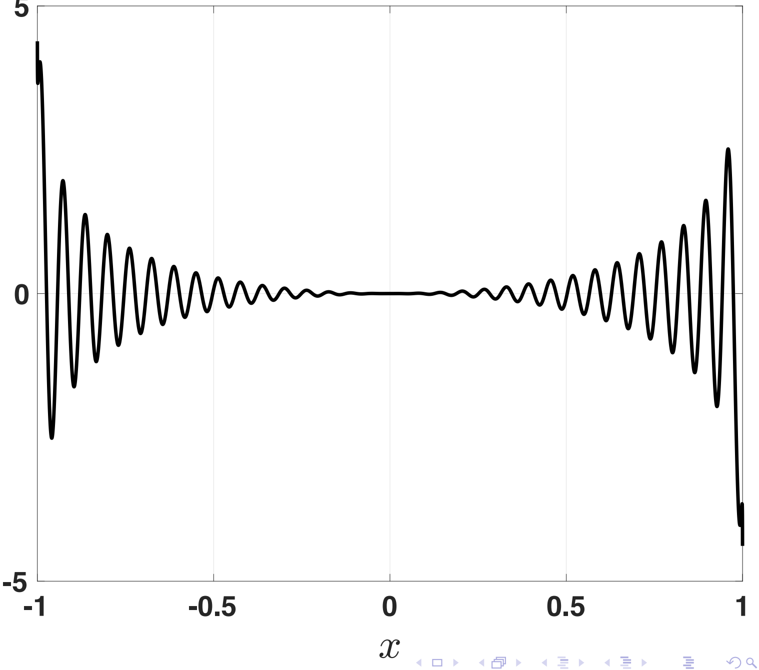
A point $x_0 \in (a, b)$ is a local minimum if there exists a sufficiently small $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subseteq (a, b) \text{ and } f(x) \geq f(x_0), \forall x \in (x_0 - \delta, x_0 + \delta).$$

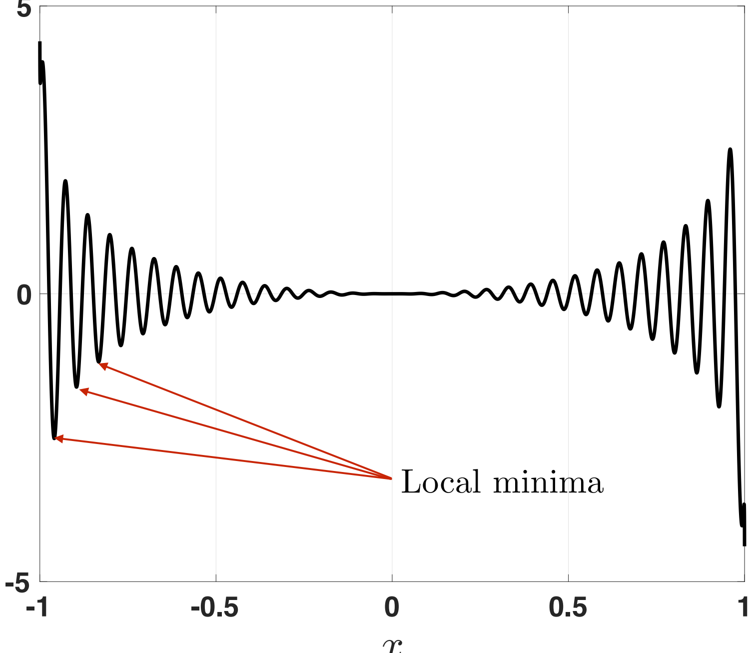
A point $x_0 \in (a, b)$ is a local maximum if there exists a sufficiently small $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subseteq (a, b) \text{ and } f(x) \leq f(x_0), \forall x \in (x_0 - \delta, x_0 + \delta).$$

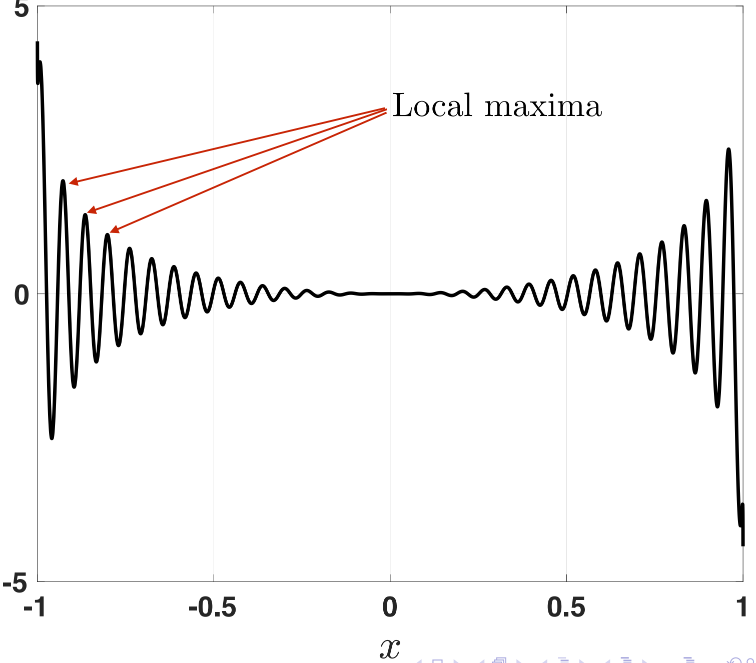
$$f(x) = \sin(100 * x)\ \ln(1 - x^2)$$



$f(x) = \sin(100 * x)\ \ln(1 - x^2)$



$$f(x) = \sin(100 * x)\ \ln(1 - x^2)$$



Theorem

Let $f:[a,b]\to\mathbb{R}$ and $x_0\in(a,b)$. If f attains a local min or max in x_0 then either

$$f'(x_0)=0$$

or

$$\nexists f'(x_0).$$

Proof. If x_0 is local min. then for suff. small h we have $f(x_0+h)\geq f(x_0)$. If f is differentiable in x_0 then

$$\lim_{h\to0^-}\frac{\overbrace{f(x_0+h)-f(x_0)}^{\geq0}}{\underbrace{h}_{<0}}=f\left(x_0^-\right)\leq0,\ \lim_{h\to0^+}\frac{\overbrace{f(x_0+h)-fx_0}^{\geq0}}{\underbrace{h}_{>0}}=f\left(x_0^+\right)\geq0$$

but since $\exists f'(x_0)$ then $f'(x_0)=0$. The only option left is that $\nexists f'(x_0)$.

Fermat's Theorem on local extrema

Theorem

Let $f:[a,b]\to\mathbb{R}$ and $x_0\in(a,b)$. If f attains a local min or max in x_0 then either

$$f'(x_0)=0$$

or

$$\nexists f'(x_0).$$

Proof. If x_0 is local max. then for suff. small h we have $f(x_0+h)\le f(x_0)$. If f is differentiable in x_0 then

$$\lim_{h\to 0^-}\frac{\overbrace{f(x_0+h)-f(x_0)}^{\le 0}}{\underbrace{h}_{<0}}=f\left(x_0^-\right)\ge 0,\ \lim_{h\to 0^+}\frac{\overbrace{f(x_0+h)-f x_0}^{\le 0}}{\underbrace{h}_{>0}}=f\left(x_0^+\right)\le 0$$

but since $\exists f'(x_0)$ then $f'(x_0)=0$. The only option left is that $\nexists f'(x_0)$.

Remark

The converse of the Fermat's Theorem on local extrema is not true!

$f'(x_0) = 0$ is a **necessary but not sufficient** condition to have a local max/min in x_0.

As always we have to find a counter-example ...

$$f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f'(0) = 0.$$

Nevertheless for $x < 0$ trivially $x^3 < 0$ and for $x > 0$ trivially $x^3 > 0$.

Definition

Let $f : [a, b] \to \mathbb{R}$ be differentiable in x_0. If $f'(x_0) = 0$ the point x_0 is called a critical point.

Exercize

Find all the critical points of $f(x) = x^x : (0, +\infty) \to (0, +\infty)$.

Solution. Define

$$g(x) = \ln(f(x)) = \ln(x^x) = x \ln(x).$$

Whence

$$g'(x) = (x \ln(x))' = (x)' \ln(x) + x (\ln(x))' = 1 \cdot \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1.$$

but also

$$g'(x) = (\ln(f(x)))' = \frac{1}{f(x)} f'(x).$$

So that

$$\frac{1}{f(x)} f'(x) = \ln(x) + 1 \Rightarrow f'(x) = f(x) (\ln(x) + 1) = x^x (\ln(x) + 1)$$

$$f'(x) = 0 \Leftrightarrow x^x (\ln(x) + 1) = 0 \Leftrightarrow \ln(x) + 1 = 0 \Leftrightarrow x = e^{-1},$$

which is thus the unique critical point.

Exercize

Establish if the critical points of $f(x) = x^x : (0,+\infty) \to (0,+\infty)$ *are local min or local max or neither local min nor local max.*

Solution. Since

$$f'(x) = (x^x)' = \underbrace{x^x}_{>0} (\ln(x) + 1)$$

then

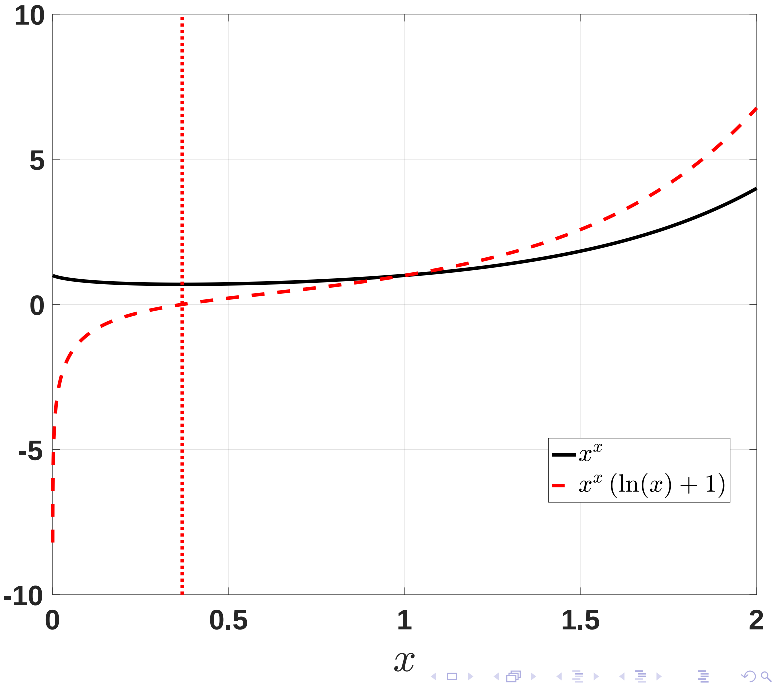
$$\operatorname{sign}\left(f'(x)\right) = \operatorname{sign}(\ln(x) + 1).$$

Whence

$$f'(x) > 0 \Leftrightarrow \ln(x) + 1 > 0 \Leftrightarrow \ln(x) > -1 \Leftrightarrow x > e^{-1} \text{ Increasing}$$

$$f'(x) < 0 \Leftrightarrow \ln(x) + 1 < 0 \Leftrightarrow \ln(x) < -1 \Leftrightarrow x < e^{-1} \text{ Decreasing}$$

$x = e^{-1}$ is a local minimum.



10
5
0
-5
-10
0
0.5
1
1.5
2
x
x^x
$x^x\,(\ln(x)+1)$

Exercize

Find a function f such that $f(x_0)$ is a local min. or local max., while $f'(x_0)$ does not exist.

Solution. Consider

$$f(x) = |x| \geq 0 = f(0) \quad \forall x \in \mathbb{R}.$$

Global minimum in $x = 0$, nevertheless:

$$\begin{aligned} f'(0^+) &= \lim_{h \to 0^+} \frac{|0+h|-0}{h} = 1 \\ f'(0^-) &= \lim_{h \to 0^+} \frac{|0+h|-0}{h} = -1, \end{aligned}$$

i.e. $f'(0)$ does not exist.

Exercize

Find all the critical points of $f(x) = x^{\frac{1}{x}} : (0, +\infty) \to (0, +\infty)$.

Solution. Define

$$g(x) = \ln(f(x)) = \ln\left(x^{\frac{1}{x}}\right) = \frac{1}{x}\ln(x).$$

Whence

$$g'(x) = \left(\frac{\ln(x)}{x}\right)' = \left(\frac{1}{x}\right)' \ln(x) + \frac{1}{x}(\ln(x))' = -\frac{\ln(x)}{x^2} + \frac{1}{x}\frac{1}{x} = \frac{1-\ln(x)}{x^2}.$$

but also

$$g'(x) = (\ln(f(x)))' = \frac{1}{f(x)} f'(x).$$

So that

$$\frac{1}{f(x)} f'(x) = \frac{1-\ln(x)}{x^2} \Rightarrow f'(x) = f(x)\frac{1-\ln(x)}{x^2} = x^{\frac{1}{x}}\frac{1-\ln(x)}{x^2}$$

$$f'(x) = 0 \Leftrightarrow x^{\frac{1}{x}}\frac{1-\ln(x)}{x^2} = 0 \Leftrightarrow \ln(x) = 1 \Leftrightarrow x = e,$$

which is thus the unique critical point.

Exercize

Establish if the critical points of $f(x) = x^{\frac{1}{x}} : (0, +\infty) \to (0, +\infty)$ *are local min or local max or neither local min nor local local max.*

Solution. Since

$$f'(x) = \left(x^{\frac{1}{x}}\right)' = \underbrace{x^{\frac{1}{x}}}_{>0} \left(\frac{1 - \ln(x)}{x^2}\right)$$

then

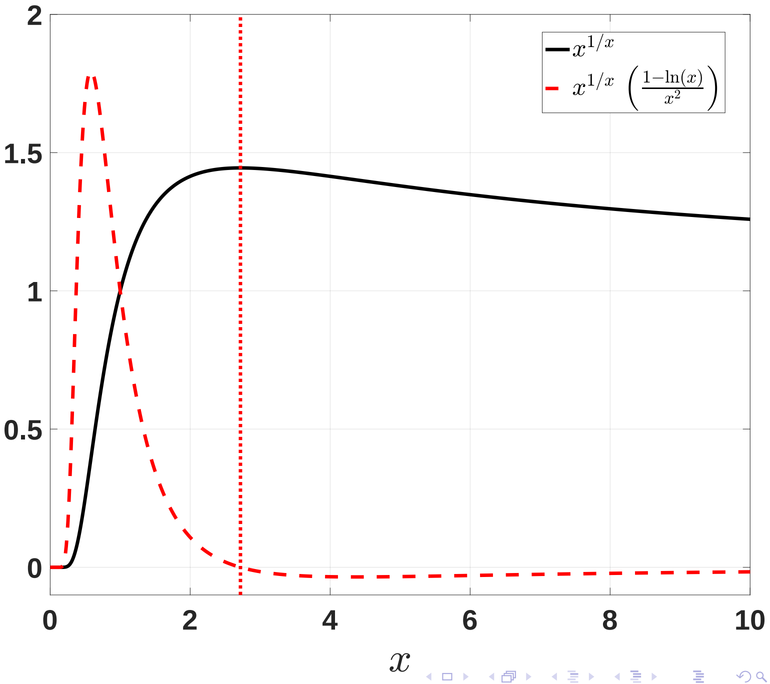
$$\mathrm{sign}\left(f'(x)\right) = \mathrm{sign}\left(\frac{1 - \ln(x)}{x^2}\right).$$

Whence

$$f'(x) > 0 \Leftrightarrow \frac{1 - \ln(x)}{x^2} > 0 \Leftrightarrow \ln(x) < 1 \Leftrightarrow x < e \text{ Increasing}$$

$$f'(x) < 0 \Leftrightarrow \frac{1 - \ln(x)}{x^2} < 0 \Leftrightarrow \ln(x) > 1 \Leftrightarrow x > e \text{ Decreasing}$$

$x = e$ is a local maximum.



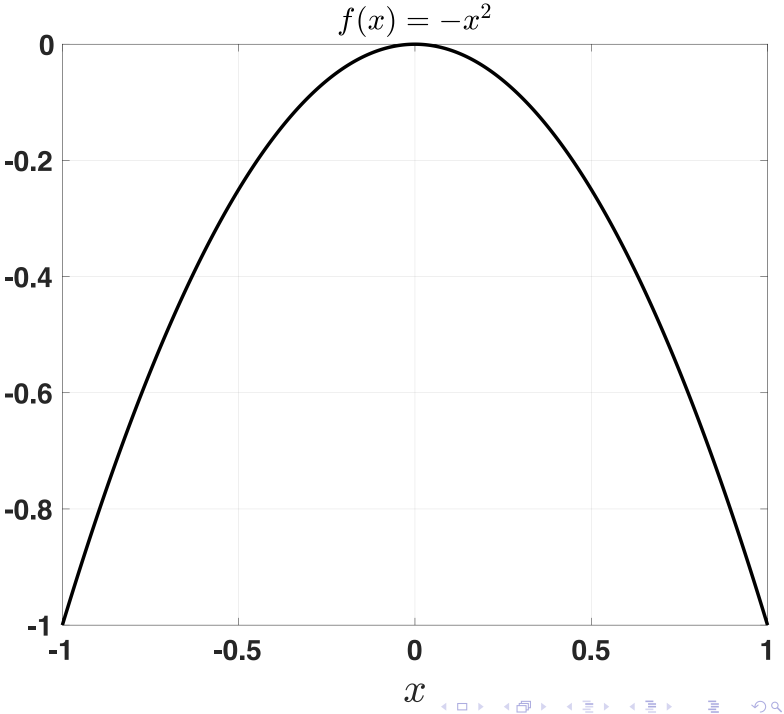
$x^{1/x}$
$x^{1/x}\left(\frac{1-\ln(x)}{x^2}\right)$
2
1.5
1
0.5
0
0
2
4
6
8
10
x

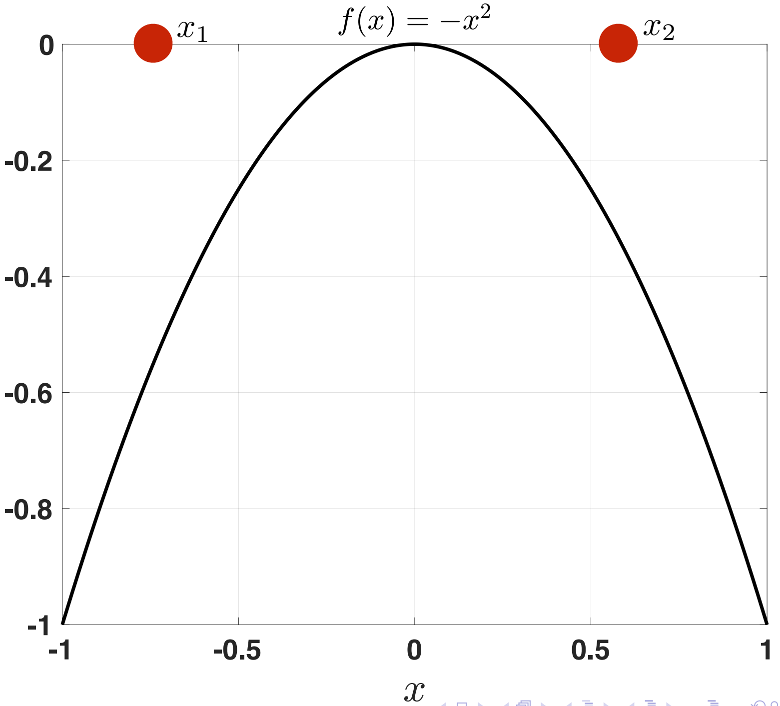
Definition

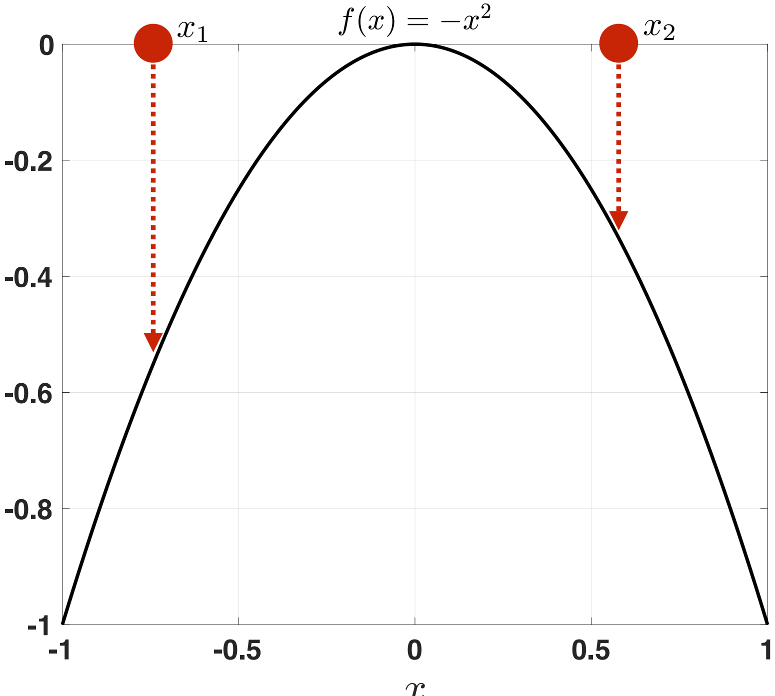
A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is said to be concave in D if for all x_1 and x_2 in D it holds that

$$f\left(\left(1-\alpha\right)\,x_1+\alpha\,x_2\right) \geq \left(1-\alpha\right)\,f\left(x_1\right)+\alpha\,f\left(x_2\right),\forall\alpha\in[0,1]$$

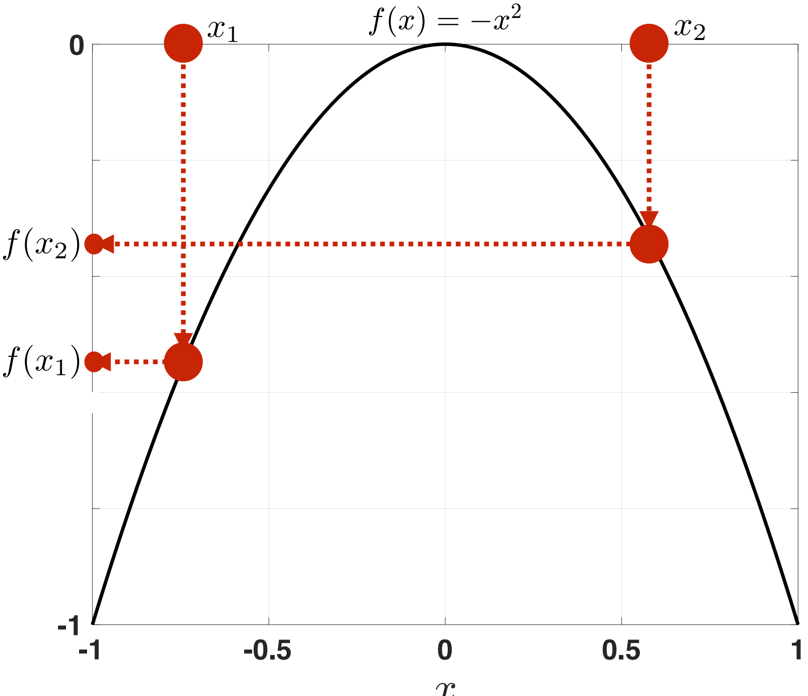
i.e. if the graph of the function is above the segment that joins $(x_1, f(x_1))$ with $(x_2, f(x_2))$.


$f(x) = -x^2$
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
x

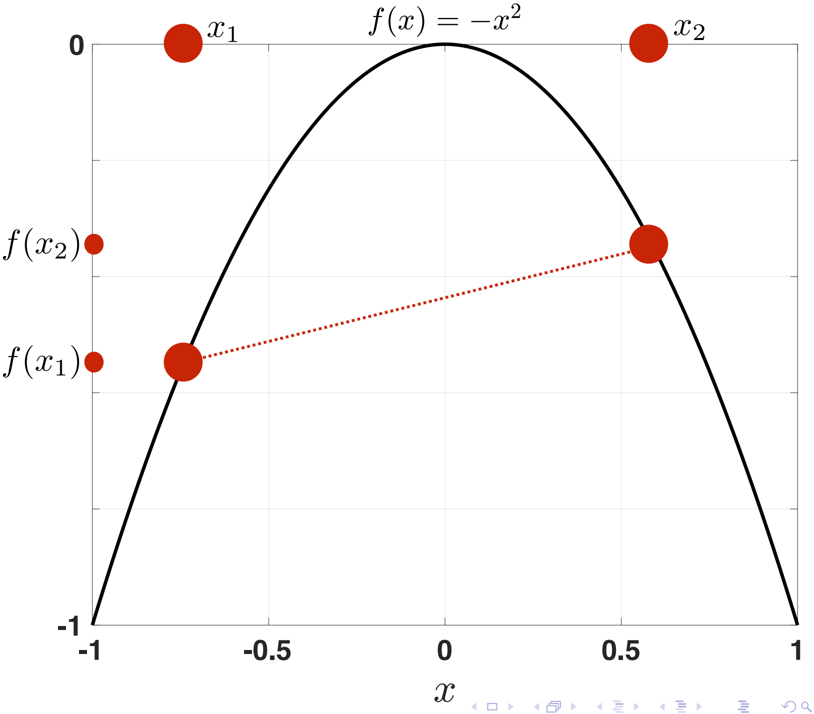

$f(x) = -x^2$
x_1
x_2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
x



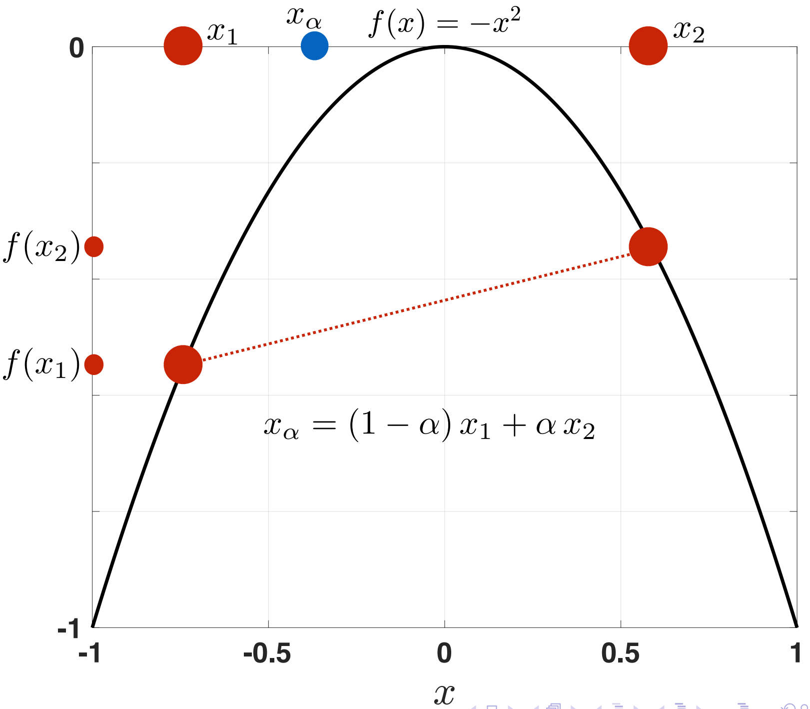
$f(x) = -x^2$
x_1
x_2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.5
0
0.5
1
x

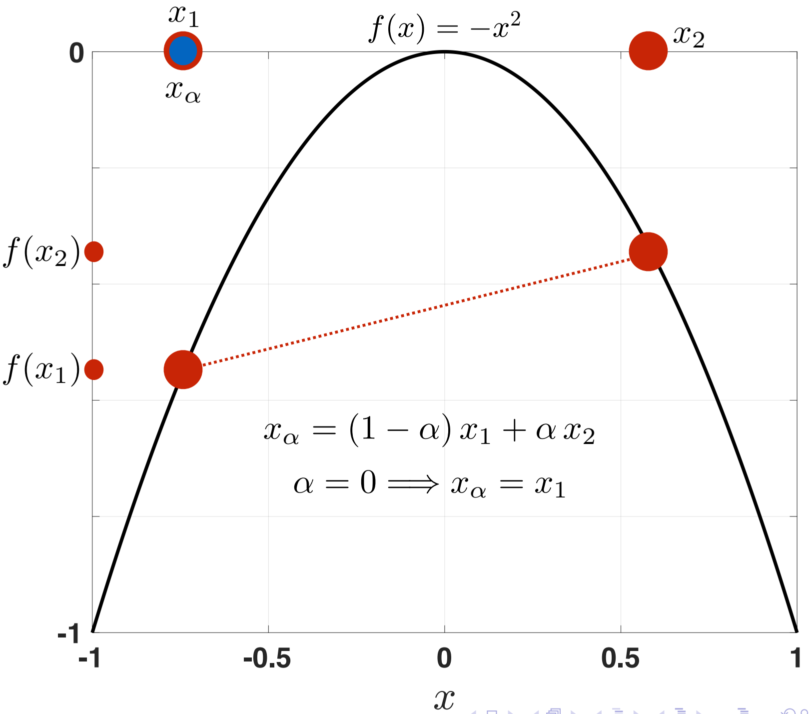


$f(x) = -x^2$
x_1
x_2
0
-1
$f(x_2)$
$f(x_1)$
-1
-0.5
0
0.5
1
x

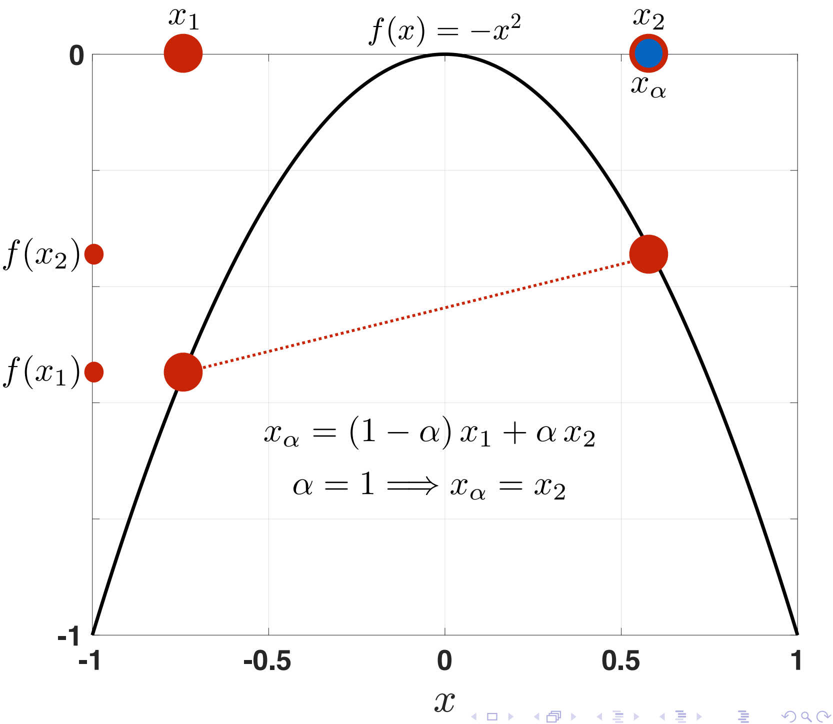


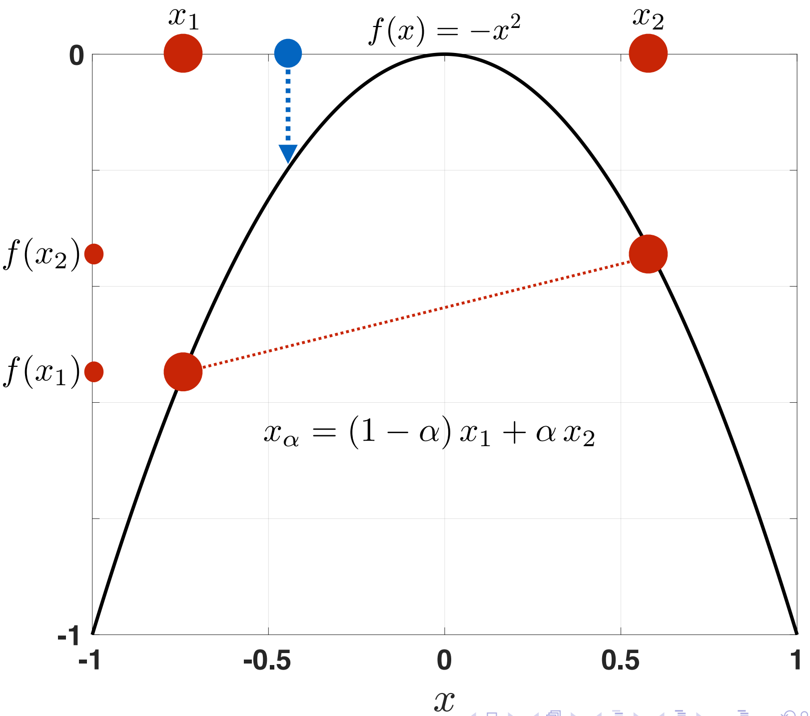
$f(x) = -x^2$
x_1
x_2
0
$f(x_2)$
$f(x_1)$
-1
-1
-0.5
0
0.5
1
x


x_1
x_α
$f(x) = -x^2$
x_2
0
$f(x_2)$
$f(x_1)$
$x_\alpha = (1-\alpha)\, x_1 + \alpha\, x_2$
-1
-1
-0.5
0
0.5
1
x

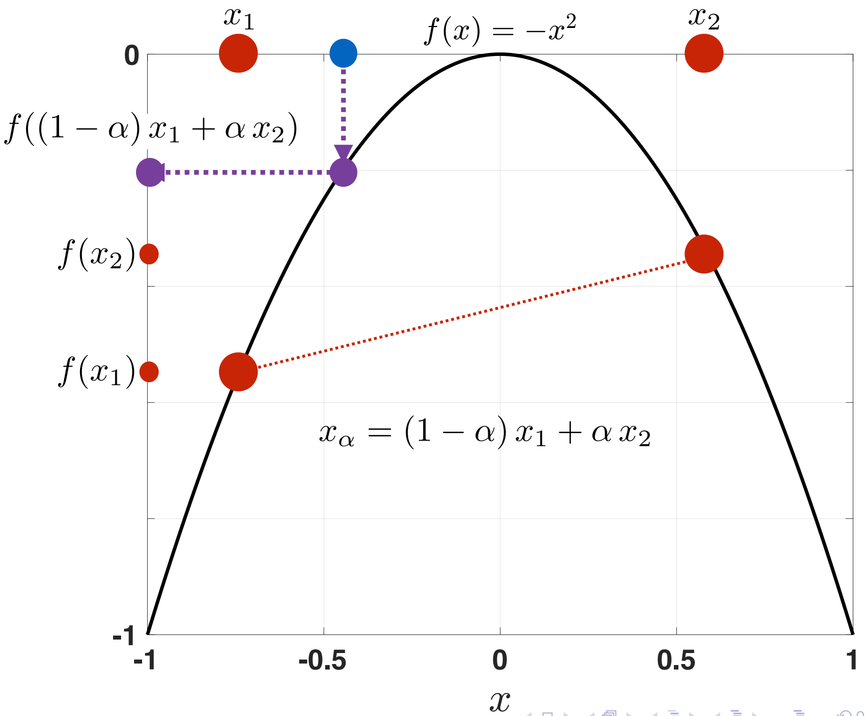


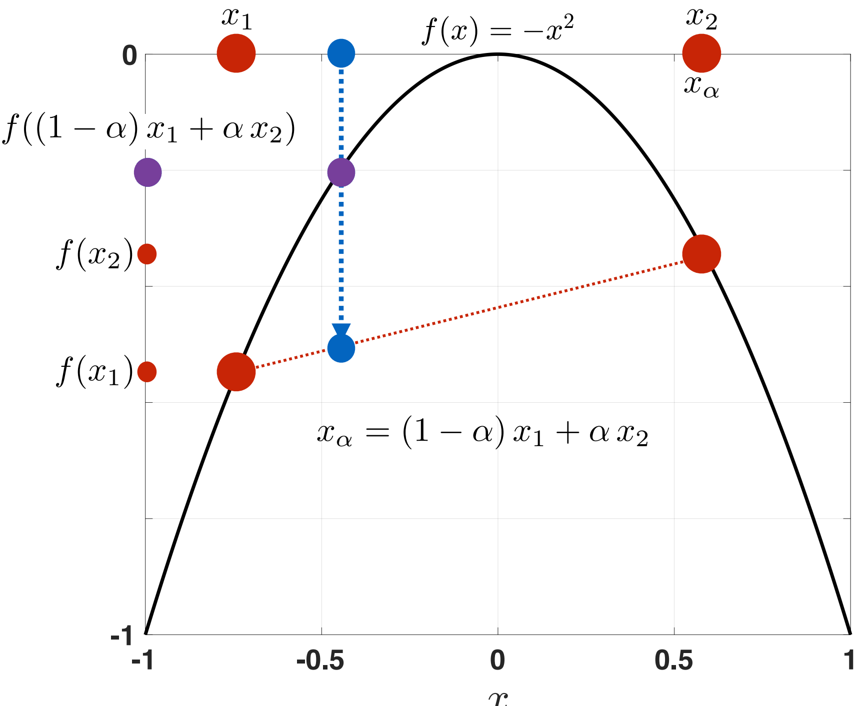
x_1
$f(x) = -x^2$
x_2
0
x_α
$f(x_2)$
$f(x_1)$
$x_\alpha = (1-\alpha)\, x_1 + \alpha\, x_2$
$\alpha = 0 \Longrightarrow x_\alpha = x_1$
-1
-1
-0.5
0
0.5
1
x


x_1
$f(x) = -x^2$
x_2
x_α
0
$f(x_2)$
$f(x_1)$
$x_\alpha = (1-\alpha)\, x_1 + \alpha\, x_2$
$\alpha = 1 \Longrightarrow x_\alpha = x_2$
-1
-1
-0.5
0
0.5
1
x

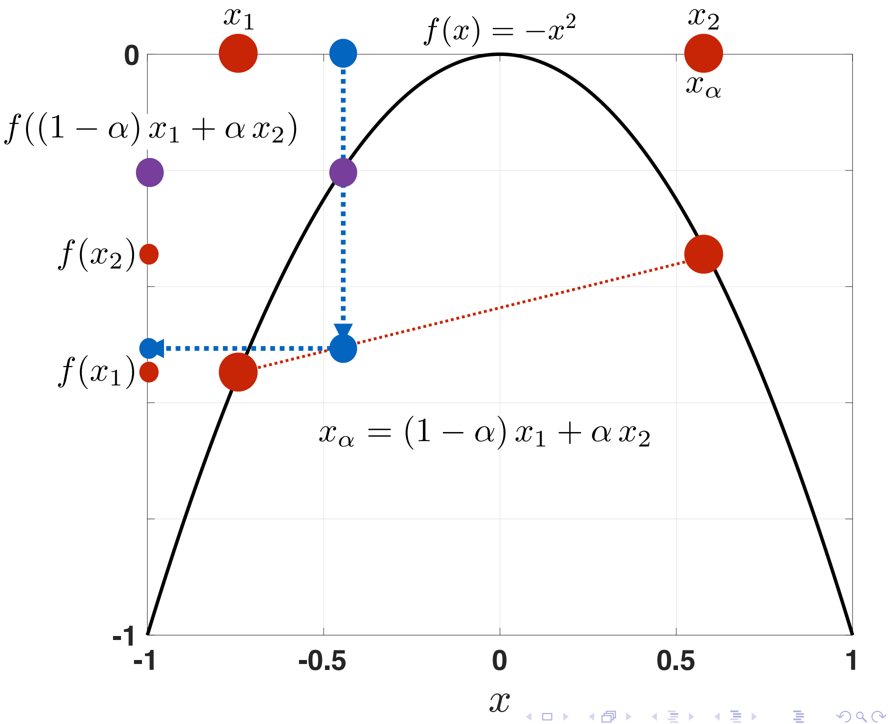


x_1
x_2
$f(x) = -x^2$
0
$f(x_2)$
$f(x_1)$
$x_\alpha = (1-\alpha)\,x_1 + \alpha\,x_2$
-1
-1
-0.5
0
0.5
1
x


x_1
x_2
$f(x) = -x^2$
0
$f((1-\alpha)\,x_1 + \alpha\,x_2)$
$f(x_2)$
$f(x_1)$
$x_\alpha = (1-\alpha)\,x_1 + \alpha\,x_2$
-1
-1
-0.5
0
0.5
1
x



x_1
x_2
$f(x) = -x^2$
x_α
$f((1-\alpha)\,x_1 + \alpha\,x_2)$
$f(x_2)$
$f(x_1)$
$x_\alpha = (1-\alpha)\,x_1 + \alpha\,x_2$
0
-1
-1
-0.5
0
0.5
1
x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0
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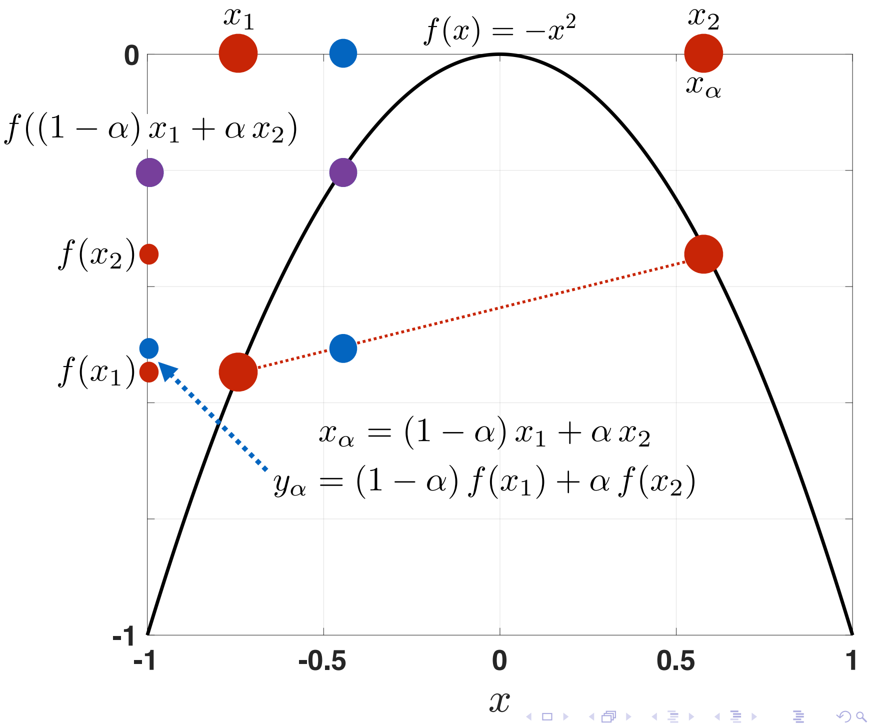
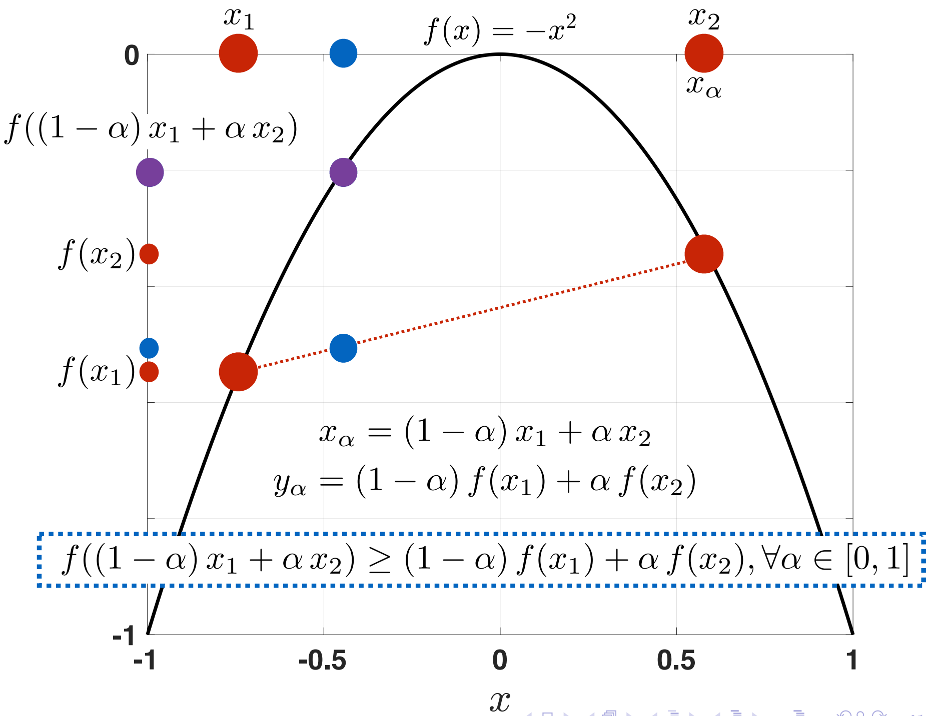
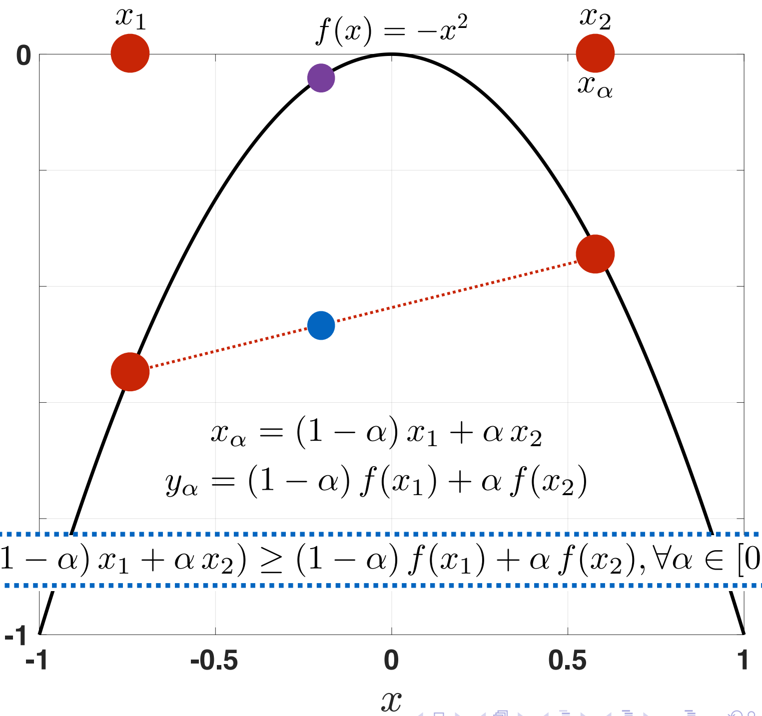
x

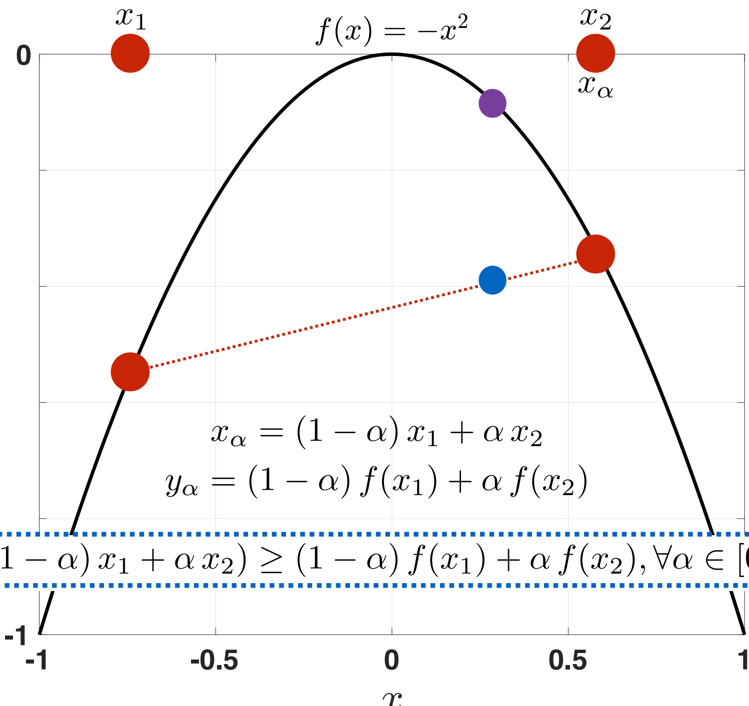


x_1
x_2
$f(x) = -x^2$
x_α
$f((1-\alpha)\,x_1 + \alpha\,x_2)$
$f(x_2)$
$f(x_1)$
$x_\alpha = (1-\alpha)\,x_1 + \alpha\,x_2$
$y_\alpha = (1-\alpha)\,f(x_1) + \alpha\,f(x_2)$
0
-1
-1
-0.5
0
0.5
1
x


x_1
$f(x) = -x^2$
x_2
x_α
0
$f((1-\alpha)\,x_1 + \alpha\,x_2)$
$f(x_2)$
$f(x_1)$
$x_\alpha = (1-\alpha)\,x_1 + \alpha\,x_2$
$y_\alpha = (1-\alpha)\,f(x_1) + \alpha\,f(x_2)$
$f((1-\alpha)\,x_1 + \alpha\,x_2) \geq (1-\alpha)\,f(x_1) + \alpha\,f(x_2), \forall \alpha \in [0,1]$
-1
-1
-0.5
0
0.5
1
x



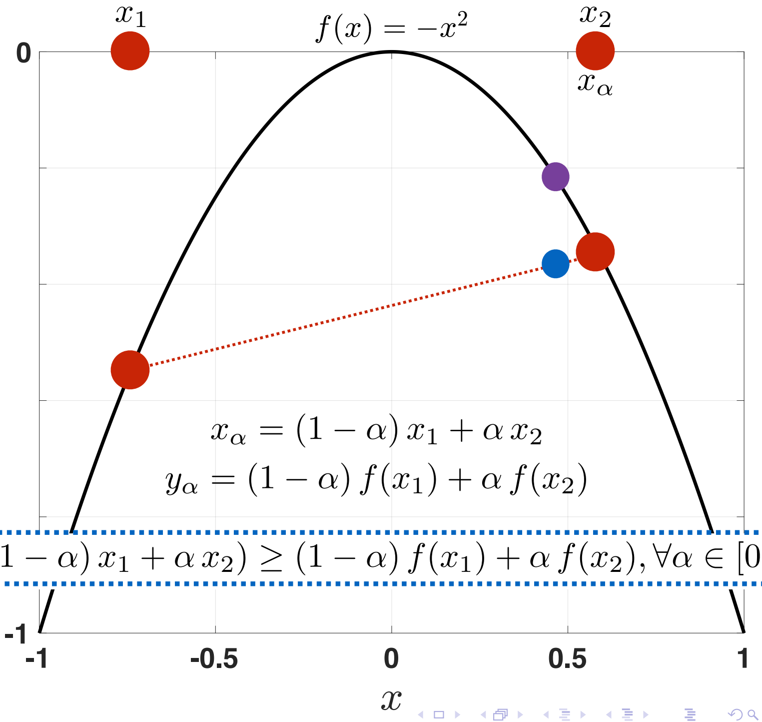
$$\boxed{f((1-\alpha)\, x_1 + \alpha\, x_2) \geq (1-\alpha)\, f(x_1) + \alpha\, f(x_2), \forall \alpha \in [0,1]}$$



$$x_\alpha = (1-\alpha)\, x_1 + \alpha\, x_2$$

$$y_\alpha = (1-\alpha)\, f(x_1) + \alpha\, f(x_2)$$

$$\boxed{f((1-\alpha)\, x_1 + \alpha\, x_2) \geq (1-\alpha)\, f(x_1) + \alpha\, f(x_2), \forall \alpha \in [0,1]}$$



$$x_\alpha = (1-\alpha)\,x_1 + \alpha\,x_2$$

$$y_\alpha = (1-\alpha)\,f(x_1) + \alpha\,f(x_2)$$

$$\boxed{f((1-\alpha)\,x_1 + \alpha\,x_2) \geq (1-\alpha)\,f(x_1) + \alpha\,f(x_2), \forall \alpha \in [0,1]}$$

Definition

A function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex in D if for all x_1 and x_2 in D it holds that

$$f\left((1-\alpha)\, x_1 + \alpha\, x_2\right) \leq (1-\alpha)\, f\left(x_1\right) + \alpha\, f\left(x_2\right), \forall \alpha \in [0,1]$$

i.e. if the graph of the function is above the segment that joins $(x_1, f(x_1))$ with $(x_2, f(x_2))$.

Theorem

A function f is convex on D if and only if

$$\forall x_1, x_2, x_3 \in D : x_1 < x_2 < x_3 \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

or, equivalently, if and only if

$$\forall x_1, x_2, x_3 \in D : x_1 < x_2 < x_3 \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}.$$

Proof. By definition of convexity

$$\forall x, y \in D, \forall \alpha \in [0,1] \Rightarrow f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) \quad (\triangle).$$

Since $x_3 - x_2 < x_3 - x_1$ define $\alpha = \frac{x_3 - x_2}{x_3 - x_1} \in (0,1)$. Then put α in the definition $(\triangle)$ with $x = x_1$ and $y = x_3$

$$\begin{aligned} f(\alpha x + (1-\alpha) y) &= f\left(\frac{x_3 - x_2}{x_3 - x_1} x_1 + \left(1 - \frac{x_3 - x_2}{x_3 - x_1}\right) x_3\right) \\ &= f\left(\frac{x_3 - x_2}{x_3 - x_1} x_1 + \frac{x_2 - x_1}{x_3 - x_1} x_3\right) \\ &= f\left(\frac{\cancel{x_3 x_1} - x_2 x_1 + x_2 x_3 - \cancel{x_1 x_3}}{x_3 - x_1}\right) = f\left(x_2 \frac{x_3 - x_1}{x_3 - x_1}\right) = f(x_2). \end{aligned}$$

Hence we can say that

$$f(x_2) = f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha)\, f(y) = \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3)$$

Summary:

$$f(x_2) = f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y) = \frac{x_3 - x_2}{x_3 - x_1} f(x_1) + \frac{x_2 - x_1}{x_3 - x_1} f(x_3)$$

which implies

$$\begin{aligned}
& (x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_2 - x_1) f(x_3) \\
\Leftrightarrow \quad & (x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (+x_3 - x_3 + x_2 - x_1) f(x_3) \\
\Leftrightarrow \quad & (x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_3 - x_1 - (x_3 - x_2)) f(x_3) \\
\Leftrightarrow \quad & (x_3 - x_2) f(x_3) + (x_3 - x_1) f(x_2) \leq (x_3 - x_2) f(x_1) + (x_3 - x_1) f(x_3) \\
\Leftrightarrow \quad & (x_3 - x_2) f(x_3) - (x_3 - x_2) f(x_1) \leq (x_3 - x_1) f(x_3) - (x_3 - x_1) f(x_2) \\
\Leftrightarrow \quad & \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.
\end{aligned}$$

Now we do similar computations as before

$$
\begin{aligned}
& (x_3 - x_1)\; f(x_2) \leq (x_3 - x_2)\, f(x_1) + (x_2 - x_1)\; f(x_3) \\
\Leftrightarrow\quad & (x_3 - x_1)\; f(x_2) \leq (x_1 - x_1 + x_3 - x_2)\, f(x_1) + (x_2 - x_1)\; f(x_3) \\
\Leftrightarrow\quad & (x_3 - x_1)\; f(x_2) \leq (x_3 - x_1 - (x_2 - x_1))\, f(x_1) + (x_2 - x_1)\; f(x_3) \\
\Leftrightarrow\quad & (x_3 - x_1)\; f(x_2) - (x_3 - x_1)\; f(x_1) \leq -(x_2 - x_1) f(x_1) + (x_2 - x_1)\; f(x_3) \\
\Leftrightarrow\quad & (x_3 - x_1)\; (f(x_2) - f(x_1)) \leq (x_2 - x_1)\; (f(x_3) - f(x_1)) \\
\Leftrightarrow\quad & \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1},
\end{aligned}
$$

so summing up

$$
\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.
$$

Since we used only double implications $\Leftrightarrow$, the argument reverses throughout. $\square$

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function which is differentiable on (a, b). Then f is convex on (a, b) $\Leftrightarrow$ f' is increasing on (a, b).

Proof. $\Rightarrow$. Consider four points $a < x_1 < x_2 < x_3 < x_4 < b$. By the property of convex functions (used two times)

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2} \le \frac{f(x_4) - f(x_3)}{x_4 - x_3}.$$

Now let $x_2 \to x_1^+$ and $x_3 \to x_4^-$ obtaining (since f is differentiable!)

$$f'(x_1) \le f'(x_4),$$

the arbitrariness of x_1 and x_4 $\Rightarrow$ f' is increasing on (a, b). $\square$

Theorem

Let $f : [a,b] \to \mathbb{R}$ be a function which is differentiable on (a,b). Then f is convex on (a,b) $\Leftrightarrow$ f' is increasing on (a,b).

Proof. $\Leftarrow$. Consider three points $a < x_1 < x_2 < x_3 < b$. By the mean value theorem

$$\exists \alpha \in (x_1, x_2) : \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\alpha)$$

and

$$\exists \beta \in (x_2, x_3) : \frac{f(x_3) - f(x_2)}{x_3 - x_2} = f'(\beta).$$

Since $\alpha < \beta$ then $f'(\alpha) \leq f'(\beta)$ and hence

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

which is the equivalent condition for convexity.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function which is differentiable on (a, b). Then f is convex on (a, b) $\Leftrightarrow$ f' is increasing on (a, b).

Corollary

Let $f : [a, b] \to \mathbb{R}$ be a function which is **twice** differentiable on (a, b). Then f is convex on (a, b) $\Leftrightarrow$ $f''(x) \geq 0$ for all $x \in (a, b)$.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function which is differentiable on (a, b). Then f is concave on (a, b) $\Leftrightarrow$ f' is decreasing on (a, b).

Corollary

Let $f : [a, b] \to \mathbb{R}$ be a function which is **twice** differentiable on (a, b). Then f is concave on (a, b) $\Leftrightarrow$ $f''(x) \leq 0$ for all $x \in (a, b)$.

Theorem

Let $f : [a, b] \to \mathbb{R}$ be twice differentiable on (a, b) and let $x_0 \in (a, b)$ be such that $f'(x_0) = 0$.

- *If $f''(x_0) > 0$ then x_0 is a local minimum.*
- *If $f''(x_0) < 0$ then x_0 is a local maximum.*

Proof. Assume $f''(x_0) > 0$. Since $f'(x_0) = 0$, we have

$$\lim_{h \to 0^+} \frac{f'(x_0 + h) - f'(x_0)}{h} = \lim_{h \to 0^+} \frac{f'(x_0 + h)}{h} = f''(x_0) > 0$$

Which means that I can find an ε such that, for $h > 0$ and sufficiently small

$$0 < f''(x_0) - \varepsilon < \frac{f'(x_0 + h)}{h} < f''(x_0) + \varepsilon \Rightarrow f'(x_0 + h) > h\left(f''(x_0) - \varepsilon\right) > 0$$

hence the function is increasing in a right neighborhood of x_0. Similarly, for $h < 0$ and sufficiently small we get

$$f'(x_0 + h) < h\left(f''(x_0) - \varepsilon\right) < 0,$$

hence the function is decreasing in a left neighborhood of x_0, and hence x_0 is a local minimum.

Theorem

Let $f : [a,b] \to \mathbb{R}$ be twice differentiable on (a,b) and let $x_0 \in (a,b)$ be such that $f'(x_0) = 0$.

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Which means that I can find an ε such that, for $h > 0$ and sufficiently small

$$f''(x_0) - \varepsilon < \frac{f'(x_0+h)}{h} < f''(x_0) + \varepsilon < 0 \Rightarrow f'(x_0+h) < h\left(f''(x_0) + \varepsilon\right) < 0$$

hence the function is decreasing in a right neighborhood of x_0. Similarly, for $h < 0$ and sufficiently small we get

$$f'(x_0+h) > h\left(f''(x_0) + \varepsilon\right) > 0,$$

hence the function is increasing in a left neighborhood of x_0, and hence x_0 is a local maximum.

Exercize

Find minima/maxima of the following function

$$f(x) = \frac{\ln(x)}{x}$$

Solution. The function is defined in $D = \{x \in \mathbb{R} \mid x > 0\}$. The first derivative is

$$f'(x) = (\ln(x))' \frac{1}{x} + \ln(x) \left(\frac{1}{x}\right)' = \frac{1}{x^2} - \frac{\ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}.$$

Hence $f'(x) = 0$ if and only if $x = e$. Besides since

$$f''(x) = (1 - \ln(x))' \frac{1}{x^2} + (1 - \ln(x)) \left(\frac{1}{x^2}\right)' = -\frac{1}{x^3} - 2\,\frac{1 - \ln(x)}{x^3} = -\frac{1 + 2\,\ln(x)}{x^3},$$

we have that

$$f''(e) = -\frac{3}{e^3} < 0 \Rightarrow x_0 = e \text{ is a local maximum.}$$

Exercize

Find minima/maxima of the following function

$$f(x) = \ln(1 - \ln(x)) - \ln(x).$$

Solution. The domain of the function is determined by the two conditions

$$\begin{cases} x > 0 \\ 1 - \ln(x) > 0 \Rightarrow \ln(x) < 1 \Rightarrow x < e \end{cases},$$

whence $D = (0, e)$. The first derivative is

$$f'(x) = (\ln(1 - \ln(x)))' - (\ln(x))' = \frac{1}{1 - \ln(x)}\left(-\frac{1}{x}\right) - \frac{1}{x} = -\frac{2 - \ln(x)}{x(1 - \ln(x))},$$

hence $f'(x) = 0 \Leftrightarrow x = e^2$. Nevertheless $e^2 \notin D \Rightarrow$ the function has no minimum no maximum in $(0, e)$.

Exercize

Find minima/maxima of the following function

$$f(x) = \frac{\ln(x)}{x}$$

Solution. The domain of the function is $D = (0, +\infty)$. The first derivative is

$$f'(x) = (\ln(x))' \frac{1}{x} + \ln(x) \left(\frac{1}{x}\right)' = \frac{1}{x^2} - \frac{\ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2},$$

hence $f'(x) = 0 \Leftrightarrow x = e$.

$$\begin{aligned} f''(x) &= (1 - \ln(x))' \frac{1}{x^2} + (1 - \ln(x)) \left(\frac{1}{x^2}\right)' = -\frac{1}{x^3} - 2\frac{1 - \ln(x)}{x^3} \\ &= -\frac{3 - 2\ln(x)}{x^3} \Rightarrow f''(e) = -\frac{3 - 2\ln(e)}{x^3} = -\frac{1}{e^3} < 0, \end{aligned} \tag{0.2}$$

whence $x = e$ is a local maximum.

Exercize

Find minima/maxima of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. The domain of the function is determined by the two conditions

$$\begin{cases} x > 0 \\ 1 + \ln(x) > 0 \Rightarrow \ln(x) > -1 \Rightarrow x > e^{-1} = \frac{1}{e} \end{cases},$$

whence $D = \left(\frac{1}{e}, +\infty\right)$.

Exercize

Find minima/maxima of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. $D = \left(\frac{1}{e}, +\infty\right)$. The first derivative is

$$\begin{aligned} f'(x) &= (\ln(1 + \ln(x)))' - (\ln(x))' = \frac{1}{x(1 + \ln(x))} - \frac{1}{x} \\ &= \frac{1 - 1 - \ln(x)}{x(1 + \ln(x))} = -\frac{\ln(x)}{x(1 + \ln(x))}, \end{aligned}$$

hence $f'(x) = 0 \Leftrightarrow x = 1$.

Exercize

Find minima/maxima of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. $D = \left(\frac{1}{e}, +\infty\right)$. $f'(x) = -\frac{\ln(x)}{x\,(1+\ln(x))}$. The second derivative is

$$\begin{aligned}
f''(x) &= -(\ln(x))' \, \frac{1}{x\,(1+\ln(x))} - \ln(x) \left(\frac{1}{x\,(1+\ln(x))}\right)' = \\
&= -\frac{1}{x^2\,(1+\ln(x))} + \frac{\ln(x)}{x^2\,(1+\ln(x))^2}\,(x\,(1+\ln(x)))' \\
&= -\frac{1}{x^2\,(1+\ln(x))} + \frac{\ln(x)}{x^2\,(1+\ln(x))^2}\,(1+\ln(x)+1) \\
&= \frac{-(1+\ln(x)) + \ln(x)\,(2+\ln(x))}{x^2\,(1+\ln(x))^2} = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2\,(1+\ln(x))^2}.
\end{aligned}$$

Exercize

Find minima/maxima of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. Summary

$$D = \left(\frac{1}{e}, +\infty\right), \quad f'(x) = -\frac{\ln(x)}{x(1 + \ln(x))}, \quad f''(x) = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2 (1 + \ln(x))^2}.$$

Whence

$$f''(1) = \frac{-1}{1} = -1 < 0 \Rightarrow x = 1 \text{ is a local maximum.}$$

Exercize

Find concavity/convexity regions of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. Summary

$$D = \left(\frac{1}{e}, +\infty\right), \quad f'(x) = -\frac{\ln(x)}{x(1+\ln(x))}, \quad f''(x) = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2 (1+\ln(x))^2}.$$

Whence

$$f''(x) \geq 0 \Leftrightarrow (\ln(x))^2 + \ln(x) - 1 \geq 0.$$

Call $t = \ln(x)$. We have $f''(x) \geq 0 \Leftrightarrow t^2 + t - 1 \geq 0$. The roots of the polynomial $t^2 + t - 1$ are $t_{1,2} = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$. Whence

$$f''(x) \geq 0 \Leftrightarrow t^2 + t - 1 \geq 0 \Leftrightarrow t \leq -\frac{1+\sqrt{5}}{2} \text{ or } t \geq \frac{\sqrt{5}-1}{2}.$$

Exercize

Find concavity/convexity regions of the following function

$$f(x) = \ln(1+\ln(x)) - \ln(x).$$

Solution. Summary

$$D = \left(\frac{1}{e}, +\infty\right), \quad f'(x) = -\frac{\ln(x)}{x\,(1+\ln(x))}, \quad f''(x) = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2\,(1+\ln(x))^2}.$$

$$f''(x) \geq 0 \Leftrightarrow t^2 + t - 1 \geq 0 \Leftrightarrow t \leq -\frac{1+\sqrt{5}}{2} \text{ or } t \geq \frac{\sqrt{5}-1}{2}.$$

Nevertheless $t = \ln(x)$ whence

$$t \leq -\frac{1+\sqrt{5}}{2} \Rightarrow \ln(x) \leq -\frac{1+\sqrt{5}}{2} \Rightarrow x \leq e^{-\frac{1+\sqrt{5}}{2}},$$

Nevertheless $e^{-\frac{1+\sqrt{5}}{2}} < e^{-1}$ hence $e^{-\frac{1+\sqrt{5}}{2}} \notin D$, whence

$$f''(x) \geq 0 \Leftrightarrow x \geq e^{\frac{\sqrt{5}-1}{2}}, \text{ and } f''(x) \leq 0 \Leftrightarrow \frac{1}{e} < x \leq e^{\frac{\sqrt{5}-1}{2}}.$$

Exercize

Find concavity/convexity regions of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution. Summary

$$D = \left(\frac{1}{e}, +\infty\right), \quad f'(x) = -\frac{\ln(x)}{x\,(1 + \ln(x))}, \quad f''(x) = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2\,(1 + \ln(x))^2}.$$

$$f''(x) \geq 0 \Leftrightarrow x \geq e^{\frac{\sqrt{5}-1}{2}}, \text{ and } f''(x) \leq 0 \Leftrightarrow \frac{1}{e} < x \leq e^{\frac{\sqrt{5}-1}{2}}.$$

Which is the relative position of $e^{\frac{\sqrt{5}-1}{2}}$ with respect to the critical point $x = 1$? Consider that $\sqrt{5} \approx 2.2361$ then $\sqrt{5} - 1 > 0$ and hence

$$e^{\frac{-1+\sqrt{5}}{2}} > 1.$$

Exercize

Find concavity/convexity regions of the following function

$$f(x) = \ln(1 + \ln(x)) - \ln(x).$$

Solution.

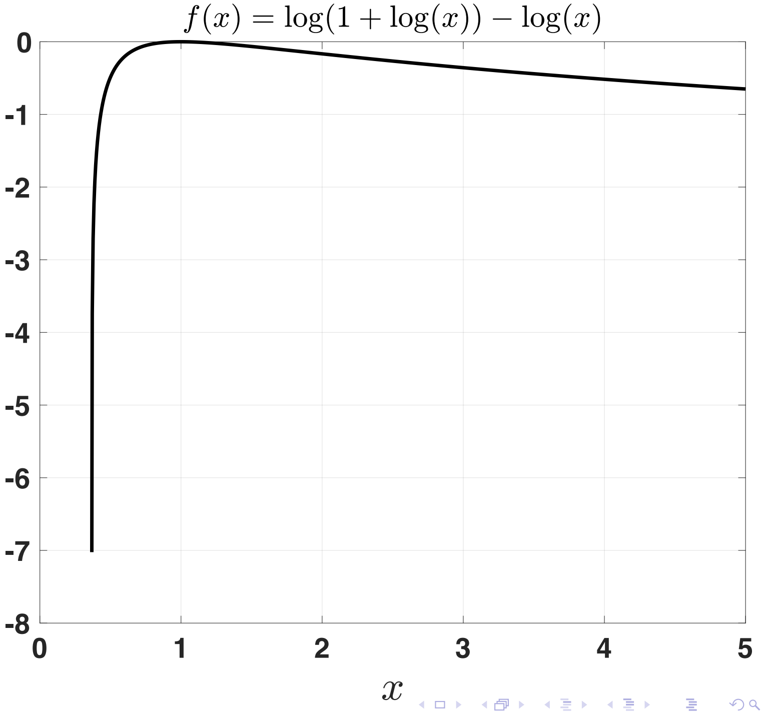
Final summary

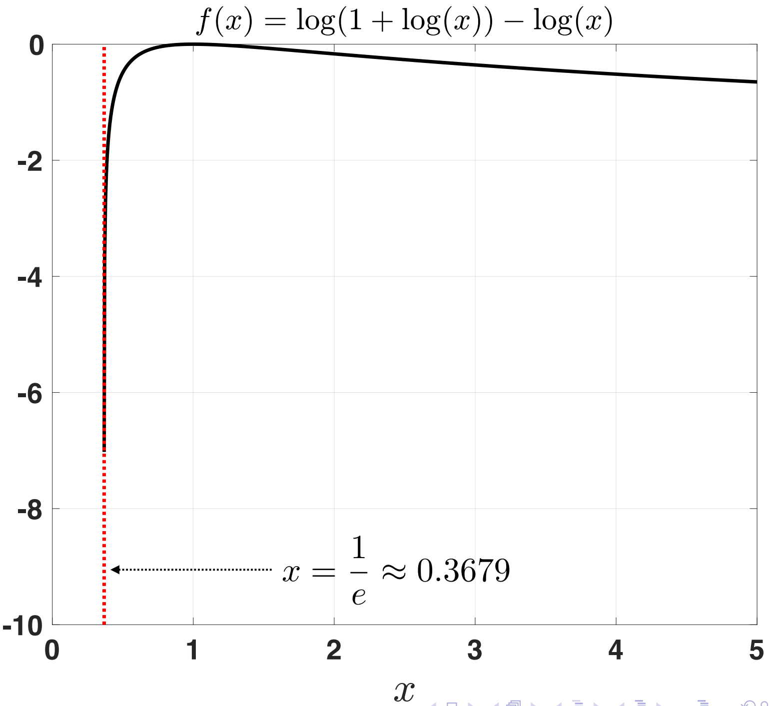
$$D = \left(\frac{1}{e}, +\infty\right), \quad f'(x) = -\frac{\ln(x)}{x\,(1+\ln(x))}, \quad f''(x) = \frac{(\ln(x))^2 + \ln(x) - 1}{x^2\,(1+\ln(x))^2}.$$

$$x_0 = 1 \text{ is a local maximum}$$

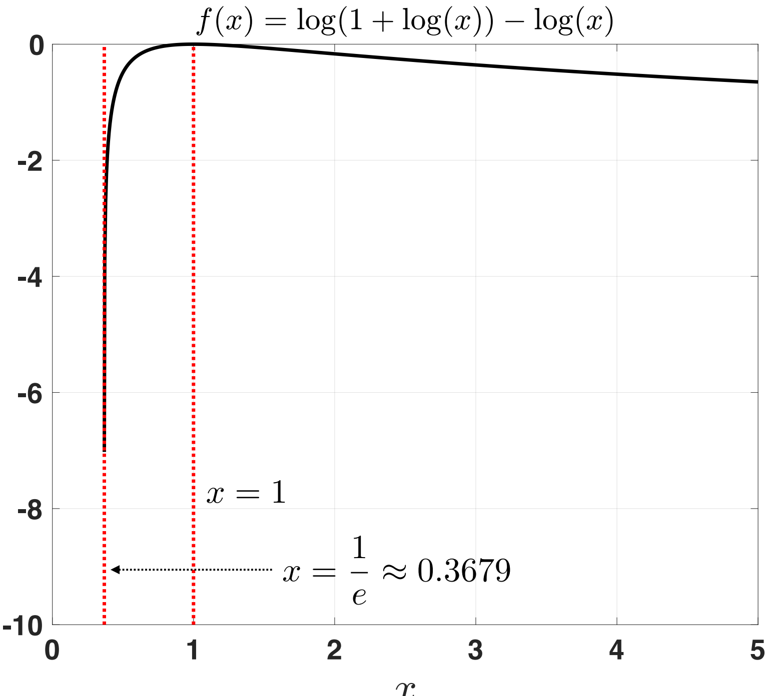
$$f''(x) \geq 0 \Leftrightarrow x \geq e^{\frac{\sqrt{5}-1}{2}} \text{ and } f''(x) \leq 0 \Leftrightarrow \frac{1}{e} < x \leq e^{\frac{\sqrt{5}-1}{2}}.$$

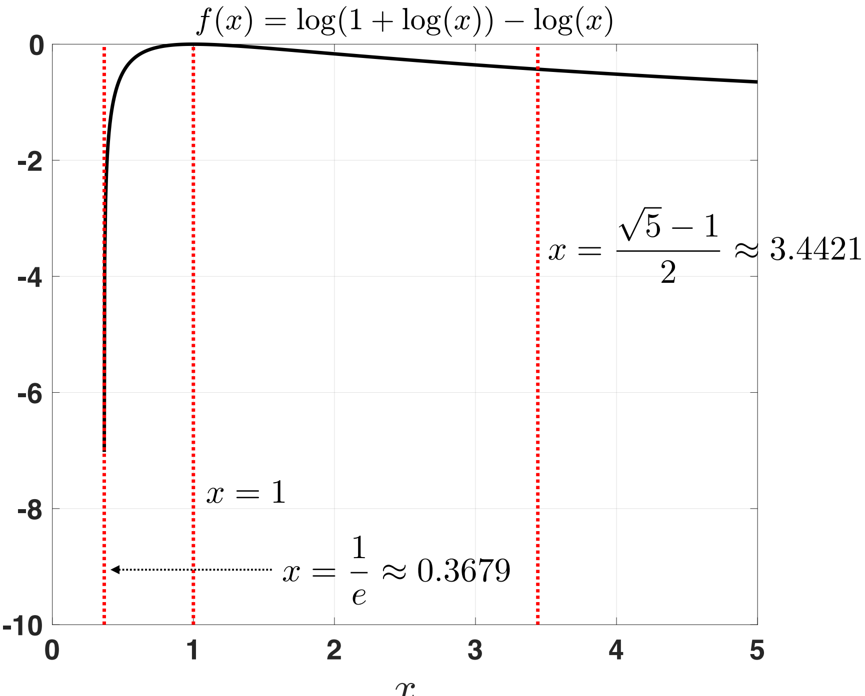
$$e^{\frac{-1+\sqrt{5}}{2}} > 1$$


$f(x) = \log(1 + \log(x)) - \log(x)$
0
-1
-2
-3
-4
-5
-6
-7
-8
0
1
2
3
4
5
x

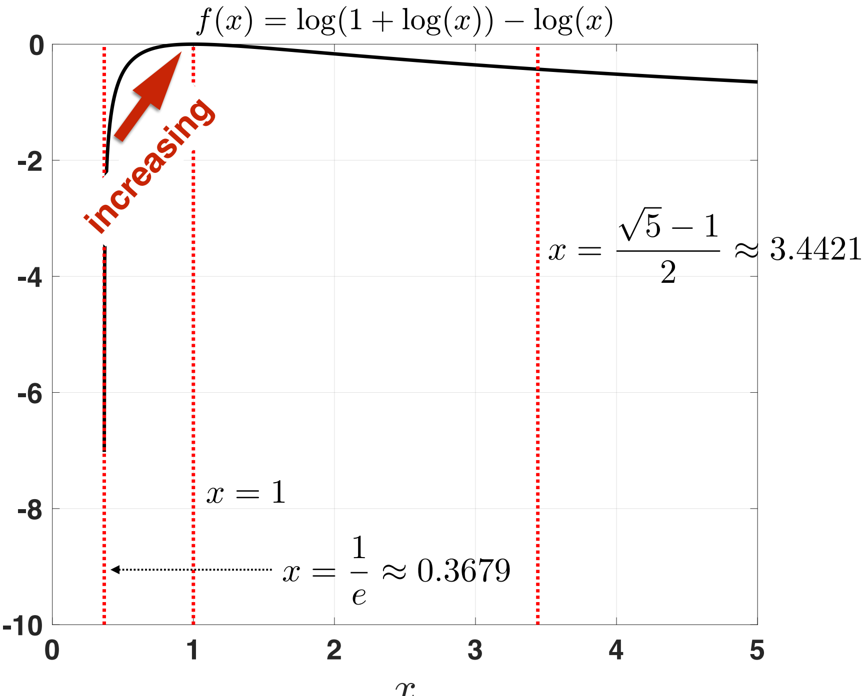


$f(x) = \log(1 + \log(x)) - \log(x)$
0
-2
-4
-6
-8
-10
0
1
2
3
4
5
x
$x = \frac{1}{e} \approx 0.3679$

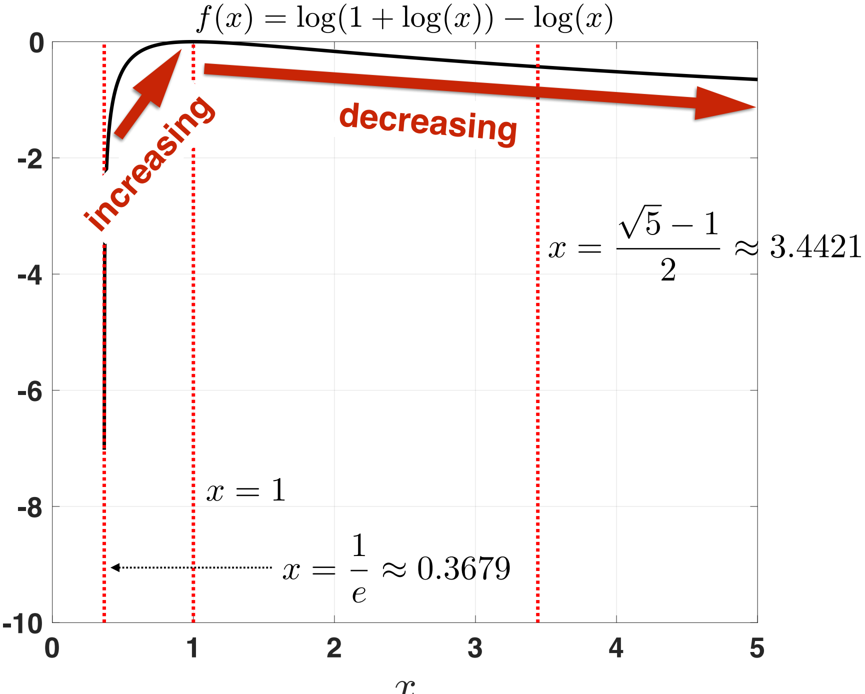

$f(x) = \log(1 + \log(x)) - \log(x)$
0
-2
-4
-6
-8
-10
0
1
2
3
4
5
x
$x = 1$
$x = \frac{1}{e} \approx 0.3679$



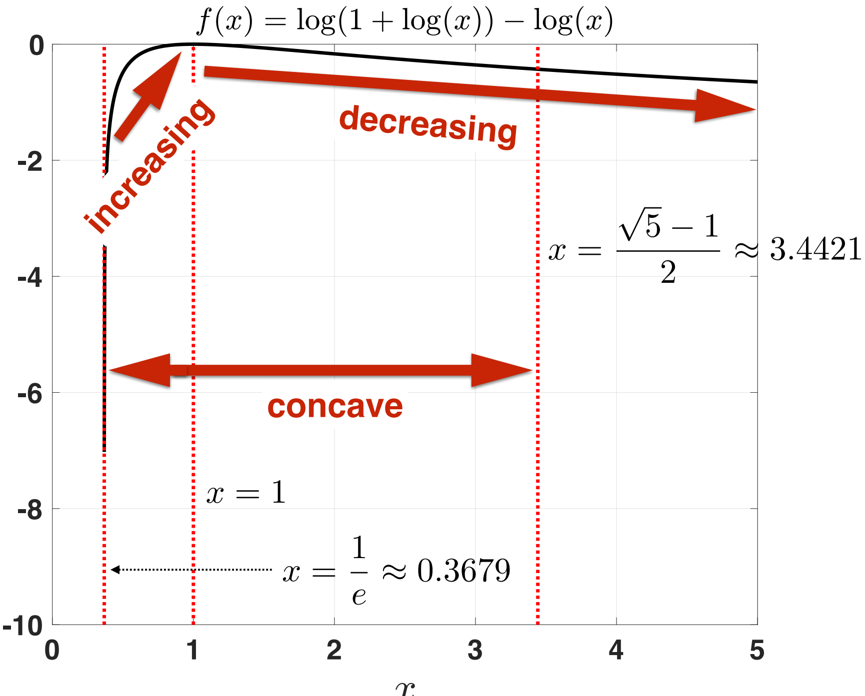
$f(x) = \log(1 + \log(x)) - \log(x)$
0
-2
-4
-6
-8
-10
0
1
2
3
4
5
x
$x = \frac{\sqrt{5}-1}{2} \approx 3.4421$
$x = 1$
$x = \frac{1}{e} \approx 0.3679$



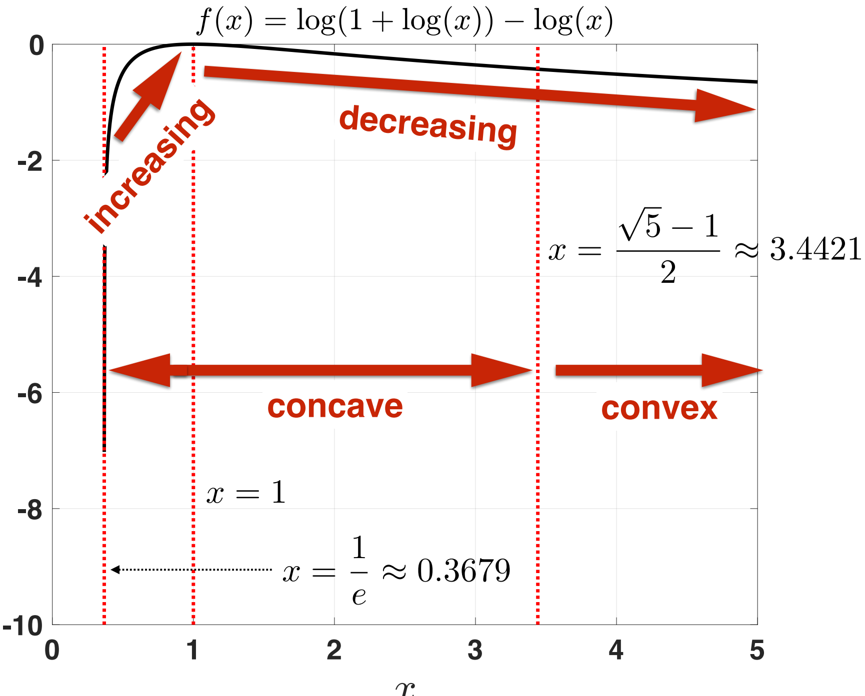
$f(x) = \log(1 + \log(x)) - \log(x)$
0
-2
-4
-6
-8
-10
0
1
2
3
4
5
x
increasing
$x = \frac{\sqrt{5}-1}{2} \approx 3.4421$
$x = 1$
$x = \frac{1}{e} \approx 0.3679$



$f(x) = \log(1 + \log(x)) - \log(x)$
0
-2
-4
-6
-8
-10
0
1
2
3
4
5
x
increasing
decreasing
$x = \frac{\sqrt{5}-1}{2} \approx 3.4421$
$x = 1$
$x = \frac{1}{e} \approx 0.3679$



$f(x) = \log(1 + \log(x)) - \log(x)$
increasing
decreasing
concave
$x = \frac{\sqrt{5}-1}{2} \approx 3.4421$
$x = 1$
$x = \frac{1}{e} \approx 0.3679$
0
-2
-4
-6
-8
-10
0
1
2
3
4
5
x


$f(x) = \log(1 + \log(x)) - \log(x)$
increasing
decreasing
concave
convex
$x = \frac{\sqrt{5}-1}{2} \approx 3.4421$
$x = 1$
$x = \frac{1}{e} \approx 0.3679$
0
-2
-4
-6
-8
-10
0
1
2
3
4
5
x

Exercize

Monopolistic manufacturer.

$$\begin{aligned} p(x) &= p_0 - x = \text{ Price per unit to sell } x \text{ units} \\ c(x) &= c_0 + \alpha x = \text{ Cost to produce } x \text{ units} \\ R(x) &= x\, p(x) = x\, p_0 - x^2 = \text{ Total revenues} \\ \Pi(x) &= R(x) - c(x) = x\,(p_0 - \alpha) - x^2 - c_0 = \text{ Net profit.} \end{aligned}$$

Remark. *Only* $0 \le x \le p_0$ *are admissible.*

1. *Determine the* x^* *that maximizes profit. Does such an* x^* *exist for any* p_0 *and* α*?*
2. *What is the maximum profit?*
3. *Which is the maximum value for* c_0 *that guarantees a positive maximum profit?*
4. *What price per unit must be charged in order to maximize the profit?*

$$\begin{aligned} p(x) &= p_0 - x = \text{ Price per unit to sell } x \text{ units} \\ c(x) &= c_0 + \alpha\, x = \text{ Cost to produce } x \text{ units} \\ R(x) &= x\, p(x) = x\, p_0 - x^2 = \text{ Total revenues} \\ \Pi(x) &= R(x) - c(x) = x\,(p_0 - \alpha) - x^2 - c_0 = \text{ Net profit.} \end{aligned}$$

Determine x^* that maximizes profit. Does such an $x^* \;\exists\; \forall\; p_0$ and α?

$$\Pi'(x) = p_0 - \alpha - 2\,x \Rightarrow \Pi'(x^*) = 0 \Leftrightarrow p_0 - \alpha - 2\,x^* = 0 \Leftrightarrow x^* = \frac{p_0 - \alpha}{2}$$

Besides

$$\Pi''(x) = -2 \Rightarrow x^* \text{ is a maximum.}$$

Finally

$$x^* \in [0, p_0] \Leftrightarrow p_0 \geq \alpha.$$

Optimization

$$\begin{aligned} p(x) &= p_0 - x = \text{ Price per unit to sell } x \text{ units} \\ c(x) &= c_0 + \alpha\, x = \text{ Cost to produce } x \text{ units} \\ R(x) &= x\, p(x) = x\, p_0 - x^2 = \text{ Total revenues} \\ \Pi(x) &= R(x) - c(x) = x\,(p_0 - \alpha) - x^2 - c_0 = \text{ Net profit.} \end{aligned}$$

What is the maximum profit?

$$\begin{aligned} x^* &= \frac{p_0 - \alpha}{2} \Rightarrow \\ \Pi(x^*) &= \frac{p_0 - \alpha}{2}\,(p_0 - \alpha) - \frac{(p_0 - \alpha)^2}{4} - c_0 = \frac{(p_0 - \alpha)^2}{4} - c_0. \end{aligned}$$

Which is the max. c_0 that guarantees a positive maximum profit?

$$\Pi(x^*) > 0 \Leftrightarrow c_0 < \frac{(p_0 - \alpha)^2}{4}.$$

If $c_0 =$ cost of the production plant $\geq \frac{(p_0-\alpha)^2}{4}$ no production.

$$
\begin{aligned}
p(x) &= p_0 - x = \text{ Price per unit to sell } x \text{ units} \\
c(x) &= c_0 + \alpha\, x = \text{ Cost to produce } x \text{ units} \\
R(x) &= x\, p(x) = x\, p_0 - x^2 = \text{ Total revenues} \\
\Pi(x) &= R(x) - c(x) = x\,(p_0 - \alpha) - x^2 - c_0 = \text{ Net profit.}
\end{aligned}
$$

What price per unit must be charged in order to maximize the profit?

$$x^* = \frac{p_0 - \alpha}{2} \Rightarrow p(x^*) = p_0 - \frac{p_0 - \alpha}{2} = \frac{1}{2}(p_0 + \alpha)\,.$$

Exercize

Let $\alpha_n > 0$ *be a positive sequence.*

$$\begin{aligned} c_n(x) &= \ln\left(\alpha_n + x^2\right) = \text{ cost of producing } x \text{ units at time } t = n \\ r(x) &= \ln(x) = \text{ revenues} \\ \pi_n(x) &= r(x) - c_n(x) = \ln(x) - \ln\left(\alpha_n + x^2\right) = \text{ Net profit.} \end{aligned}$$

- *Determine, at each time* n*, the optimal amount* x_n *of units that must be produced.*
- *Assume* $\alpha_n = \frac{1}{4^n}$.
  Which is the first date (i.e. the first n*) in which the producer faces a strictly positive optimal profit?*
  Which is the total amount produced from the initial time $(n = 0)$ *to infinity* $(n = \infty)$*?*

Optimization

$$\begin{aligned} c_n(x) &= \ln\left(\alpha_n + x^2\right) = \text{ cost of producing } x \text{ units at time } t = n \\ r(x) &= \ln(x) = \text{ revenues} \\ \pi_n(x) &= r(x) - c_n(x) = \ln(x) - \ln\left(\alpha_n + x^2\right) = \text{ Net profit.} \end{aligned}$$

Determine, at each time n, the optimal amount x_n.

$$\pi_n'(x) = \frac{1}{x} - \frac{2x}{\alpha_n + x^2} = \frac{\alpha_n + x^2 - 2x^2}{x\left(\alpha_n + x^2\right)} = \frac{\alpha_n - x^2}{x\left(\alpha_n + x^2\right)} = 0 \Leftrightarrow x_n = \pm\sqrt{\alpha_n},$$

only $x_n = +\sqrt{\alpha_n}$ is admissible. Since

$$\pi_n''(x) = \frac{-2x}{x\left(\alpha_n + x^2\right)} - \frac{\alpha_n - x^2}{x^2\left(\alpha_n + x^2\right)^2}\left(\alpha_n + x^2 + 2x^2\right),$$

we get

$$\pi_n''(x_n) = -\frac{2\sqrt{\alpha_n}}{2\alpha_n\sqrt{\alpha_n}} < 0,$$

whence $x_n = \sqrt{\alpha_n}$ is a maximum.

Assume $\alpha_n = \frac{1}{4^n}$

$$\begin{aligned} c_n(x) &= \ln\left(\frac{1}{4^n} + x^2\right) = \text{ cost of producing } x \text{ units at time } t = n \\ r(x) &= \ln(x) = \text{ revenues} \\ \pi_n(x) &= r(x) - c_n(x) = \ln(x) - \ln\left(\frac{1}{4^n} + x^2\right) = \text{ Net profit.} \end{aligned}$$

Which is the first date in which the producer faces a strictly positive optimal profit?

$$\pi_n(x_n) = \ln\left(\frac{2^n}{2}\right) \Rightarrow \begin{cases} \pi_0(x_0) = \ln(1/2) < 0 \\ \pi_1(x_1) = \pi_n(x_1) = \ln(1) = 0 \\ \pi_2(x_1) = \ln(2) > 0. \end{cases}$$

so the answer is $n = 2$.

Assume $\alpha_n = \frac{1}{4^n}$

$$\begin{aligned}
c_n(x) &= \ln\left(\frac{1}{4^n} + x^2\right) = \text{ cost of producing } x \text{ units at time } t = n \\
r(x) &= \ln(x) = \text{ revenues} \\
\pi_n(x) &= r(x) - c_n(x) = \ln(x) - \ln\left(\frac{1}{4^n} + x^2\right) = \text{ Net profit.}
\end{aligned}$$

Which is the total amount produced from the initial time $(n = 0)$ to infinity $(n = \infty)$?

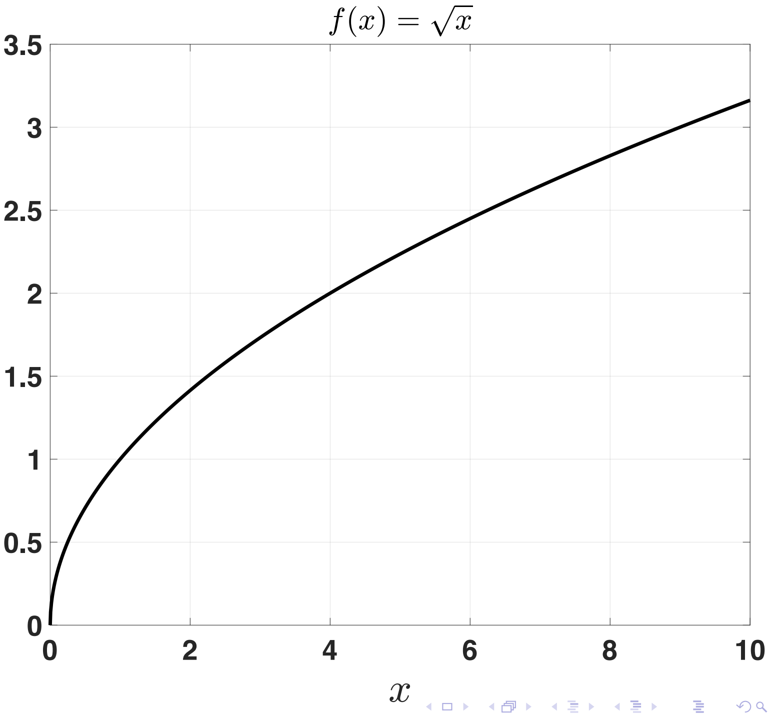
Remember that $x_n = \sqrt{\alpha_n} = \frac{1}{2^n}$.

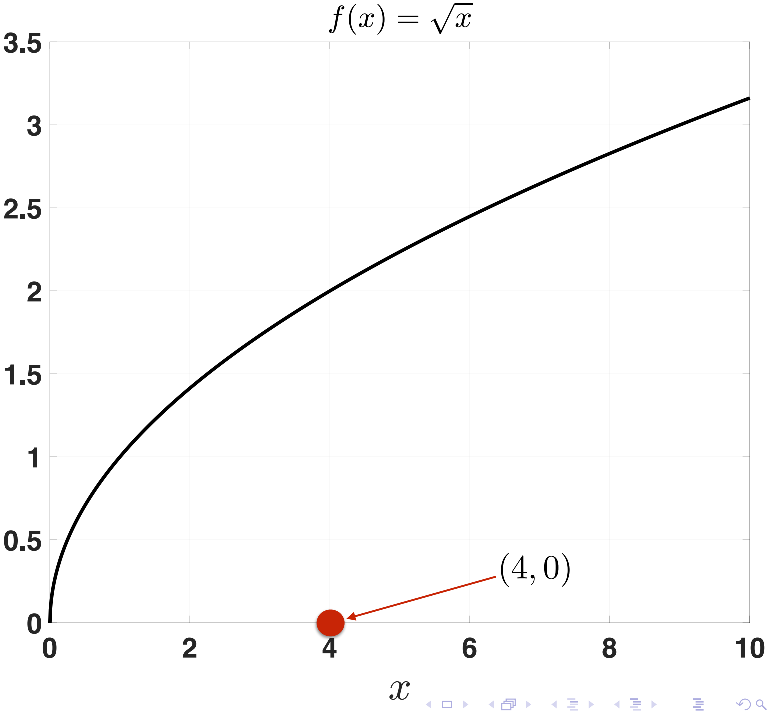
The total amount produced is

$$\sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2.$$

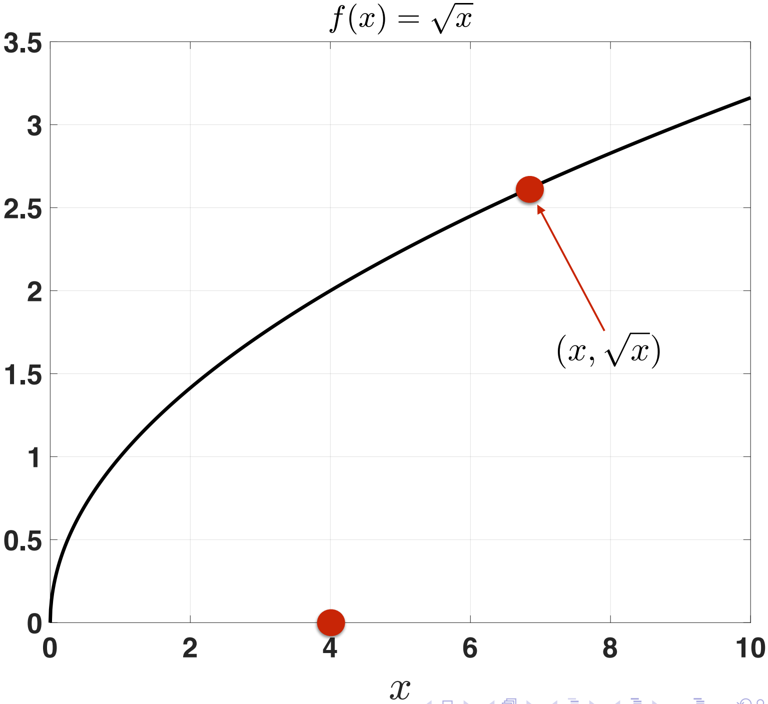
Exercize

Find the point on the graph of $y = \sqrt{x}$ *nearest to the point* $(4, 0)$.

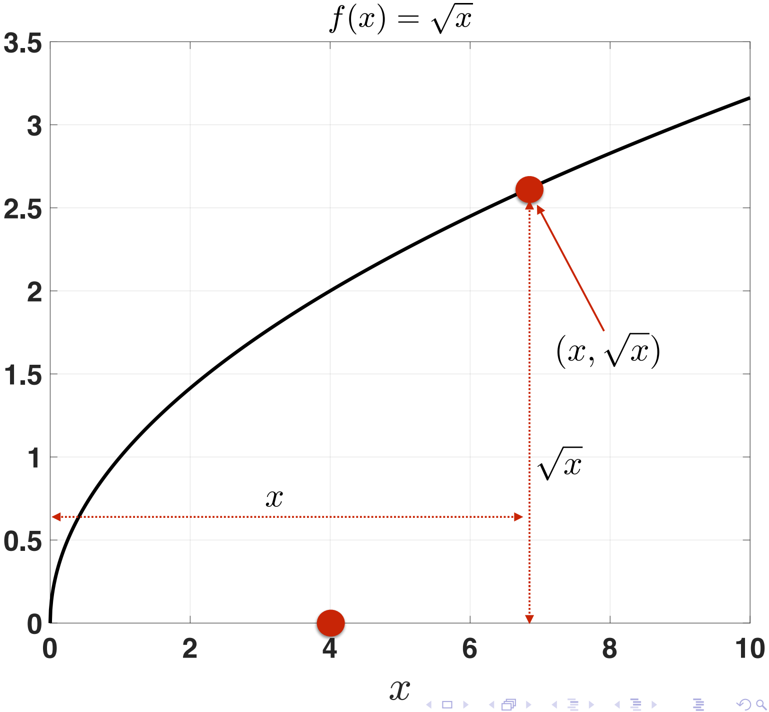

$f(x) = \sqrt{x}$
3.5
3
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
x

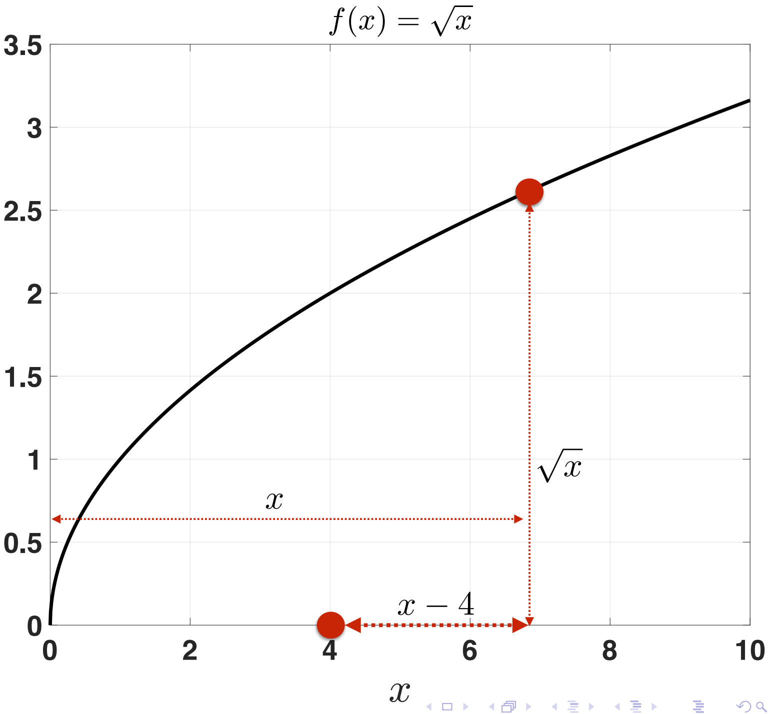


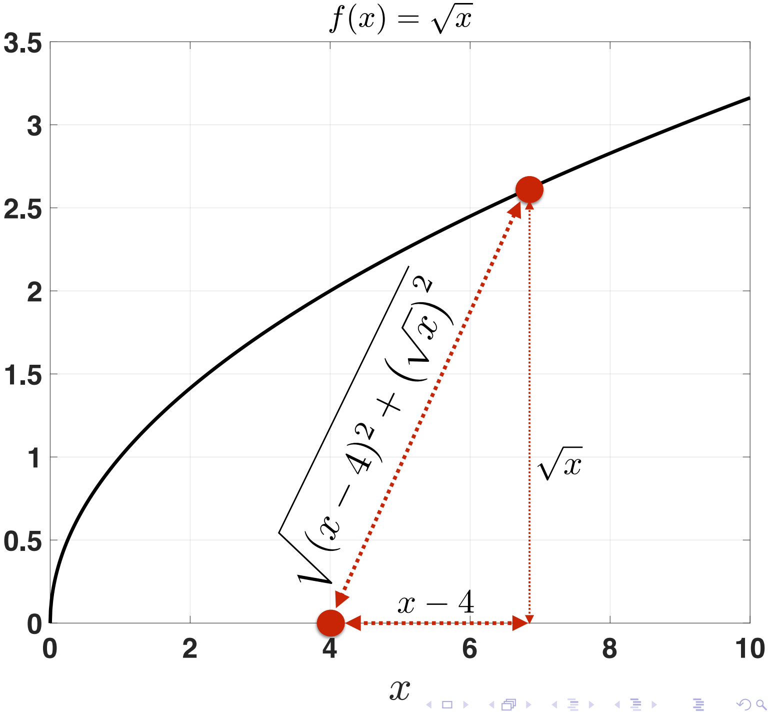
$f(x) = \sqrt{x}$
3.5
3
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
$(4, 0)$
x



$f(x) = \sqrt{x}$
3.5
3
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
$(x, \sqrt{x})$
x


$f(x) = \sqrt{x}$
3.5
3
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
x
$(x, \sqrt{x})$
$\sqrt{x}$
x


$f(x) = \sqrt{x}$
3.5
3
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
x
$\sqrt{x}$
x
$x - 4$


$f(x) = \sqrt{x}$
3.5
3
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
x
$\sqrt{(x-4)^2 + (\sqrt{x})^2}$
$\sqrt{x}$
$x - 4$

Exercize

Find the point on the graph of $y = \sqrt{x}$ nearest to the point $(4, 0)$.

Solution. We have to find, if it exists, the minimum of

$$f(x) = \sqrt{(x-4)^2 + (\sqrt{x})^2} = \sqrt{(x-4)^2 + x} = \left((x-4)^2 + x\right)^{\frac{1}{2}}.$$

By the rule of derivation of composite functions:

$$(g(x)^{\alpha})' = \alpha\,(g(x))^{\alpha-1}\,g'(x) \Rightarrow \left(g(x)^{\frac{1}{2}}\right)' = \frac{1}{2}\,(g(x))^{-\frac{1}{2}}\,g'(x).$$

Hence

$$f'(x) = \frac{\left((x-4)^2 + x\right)'}{2\sqrt{(x-4)^2 + x}} = \frac{(2\,(x-4) + 1)}{2\sqrt{(x-4)^2 + x}} = \frac{(2\,x - 7)}{2\sqrt{(x-4)^2 + x}}$$

Exercize

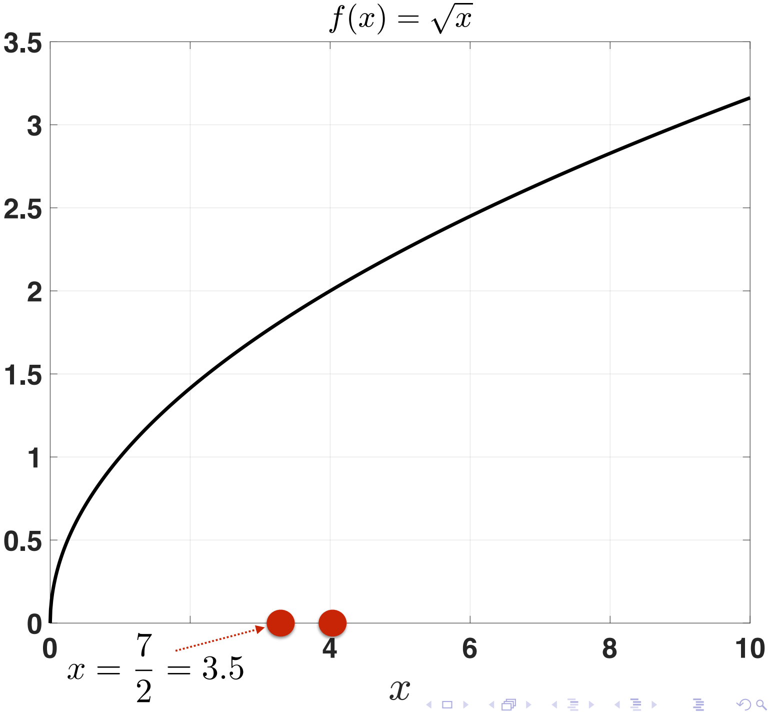
Find the point on the graph of $y = \sqrt{x}$ nearest to the point $(4, 0)$.

Solution. Summary

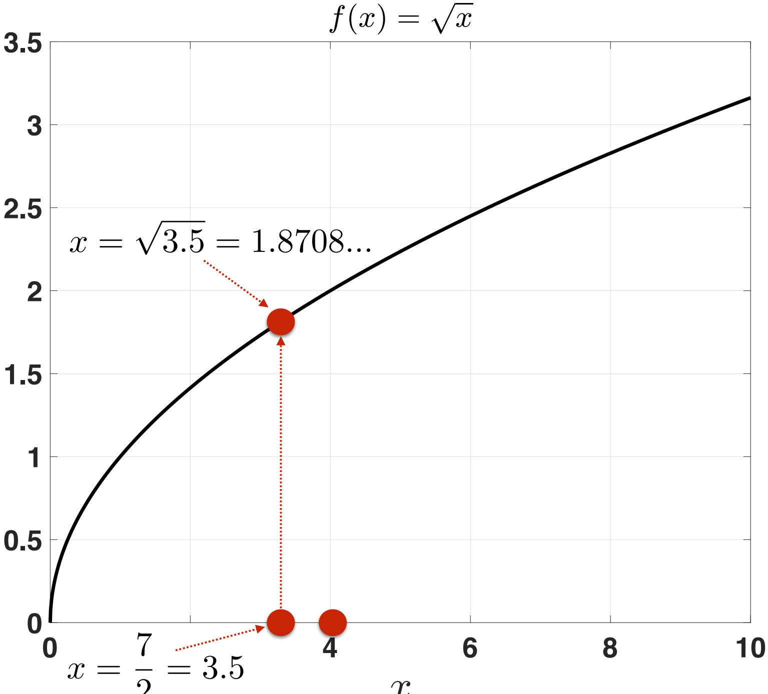
$$f(x) = \left((x-4)^2 + x\right)^{\frac{1}{2}}, \quad f'(x) = \frac{(2x-7)}{2\sqrt{(x-4)^2 + x}}.$$

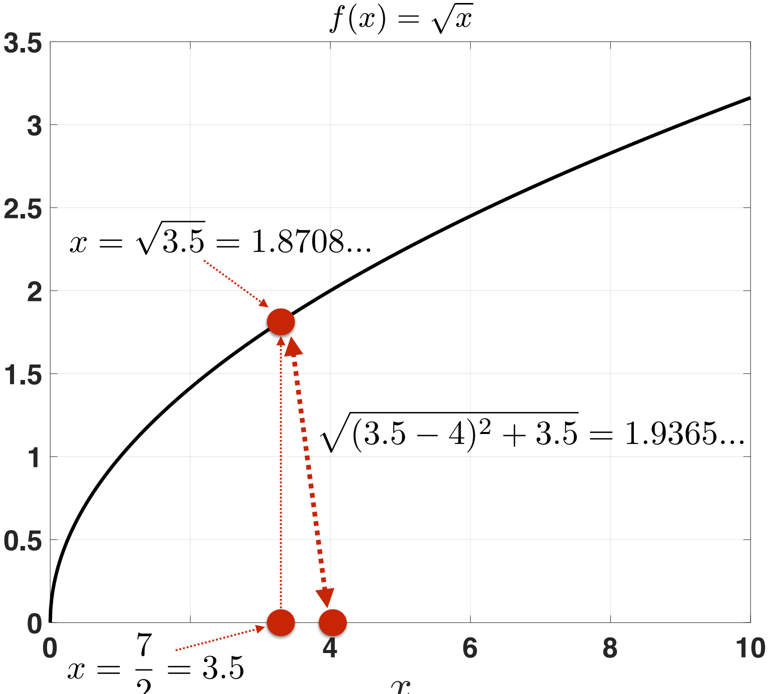
Whence $f'(x) = 0 \Leftrightarrow x = \frac{7}{2}$. Is it a minimum?

$$\begin{aligned} f''(x) &= (2x-7)' \frac{1}{2\sqrt{(x-4)^2 + x}} + (2x-7) \left(\frac{1}{2\sqrt{(x-4)^2 + x}}\right)' \\ &= \frac{2}{2\sqrt{(x-4)^2 + x}} - \frac{1}{2} \frac{(2x-7)}{2\left((x-4)^2 + x\right)^{3/2}} \Rightarrow f''\left(\frac{7}{2}\right) > 0 \end{aligned}$$



$f(x) = \sqrt{x}$
3.5
3
2.5
2
1.5
1
0.5
0
0
4
6
8
10
$x = \frac{7}{2} = 3.5$
x


$f(x) = \sqrt{x}$
3.5
3
2.5
2
1.5
1
0.5
0
0
4
6
8
10
x
$x = \sqrt{3.5} = 1.8708...$
$x = \frac{7}{2} = 3.5$



$f(x) = \sqrt{x}$
3.5
3
2.5
2
1.5
1
0.5
0
0
4
6
8
10
x
$x = \sqrt{3.5} = 1.8708...$
$\sqrt{(3.5-4)^2 + 3.5} = 1.9365...$
$x = \frac{7}{2} = 3.5$

Exercize

Find the minimum distance between the point $(0,0)$ *and the graph of the function* $g(x) = \frac{1}{\sqrt{x}}$.

Solution. The (squared) distance between $(0,0)$ and $\left(x, \frac{1}{\sqrt{x}}\right)$ is

$$f(x) = (x-0)^2 + \left(\frac{1}{\sqrt{x}} - 0\right)^2 = x^2 + \frac{1}{x}.$$

whence

$$f'(x) = 2x - \frac{1}{x^2} \Rightarrow f'(x) = 0 \Leftrightarrow 2x^3 - 1 = 0 \Leftrightarrow x^3 = \frac{1}{2} \Leftrightarrow x = \frac{1}{2^{1/3}}$$

The second derivative is

$$f''(x) = 2 + \frac{2}{x^3} \Rightarrow f''\left(\frac{1}{2^{1/3}}\right) > 0 \Rightarrow x = \frac{1}{2^{1/3}} \text{ is a minimum.}$$

The minimum distance is $\sqrt{f\left(\frac{1}{2^{1/3}}\right)} = \sqrt{\frac{1}{2^{2/3}} + 2^{1/3}}$.

Utility function

From **WIKIPEDIA**: Consider a set of alternatives facing an individual, and over which the individual has a preference ordering.

A utility function is able to represent those preferences if it is possible to assign a real number to each alternative, in such a way that alternative a is assigned a number greater than alternative b if, and only if, the individual prefers alternative a to alternative b.

In a rational choice framework every consumer decides to consume the amount of good x that maximizes the utility $U(x)$.

$$a \text{ is preferred to } b \Leftrightarrow U(a) > U(b)$$

Exercize

Let $u_0 > 0$ and $u_1 > 0$:

$$U(x) = u_0 \ln\left(x^2\right) - u_1\, x = \text{ utility of buying } x \text{ units of good.}$$

For which value of u_0 and u_1 will the consumer buy an amount of good larger than 1?

For which value of u_0 and u_1 the optimal utility is positive?

Solution. The consumer has to maximize the utility $U(x)$:

$$U'(x) = \frac{2\,u_0}{x} - u_1 = 0 \Leftrightarrow x = \frac{2\,u_0}{u_1} = x_0.$$

Note that

$$U''(x) = -\frac{2\,u_0}{x^2} < 0, \forall x \in \mathbb{R}$$

so the function is concave everywhere, whence x_0 is a maximum. Finally

$$x_0 > 1 \Leftrightarrow u_1 < 2\,u_0.$$

$$U(x_0) = 2\,u_0 \ln\left(\frac{2\,u_0}{u_1}\right) - 2\,u_0 = 2\,u_0\left(\ln\left(\frac{2\,u_0}{u_1}\right) - 1\right) > 0 \Leftrightarrow \ln\left(\frac{2\,u_0}{u_1}\right) > 1 \Leftrightarrow \frac{2\,u_0}{u_1} > e.$$

Optimization

Exercize

Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is maximal.

Solution. We have to find $x \geq 0$ and $y \geq 0$ such that $x + y = 9$ and such that

$$F(x, y) = x\,y^2$$

is maximal. Since $y = 9 - x$ we have to find the maximum of

$$f(x) = x\,(9-x)^2 \Rightarrow f'(x) = (9-x)^2 - 2\,x\,(9-x) = (9-x)\,(9-x-2\,x).$$

$$f'(x) = (9-x)\,(9-3\,x) = 0 \Leftrightarrow x = 9 \text{ or } x = 3.$$

$$f''(x) = -2\,(9-x) + 2\,x - 2\,(9-x) = -4\,(9-x) + 2\,x$$

$f''(3) = -18 < 0$ and $f''(9) = 18 > 0$ so $x = 9$ is the minimum and $x = 3$ the maximum. The final answer is thus

$$x = 3, \quad y = 9 - 3 = 6.$$

Theorem

Let $f : [a, b] \to \mathbb{R}$ be differentiable in (a, b). Assume f is invertible and call $f^{(-1)} : I_f \subseteq \mathbb{R} \to \mathbb{R}$ the inverse function, where I_f denotes the image of f. Then $f^{(-1)}$ is differentiable in I_f and

$$\left[f^{(-1)}\right]'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}.$$

for all $y \in I_f$.

Proof. Take a point $x_0 \in (a, b)$ and call $y_0 = f(x_0)$, that is $x_0 = f^{-1}(y_0)$. Hence

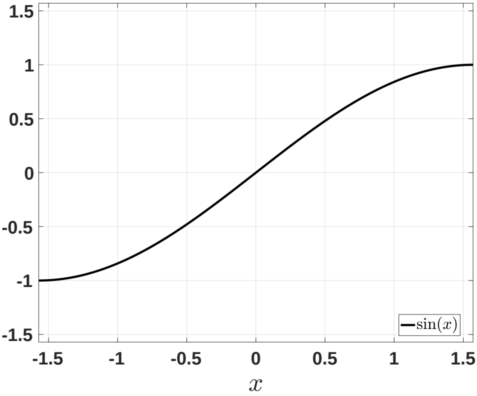
$$\lim_{y \to y_0} \frac{f^{(-1)}(y) - f^{(-1)}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)},$$

whence

$$\left[f^{(-1)}\right]'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'\left(f^{-1}(y_0)\right)}. \qquad \square$$

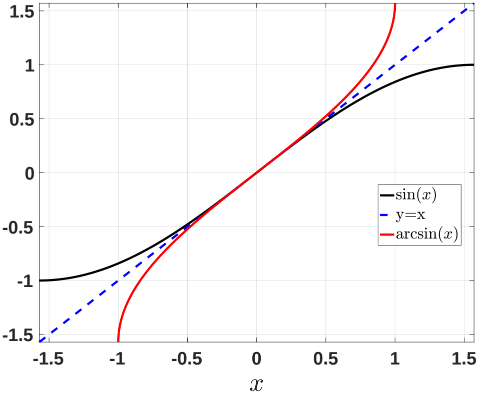
Definition

The function $\sin(x)$ *is strictly monotonic and increasing in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *so it can be inverted and the inverse is called* $\arcsin(x)$ *and it is defined in* $[-1, 1]$ *with values in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Definition

The function $\sin(x)$ *is strictly monotonic and increasing in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *so it can be inverted and the inverse is called* $\arcsin(x)$ *and it is defined in* $[-1, 1]$ *with values in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



The arcsin function: some important values.

Remember that, by definition,

$$x_0 \xrightarrow{f} y_0 = f(x_0) \Leftrightarrow y_0 \xrightarrow{f^{(-1)}} f^{(-1)}(y_0) = x_0.$$

$$\begin{array}{ccccccc}
\sin(0) & = & 0 & \Rightarrow & \arcsin(0) & = & 0 \\
\sin\left(\frac{\pi}{2}\right) & = & 1 & \Rightarrow & \arcsin(1) & = & \frac{\pi}{2} \\
\sin\left(-\frac{\pi}{2}\right) & = & -1 & \Rightarrow & \arcsin(-1) & = & -\frac{\pi}{2} \\
\sin\left(\frac{\pi}{6}\right) & = & \frac{1}{2} & \Rightarrow & \arcsin\left(\frac{1}{2}\right) & = & \frac{\pi}{6} \\
\sin\left(-\frac{\pi}{6}\right) & = & -\frac{1}{2} & \Rightarrow & \arcsin\left(-\frac{1}{2}\right) & = & -\frac{\pi}{6} \\
\sin\left(\frac{\pi}{4}\right) & = & \frac{\sqrt{2}}{2} & \Rightarrow & \arcsin\left(\frac{\sqrt{2}}{2}\right) & = & \frac{\pi}{4} \\
\sin\left(-\frac{\pi}{4}\right) & = & -\frac{\sqrt{2}}{2} & \Rightarrow & \arcsin\left(-\frac{\sqrt{2}}{2}\right) & = & -\frac{\pi}{4} \\
\sin\left(\frac{\pi}{3}\right) & = & \frac{\sqrt{3}}{2} & \Rightarrow & \arcsin\left(\frac{\sqrt{3}}{2}\right) & = & \frac{\pi}{3} \\
\sin\left(-\frac{\pi}{3}\right) & = & -\frac{\sqrt{3}}{2} & \Rightarrow & \arcsin\left(-\frac{\sqrt{3}}{2}\right) & = & -\frac{\pi}{3}
\end{array}$$

Exercize

Compute the derivative of the $\arcsin(x)$.

Solution. Recall the formula for the derivative of the inverse

$$\left[f^{(-1)}\right]'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}.$$

In our case, for $x \in [-1, 1]$, it means that

$$(\arcsin(x))' = \frac{1}{\cos(\arcsin(x))}.$$

Since $\arcsin(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have

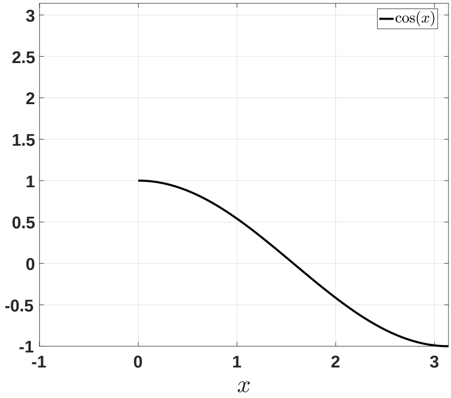
$$\cos(\arcsin(x)) = +\sqrt{1 - \sin^2(\arcsin(x))} = \sqrt{1 - x^2},$$

whence

$$(\arcsin(x))' = \frac{1}{\sqrt{1 - x^2}}.$$

Definition

The function $\cos(x)$ *is strictly monotonic and decreasing in* $[0, \pi]$ *so it can be inverted and the inverse is called* $\arccos(x)$ *and it is defined in* $[-1, 1]$ *with values in* $[0, \pi]$.



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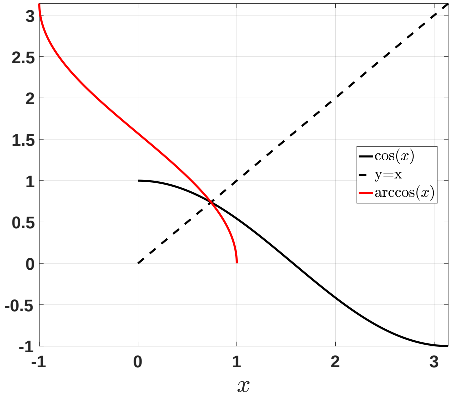


The arccos function: some important values.

Remember that, by definition,

$$x_0 \xrightarrow{f} y_0 = f\left(x_0\right) \Leftrightarrow y_0 \xrightarrow{f^{(-1)}} f^{(-1)}\left(y_0\right) = x_0.$$

$$\begin{array}{ccccccc}
\cos\left(0\right) & = & 1 & \Rightarrow & \arccos\left(1\right) & = & 0 \\
\cos\left(\frac{\pi}{2}\right) & = & 0 & \Rightarrow & \arccos\left(0\right) & = & \frac{\pi}{2} \\
\cos\left(\frac{\pi}{6}\right) & = & \frac{\sqrt{3}}{2} & \Rightarrow & \arccos\left(\frac{\sqrt{3}}{2}\right) & = & \frac{\pi}{6} \\
\cos\left(\frac{\pi}{4}\right) & = & \frac{\sqrt{2}}{2} & \Rightarrow & \arccos\left(\frac{\sqrt{2}}{2}\right) & = & \frac{\pi}{4} \\
\cos\left(\frac{\pi}{3}\right) & = & \frac{1}{2} & \Rightarrow & \arccos\left(\frac{1}{2}\right) & = & \frac{\pi}{3}.
\end{array}$$

Exercize

Compute the derivative of the $\arccos(x)$.

Solution. Recall the formula for the derivative of the inverse

$$\left[f^{(-1)}\right]'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}.$$

In our case, for $x \in [-1, 1]$, it means that

$$(\arccos(x))' = \frac{1}{-\sin(\arccos(x))}.$$

Since $\arccos(x) \in [0, \pi]$, we have

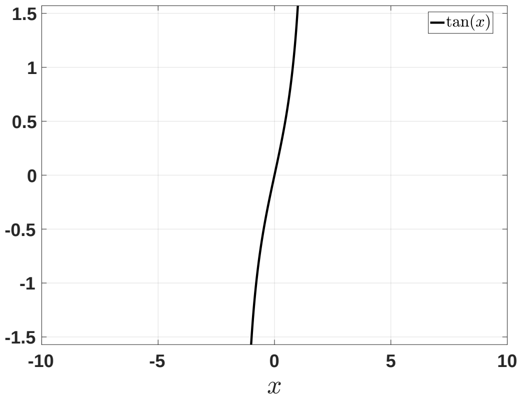
$$\sin(\arccos(x)) = +\sqrt{1 - \cos^2(\arccos(x))} = \sqrt{1 - x^2},$$

whence

$$(\arccos(x))' = -\frac{1}{\sqrt{1 - x^2}}.$$

Definition

The function $\tan(x)$ *is strictly monotonic and increasing in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *so it can be inverted and the inverse is called* $\arctan(x)$ *and it is defined in* $\mathbb{R}$ *with values in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



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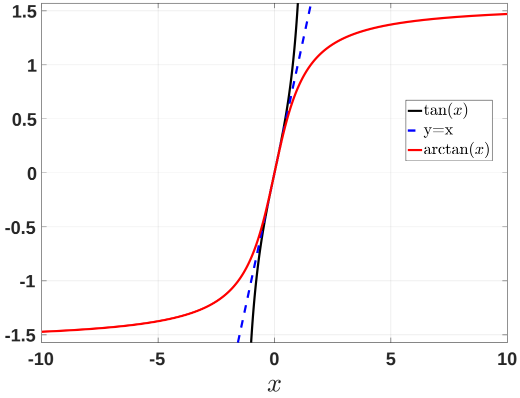
The function $\tan(x)$ *is strictly monotonic and increasing in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *so it can be inverted and the inverse is called* $\arctan(x)$ *and it is defined in* $\mathbb{R}$ *with values in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



The arctan function: some important values.

Remember that, by definition,

$$x_0 \xrightarrow{f} y_0 = f\left(x_0\right) \Leftrightarrow y_0 \xrightarrow{f^{(-1)}} f^{(-1)}\left(y_0\right) = x_0.$$

$$\begin{array}{ccccccc}
\tan(0) & = & 0 & \Rightarrow & \arctan(0) & = & 0 \\
\lim_{x\to\left(\frac{\pi}{2}\right)^-} \tan(x) & = & +\infty & \Rightarrow & \lim_{x\to+\infty} \arctan(x) & = & \left(\frac{\pi}{2}\right)^- \\
\lim_{x\to\left(\frac{\pi}{2}\right)^+} \tan(x) & = & -\infty & \Rightarrow & \lim_{x\to-\infty} \arctan(x) & = & \left(\frac{\pi}{2}\right)^+ \\
\tan\left(\frac{\pi}{6}\right) & = & \frac{\sqrt{3}}{3} & \Rightarrow & \arctan\left(\frac{\sqrt{3}}{3}\right) & = & \frac{\pi}{6} \\
\tan\left(\frac{\pi}{4}\right) & = & 1 & \Rightarrow & \arctan(1) & = & \frac{\pi}{4} \\
\tan\left(\frac{\pi}{3}\right) & = & \sqrt{3} & \Rightarrow & \arctan\left(\sqrt{3}\right) & = & \frac{\pi}{3}.
\end{array}$$

Exercize

Compute the derivative of the $\arctan(x)$.

Solution. Recall the formula for the derivative of the inverse

$$\left[f^{(-1)}\right]'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}.$$

In our case, for $x \in \mathbb{R}$, it means that

$$(\arctan(x))' = \frac{1}{\frac{1}{\cos^2(\arctan(x))}} = \cos^2(\arctan(x)).$$

Now use

$$cos^2(x) = \frac{1}{1+\tan^2(x)},$$

to have

$$(\arctan(x))' = \frac{1}{1+\tan^2(\arctan(x))} = \frac{1}{1+x^2}$$

Theorem

Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ *and* $g : D \subseteq \mathbb{R} \to \mathbb{R}$ *be continuous on* D *and let* x_0 *be a limit point of* D. *Assume that* f *and* g *are both differentiable in* $D / \{x_0\}$ *and* $g(x_0) \neq 0$. *If:*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \quad AND \quad \exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L,$$

or if

$$\lim_{x \to x_0} f(x) = \pm\infty, \qquad \lim_{x \to x_0} g(x) = \pm\infty \quad AND \quad \exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L,$$

then

$$\exists \lim_{x \to x_0} \frac{f(x)}{g(x)} = L.$$

De L'Hôpital rule

Proof. Assume

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0 \quad \text{AND} \quad \exists \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L,$$

The red condition implies

$$\forall \varepsilon > 0 \exists \delta_\varepsilon > 0 : \forall x \in D : |x - x_0| < \delta_\varepsilon \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon$$

Take $x_1 < x_2$ in $(x_0 - \delta, x_0)$. Cauchy applied to f and g in $[x_1, x_2]$ gives

$$\exists \xi \in (x_1, x_2) : \frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} = \frac{f'(\xi)}{g'(\xi)}.$$

Note that $x_0 - \delta_\varepsilon < x_1 < \xi < x_2 < x_0$ hence ξ is such that $|\xi - x_0| < \delta_\varepsilon$

$$\left| \frac{f'(\xi)}{g'(\xi)} - L \right| = \left| \frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} - L \right| < \varepsilon.$$

Now take the limit for $x_2 \to x_0^-$ and use the blue condition

$$\left| \frac{f(x_1)}{g(x_1)} - L \right| < \varepsilon.$$

Now for $\varepsilon \to 0$ we have that (remember that $x_1 \in (x_o - \delta_\varepsilon, x_0)$) $x_1 \to x_0^-$. Hence:

$$\exists \lim_{x\to x_0^-} \frac{f(x)}{g(x)} = L.$$

with an identical argument we arrive at: $\exists \lim_{x\to x_0^+} \frac{f(x)}{g(x)} = L$.

De L'Hôpital rule

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable on $\mathbb{R}$. If:

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} g(x) = 0 \quad AND \quad \exists \lim_{x \to \pm\infty} \frac{f'(x)}{g'(x)} = L,$$

or if

$$\lim_{x \to \pm\infty} f(x) = \pm\infty, \qquad \lim_{x \to \pm\infty} g(x) = \pm\infty \quad AND \quad \exists \lim_{x \to \pm\infty} \frac{f'(x)}{g'(x)} = L,$$

then

$$\exists \lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = L.$$

Exercize

Compute the limit

$$\lim_{x\to 0^+} x \ln(x)$$

$$\begin{aligned}
\lim_{x\to 0^+} x \ln(x) &= 0 \times (-\infty) \\
&= \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \\
&= \frac{-\infty}{+\infty} \overset{H}{=} \lim_{x\to 0^+} \frac{(\ln(x))'}{\left(\frac{1}{x}\right)'} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -x = 0.
\end{aligned}$$

Exercize

Compute the limit

$$\lim_{x\to+\infty} \frac{\ln(x)}{\sqrt{x}}$$

$$\begin{aligned}\lim_{x\to+\infty} \frac{\ln(x)}{\sqrt{x}} &= \frac{+\infty}{+\infty} \overset{H}{=} \lim_{x\to+\infty} \frac{(\ln(x))'}{\left(\sqrt{x}\right)'} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to+\infty} \frac{2\sqrt{x}}{x} \\ &= \lim_{x\to+\infty} \frac{2}{\sqrt{x}} = 0.\end{aligned}$$

Exercize

Compute the limit

$$\lim_{x\to\infty}\left(\arctan x-\frac{\pi}{2}\right)e^x.$$

Since the arc whose tangent is $+\infty$ is $\frac{\pi}{2}$

$$\lim_{x\to+\infty}\arctan(x)=+\frac{\pi}{2}$$

whence

$$\begin{aligned}\lim_{x\to\infty}\left(\arctan x-\frac{\pi}{2}\right)e^x &= 0\times(+\infty)=\lim_{x\to\infty}\left(\frac{\arctan x-\frac{\pi}{2}}{e^{-x}}\right)\\ &= \frac{0}{0}\overset{H}{=}\lim_{x\to\infty}\frac{\frac{1}{1+x^2}}{-e^{-x}}=-\lim_{x\to\infty}\frac{e^x}{1+x^2}\\ &= \frac{\infty}{\infty}\overset{H}{=}-\lim_{x\to\infty}\frac{e^x}{2x}=\frac{\infty}{\infty}\overset{H}{=}-\lim_{x\to\infty}\frac{e^x}{2}=-\infty.\end{aligned}$$

WARNING!

The hypothesis

$$\exists \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

is fundamental!

Example

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \frac{+\infty}{+\infty} \overset{H}{=} \lim_{x \to +\infty} \frac{1 + \cos(x)}{1} = \nexists.$$

However

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \lim_{x \to +\infty} \left(1 + \underbrace{\frac{\sin(x)}{x}}_{\to 0} \right) = 1.$$

De L'Hôpital rule

Exercize

$$\lim_{x\to 0^+}\left(\ln\left(1+e^{-1/x}\right)\right)^x = 0^0 = ???$$

Solution. Use the identity

$$\left(\ln\left(1+e^{-1/x}\right)\right)^x = e^{x\,\ln\left(\ln\left(1+e^{-1/x}\right)\right)},$$

and compute

$$\begin{aligned}
& \lim_{x\to 0^+} x\ln\left(\ln\left(1+e^{-1/x}\right)\right) \\
= & \lim_{x\to 0^+}\frac{\ln\left(\ln\left(1+e^{-1/x}\right)\right)}{\frac{1}{x}} = \frac{-\infty}{+\infty} \overset{H}{=} \lim_{x\to 0^+} -\frac{\left(\frac{1}{\ln\left(1+e^{-1/x}\right)}\right)\left(\frac{1}{1+e^{-1/x}}\right)\frac{e^{-1/x}}{x^2}}{\frac{1}{x^2}} \\
= & \lim_{x\to 0^+} -\left(\frac{1}{\ln\left(1+e^{-1/x}\right)}\right)\left(\frac{1}{1+e^{1/x}}\right) = \lim_{y\to 0^+} -\left(\frac{1}{\ln\left(1+y\right)}\right)\left(\frac{1}{1+1/y}\right) \\
= & \lim_{y\to 0^+} -\frac{1}{\ln\left(1+y\right)+\ln\left(1+y\right)^{1/y}} = -1 \Rightarrow \lim_{x\to 0^+}\left(\ln\left(1+e^{-1/x}\right)\right)^x = e^{-1} = \frac{1}{e}.
\end{aligned}$$

Exercize

Compute the limit

$$\lim_{x\to\infty} \frac{5\,x + \sin(x) + \ln\left(\sqrt{x}\right)}{3\,x}$$

Solution. Blindly applying De L'Hôpital gives

$$\lim_{x\to\infty} \frac{5\,x + \sin(x) + \ln\left(\sqrt{x}\right)}{3\,x} \overset{H}{=} \lim_{x\to\infty} \frac{5 + \cos(x) + \frac{1}{\sqrt{x}}\,\frac{1}{2\,\sqrt{x}}}{3}.$$

which does not exist! Nevertheless

$$\lim_{x\to\infty} \frac{5\,x}{3\,x} = \frac{5}{3}, \qquad \lim_{x\to\infty} \frac{\sin(x)}{3\,x} = 0.$$

$$\lim_{x\to\infty} \frac{\ln\left(\sqrt{x}\right)}{3\,x} = \frac{+\infty}{+\infty} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}\,\frac{1}{2\sqrt{x}}}{3} = 0,$$

whence

$$\lim_{x\to\infty} \frac{5\,x + \sin(x) + \ln\left(\sqrt{x}\right)}{3\,x} = \frac{5}{3}.$$

Exercize

Compute the limit

$$\lim_{x\to 1^+}\left(\frac{1}{x^2-1}-\frac{1}{\ln x}\right).$$

Solution.

$$\begin{aligned}
\lim_{x\to 1^+}\left(\frac{1}{x^2-1}-\frac{1}{\ln x}\right) &= +\infty-\infty\\
&= \lim_{x\to 1^+}\frac{\ln x-(x^2-1)}{(x^2-1)\ln x}=\frac{0}{0}\\
&\overset{H}{=} \lim_{x\to 1^+}\frac{\frac{1}{x}-2x}{2x\ln x+\frac{x^2-1}{x}}=\frac{1-2}{0}=-\infty.
\end{aligned}$$

Exercize

Establish for which values of $\alpha > 0$ the following limit exists and it is finite:

$$\lim_{x\to 1^-} \frac{\arcsin(x) - \frac{\pi}{2}}{(1-x^2)^\alpha}.$$

Solution. Recall that $\arcsin(1) = \frac{\pi}{2}$. Whence

$$\begin{aligned}
\lim_{x\to 1^-} \frac{\arcsin(x) - \frac{\pi}{2}}{(1-x^2)^\alpha} &= \frac{0}{0} \stackrel{H}{=} \lim_{x\to 1^-} \frac{\left(\arcsin(x) - \frac{\pi}{2}\right)'}{((x-1)^\alpha)'} \\
&= \lim_{x\to 1^-} \frac{1}{\sqrt{1-x^2}} \frac{1}{\alpha\,(1-x^2)^{\alpha-1}\,(-2\,x)} \\
&= \lim_{x\to 1^-} \frac{1}{\alpha\,(1-x^2)^{\alpha-\frac{1}{2}}\,(-2\,x)}.
\end{aligned}$$

so if $\alpha > 0$ it must be $\alpha \leq \frac{1}{2}$ to have a finite limit.

Exercize

Establish for which values of $\alpha > 0$ the following limit exists and it is finite:

$$\lim_{x \to +\infty} \frac{\arctan(x) - \frac{\pi}{2}}{\ln\left(1 + \frac{1}{x^\alpha}\right)}.$$

Solution. Recall that $\lim_{x \to +\infty} \arctan(x) = \frac{\pi}{2}$. Whence

$$\begin{aligned}
\lim_{x \to +\infty} \frac{\arctan(x) - \frac{\pi}{2}}{\ln\left(1 + \frac{1}{x^\alpha}\right)} &= \frac{0}{0} \overset{H}{=} \lim_{x \to +\infty} \frac{\left(\arctan(x) - \frac{\pi}{2}\right)'}{\left(\ln\left(1 + \frac{1}{x^\alpha}\right)\right)'} \\
&= \lim_{x \to +\infty} \frac{1}{1+x^2} \left(1 + \frac{1}{x^\alpha}\right) \frac{1}{-\alpha\, x^{-\alpha-1}} \\
&= \lim_{x \to +\infty} \frac{1}{1+x^2} \left(1 + \frac{1}{x^\alpha}\right) \frac{x^{\alpha+1}}{-\alpha}.
\end{aligned}$$

so, since $\alpha > 0$, the answer is $\alpha + 1 \leq 2 \Rightarrow \alpha \leq 1$.

Exercize

Compute the limit

$$\lim_{x\to 0}\frac{\arctan(x)}{\ln(1+x)}.$$

Solution.

$$\begin{aligned}\lim_{x\to 0}\frac{\arctan(x)}{\ln(1+x)} &= \frac{0}{0} \stackrel{H}{=} \lim_{x\to 0}\frac{(\arctan(x))'}{(\ln(1+x))'}\\ &= \lim_{x\to 0}\frac{\frac{1}{1+x^2}}{\frac{1}{1+x}}\\ &= \lim_{x\to 0}\frac{1+x}{1+x^2} = 1.\end{aligned}$$

Exercize

Compute the limit

$$\lim_{x\to 0} \frac{\arctan\left(x^2\right)}{\ln\left(1+x\right)}.$$

Solution.

$$\begin{aligned}
\lim_{x\to 0} \frac{\arctan\left(x^2\right)}{\ln\left(1+x\right)} &= \frac{0}{0} \overset{H}{=} \lim_{x\to 0} \frac{\left(\arctan\left(x^2\right)\right)'}{\left(\ln\left(1+x\right)\right)'} \\
&= \lim_{x\to 0} \frac{\frac{2x}{1+x^2}}{\frac{1}{1+x}} \\
&= \lim_{x\to 0} \frac{2x\left(1+x\right)}{1+x^2} = 0.
\end{aligned}$$

Exercize

Compute, as a function of α, the following limit

$$\lim_{x\to 0} \frac{\sin\left(x^2\right)}{x^\alpha}.$$

Solution.

$$\lim_{x\to 0} \frac{\sin\left(x^2\right)}{x^\alpha} = \frac{0}{0} \overset{H}{=} \lim_{x\to 0} \frac{\left(\sin\left(x^2\right)\right)'}{\left(x^\alpha\right)'} = \lim_{x\to 0} \frac{\cos\left(x^2\right)\, 2\,x}{\alpha\, x^{\alpha-1}} = \lim_{x\to 0} 2\, \frac{\cos\left(x^2\right)}{\alpha\, x^{\alpha-2}}.$$

In summary

$$\lim_{x\to 0} \frac{\sin\left(x^2\right)}{x^\alpha} = \begin{cases} 0 & \text{if } \alpha < 2 \\ 1 & \text{if } \alpha = 2 \\ +\infty & \text{if } \alpha > 2 \end{cases}$$

Exercize

Compute, as a function of α, the following limit

$$\lim_{x\to 0} \frac{\ln(1+x^\alpha)}{\arcsin(x)}.$$

Solution.

$$\lim_{x\to 0} \frac{\ln(1+x^\alpha)}{\arcsin(x)} = \frac{0}{0} \overset{H}{=} \lim_{x\to 0} \frac{(\ln(1+x^\alpha))'}{(\arcsin(x))'} = \lim_{x\to 0} \frac{\frac{\alpha\, x^{\alpha-1}}{1+x^\alpha}}{\frac{1}{(1+x^2)^{1/2}}}$$

$$= \lim_{x\to 0} \frac{\alpha\, x^{\alpha-1}\sqrt{1+x^2}}{1+x^\alpha} = \lim_{x\to 0} \frac{\alpha\sqrt{1+x^2}}{x^{1-\alpha}+x}$$

In summary

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^\alpha} = \begin{cases} +\infty & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha > 1 \end{cases}$$