1 Domains

Exercize 1. Find the domain of the following function

$$f(x) = \ln(\ln(x)).$$

Exercize 2. Find the domain of the following function

$$f(x) = \ln(\ln(\ln(x))).$$

Exercize 3. Find the domain of the following function

$$f(x) = e^{\frac{\sqrt{x}}{x-2}}.$$

Exercize 4. Find the domain of the following function

$$f(x) = \frac{\sqrt{x}+\sqrt{1-x}}{\sqrt{x-2}}.$$

2 Limits

Recall the operations with infinity

$$\begin{aligned}
\boldsymbol{a} + \infty = +\infty + \boldsymbol{a} &= +\infty, && \boldsymbol{a} \neq -\infty \\
\boldsymbol{a} - \infty = -\infty + \boldsymbol{a} &= -\infty, && \boldsymbol{a} \neq +\infty \\
\boldsymbol{a} \cdot (\pm\infty) = \pm\infty \cdot \boldsymbol{a} &= \pm\infty, && \boldsymbol{a} \in (\mathbf{0}, +\infty] \\
\boldsymbol{a} \cdot (\pm\infty) = \pm\infty \cdot \boldsymbol{a} &= \mp\infty, && \boldsymbol{a} \in [-\infty, \mathbf{0}) \\
\frac{\boldsymbol{a}}{\pm\infty} &= \mathbf{0}, && \boldsymbol{a} \in \mathbb{R} \\
\frac{\pm\infty}{\boldsymbol{a}} &= \pm\infty, && \boldsymbol{a} \in (\mathbf{0}, +\infty) \\
\frac{\pm\infty}{\boldsymbol{a}} &= \mp\infty, && \boldsymbol{a} \in (-\infty, \mathbf{0})
\end{aligned}$$

and the Figure 1 that lists the most common indeterminate forms and the transformations for applying l'Hopital's rule.

Exercize 5. Compute the limit

$$\lim_{x\to\infty}\left(\frac{x-1}{x^2}\right)^{x^2}.$$

Exercize 6. Compute the limit

$$\lim_{x\to 0}\left(1+\frac{\sin x^2}{x}\right)^{1/x}.$$

Indeterminate form	Conditions	Transformation to 0/0	Transformation to ∞/∞
0/0	$\lim_{x\to c} f(x)=0,\ \lim_{x\to c} g(x)=0$	–	$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{1/g(x)}{1/f(x)}$
∞/∞	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{1/g(x)}{1/f(x)}$	–
$0\times\infty$	$\lim_{x\to c} f(x)=0,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c} f(x)g(x)=\lim_{x\to c}\frac{f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)g(x)=\lim_{x\to c}\frac{g(x)}{1/f(x)}$
$\infty-\infty$	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c}(f(x)-g(x))=\lim_{x\to c}\frac{1/g(x)-1/f(x)}{1/(f(x)g(x))}$	$\lim_{x\to c}(f(x)-g(x))=\ln\lim_{x\to c}\frac{e^{f(x)}}{e^{g(x)}}$
0^0	$\lim_{x\to c} f(x)=0^+, \lim_{x\to c} g(x)=0$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$
1^∞	$\lim_{x\to c} f(x)=1,\ \lim_{x\to c} g(x)=\infty$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$
∞^0	$\lim_{x\to c} f(x)=\infty,\ \lim_{x\to c} g(x)=0$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{g(x)}{1/\ln f(x)}$	$\lim_{x\to c} f(x)^{g(x)}=\exp\lim_{x\to c}\frac{\ln f(x)}{1/g(x)}$

Fig. 1: Common indeterminate forms and the transformations for applying l'Hopital's rule

Exercize 7. Consider the function

$$f(x)=\begin{cases}1 & \text{if } x=0\\ x^2 & \text{if } x\neq 0\end{cases}$$

compute $\lim_{x\to 0} f(x)$.

Exercize 8. Compute the limit

$$\lim_{x\to 0}\ln\left(\left|\frac{\sin x}{x}\right|\right)$$

Exercize 9. Compute the limit

$$\lim_{x\to 0^+} x^{\sin x}$$

3 Series

Exercize 10. Establish if

$$\sum_{n=1}^{\infty} 2(\sqrt{n}-\sqrt{n-1})-\frac{1}{\sqrt{n}}$$

converges or not.

Remember that $\sum_n \frac{1}{n^p}$ *converges* $\Leftrightarrow p>1$.

Exercize 11. For which values of x the series

$$\sum_{n=1}^{\infty}\frac{n!\,x^n}{n^n} \tag{3.1}$$

converges? For which values of x it diverges?

4 Taylor's expansions

Exercize 12. Using a Taylor's expansion of $\ln(1+x)$ around $x = 0$ truncated at the third order compute an approximation for the number $\ln(2)$.

5 Graphs of functions

Exercize 13. Find the domain, vertical asymptotes, horizontal asymptotes and the intersection with the $x = 0$ and $y = 0$ axes, plus study the sign, monotonicity, maxima, minima, concavity and convexity of the function

$$f(x) = x^2 e^{-x}.$$

Exercize 14. Find the domain, vertical asymptotes, horizontal asymptotes and the intersection with the $x = 0$ and $y = 0$ axes, plus study the sign, monotonicity, maxima, minima, concavity and convexity of the function

$$f(x) = \frac{x - x^3}{1 + x^2}.$$

Exercize 15. Find the domain, vertical asymptotes, horizontal asymptotes and the intersection with the $x = 0$ and $y = 0$ axes, plus study the sign, monotonicity, maxima, minima, concavity and convexity of the function

$$f(x) = 2x + \ln\left(\frac{1 - x}{1 + x}\right).$$

Exercize 16. Find the domain, vertical asymptotes, horizontal asymptotes and the intersection with the $x = 0$ and $y = 0$ axes, plus study the sign, monotonicity, maxima, minima, concavity and convexity of the function

$$f(x) = \frac{\sqrt{x}}{1 + \ln(x)}$$

6 Solutions

6.1 Domains

Solution of Exercise 1 The function $\ln(\ln(x))$ is the composition of two functions

$$x \longrightarrow \ln(x) \longrightarrow \ln(\ln(x)).$$

This composition is defined for all x such that $\ln(x) > 0$ and hence the domain is

$$D = (1, +\infty).$$

Solution of Exercise 2 The function $\ln(\ln(\ln(x)))$ is the composition of three functions

$$x \longrightarrow \ln(x) \longrightarrow \ln(\ln(x)) \longrightarrow \ln(\ln(\ln(x))).$$

This composition is defined for all x such that $\ln(\ln(x)) > 0$ hence it must be that $\ln(x) > 1$ that is $x > e$ where e is the Neper number. So the domain is

$$D = (e, +\infty).$$

Solution of Exercise 3 The function $e^{\frac{\sqrt{x}}{x-2}}$ is defined whenever the argument of the exponential function is defined (remember that the exponential function is defined everywhere), hence it must be that $x > 0$ in order to have $\sqrt{x}$ defined and moreover it must be $x \neq 2$ in order to have the fraction $1/(x-2)$ defined. Hence the domain is

$$D = (0, 2) \bigcup (2, \infty).$$

Solution of Exercise 3 The function $\frac{\sqrt{x}+\sqrt{1-x}}{\sqrt{x-2}}$ is defined in all x such that the three functions $\sqrt{x}$, $\sqrt{1-x}$ and $1/\sqrt{x-2}$ are defined. Let's analyze them separately. The function $\sqrt{x}$ is defined for $x \geq 0$. The function $\sqrt{1-x}$ is defined for $x \leq 1$. The function $1/\sqrt{x-2}$ is defined for $x > 2$ (note that I am not writing $x \geq 2$ since the denominator must be different from zero). Hence the original function $\frac{\sqrt{x}+\sqrt{1-x}}{\sqrt{x-2}}$ is defined for all x such that $x \geq 0$ and $x \leq 1$ and $x > 2$. So the domain of the function $\frac{\sqrt{x}+\sqrt{1-x}}{\sqrt{x-2}}$ is empty.

6.2 Limits

Solution of Exercise 5 Consider that

$$\left(\frac{x-1}{x^2}\right)^{x^2} = e^{\ln\left(\left(\frac{x-1}{x^2}\right)^{x^2}\right)} = e^{x^2 \ln\left(\left(\frac{x-1}{x^2}\right)\right)}$$

Let's study the asymptotic behaviour of the argument of the logarithm that appears above

$$\lim_{x\to+\infty} \frac{x-1}{x^2} = \lim_{x\to\infty} \frac{1-\frac{1}{x}}{x} = \frac{1}{+\infty} = 0$$

hence (remember that $\lim_{y\to 0^+} \ln(y) = -\infty$)

$$\lim_{x\to+\infty} \ln\left(\frac{x-1}{x^2}\right) = -\infty$$

whence

$$\lim_{x\to+\infty} x^2 \ln\left(\frac{x-1}{x^2}\right) = (+\infty) \times (-\infty) = -\infty$$

so that

$$\lim_{x\to\infty} \left(\frac{x-1}{x^2}\right)^{x^2} = \lim_{x\to\infty} e^{x^2 \ln\left(\left(\frac{x-1}{x^2}\right)\right)} = e^{-\infty} = 0.$$

Solution of Exercise 6 Consider that

$$\lim_{x\to 0} \frac{\sin\left(x^2\right)}{x} = \lim_{x\to 0} \frac{\sin\left(x^2\right)}{x^2}\, x$$

but

$$\lim_{x\to 0} \frac{\sin\left(x^2\right)}{x^2} = \lim_{y\to 0} \frac{\sin(y)}{y} = 1$$

so that

$$\lim_{x\to 0} \frac{\sin\left(x^2\right)}{x} = \lim_{x\to 0} \frac{\sin\left(x^2\right)}{x^2}\, x = 0.$$

Hence the limit

$$\lim_{x\to 0} \left(1 + \frac{\sin\left(x^2\right)}{x}\right)^{1/x}$$

is a 1^∞ indeterminate form. Re-write the limit as

$$\lim_{x\to 0} \left(1 + \frac{\sin\left(x^2\right)}{x}\right)^{1/x} = \lim_{x\to 0} e^{\ln\left(\left(1+\frac{\sin\left(x^2\right)}{x}\right)^{1/x}\right)} = \lim_{x\to 0} e^{\frac{1}{x} \ln\left(1+\frac{\sin x^2}{x}\right)}.$$

Now consider the exponent and apply Hopital's rule

$$\lim_{x\to 0} \frac{1}{x} \ln\left(1 + \frac{\sin\left(x^2\right)}{x}\right) = \frac{0}{0} \overset{H}{=} \lim_{x\to 0} \frac{1}{1+\frac{\sin x^2}{x}} \left(\underbrace{\frac{2 \sin x \cos x}{x}}_{\to 2} - \underbrace{\frac{\sin\left(x^2\right)}{x^2}}_{\to 1}\right) = 1$$

so that

$$\lim_{x\to 0}\left(1+\frac{\sin\left(x^2\right)}{x}\right)^{1/x}=e.$$

Solution of Exercise 7 Remember that in the definition of the limit of a function

$$\lim_{x\to x_0} f(x)=L \Leftrightarrow \forall\epsilon>0\exists\delta>0 \text{ such that } \forall x \text{ such that } 0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon.$$

since it is required that $|x-x_0|>0$ the value of the function in x_0, that is $f(x_0)$, does not enter in the definition. In other words, what really matters is the behaviour of the function around x_0, irrespectively of the value of the function in x_0. So consider the function

$$f(x)=\begin{cases}1 & \text{if } x=0\\ x^2 & \text{if } x\neq 0\end{cases}$$

the idea is that, for all $x\neq 0$, if x is very close to 0 then also $f(x)=x^2$ is very close to zero, so the limit

$$\lim_{x\to 0} f(x)$$

is exactly 0. Let's try to verify this claim using the definition

$$\forall\epsilon>0\exists\delta>0 \text{ such that } \forall x \text{ such that } 0<|x|<\delta \Rightarrow \left|x^2\right|<\varepsilon,$$

which is true. In fact it is enough to take, $\forall\epsilon>0$, any $\delta<\sqrt{\epsilon}$ so that if $0<|x|<\delta$ we have $0<x^2<\delta^2<\epsilon$.

Solution of Exercise 8 First remember the notable limit

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

which, by continuity of the absolute value, implies

$$\lim_{x\to 0}\left|\frac{\sin x}{x}\right|=|1|=1$$

which, by continuity of the logarithm, implies

$$\lim_{x\to 0}\ln\left(\left|\frac{\sin x}{x}\right|\right)=\ln(1)=0.$$

Solution of Exercise 9 Note that

$$x^{\sin x}=e^{\ln\left(x^{\sin x}\right)}=e^{\sin x\,\ln x}$$

now consider that

$$\lim_{x\to 0^+}\sin x\,\ln x$$

is a $0 \times (-\infty)$ indeterminate form. Re-write $\sin x \ln x$ as

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\sin x}{\frac{1}{\ln x}} = \frac{0}{0} \overset{H}{=} \lim_{x\to 0^+} \frac{\cos x}{-\frac{1}{x(\ln x)^2}} = \lim_{x\to 0^+} -\left(x \cos x \, (\ln x)^2\right).$$

Again we have a $0 \times \infty$ indeterminate form....

$$\lim_{x\to 0^+} x\,(\ln x)^2 = \lim_{x\to 0^+} \frac{(\ln x)^2}{\frac{1}{x}} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{2\ln x}{x}}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0^+} \frac{2\ln x}{\left(-\frac{1}{x}\right)} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{2}{x}}{\frac{1}{x^2}} = \lim_{x\to 0^+} 2x = 0$$

and so

$$\lim_{x\to 0^+} \sin x \ln x = 0$$

and finally

$$\lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} e^{\sin x \ln x} = e^{\lim_{x\to 0^+} \sin x \ln x} = 1.$$

6.3 Series

Solution of Exercise 10 Consider that

$$\left(2\sqrt{n} - 2\sqrt{n-1} - \frac{1}{\sqrt{n}}\right) = \frac{1}{\sqrt{n}}\left(2n - 2\sqrt{n}\sqrt{n-1} - 1\right) = \frac{1}{\sqrt{n}}\left(\sqrt{n} - \sqrt{n-1}\right)^2,$$

hence

$$\begin{aligned}
\sum_{k=1}^{n}\left(2\sqrt{k} - 2\sqrt{k-1} - \frac{1}{\sqrt{k}}\right) &= \sum_{k=1}^{n}\frac{1}{\sqrt{k}}\left(\sqrt{k} - \sqrt{k-1}\right)^2 \\
&= \sum_{k=1}^{n}\frac{1}{\sqrt{k}}\left(\frac{(\sqrt{k} - \sqrt{k-1})(\sqrt{k} + \sqrt{k-1})}{(\sqrt{k} + \sqrt{k-1})}\right)^2 \\
&= \sum_{k=1}^{n}\frac{1}{\sqrt{k}\left(\sqrt{k} + \sqrt{k-1}\right)^2} \sim \sum_{k=1}^{n}\frac{1}{\sqrt{k}\,k} < \infty.
\end{aligned}$$

Solution of Exercise 11 The series trivially converges if $x = 0$. Assume now $x \neq 0$. Apply the ratio criterion

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!\,|x|^{n+1}}{(n+1)^{n+1}}\,\frac{n^n}{n!\,|x|^n} = |x|\left(\frac{n}{n+1}\right)^n = |x|\left(\frac{1}{\frac{n+1}{n}}\right)^n = \frac{|x|}{\left(1+\frac{1}{n}\right)^n}.$$

Hence the series absolute converges (and hence converges) if $|x| < e$. Nevertheless if $|x| \geq e$ we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|}{\left(1+\frac{1}{n}\right)^n} \geq \frac{e}{\left(1+\frac{1}{n}\right)^n} \geq 1.$$

Therefore $|a_n|$ is increasing and it cannot happen that $|a_n| \to 0$, and hence it cannot happen that $a_n \to 0$, hence the necessary condition is not satisfied. Summarizing

$$\sum_{n=1}^{\infty}\frac{n!\,x^n}{n^n} < \infty \Leftrightarrow x \in (-e, e).$$

6.4 Taylor's expansions

Solution of Exercise 12 We want to use the formula

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{1}{2!} f''(x_0)\,(x - x_0)^2 + \frac{1}{3!} f'''(x_0)\,(x - x_0)^3 + o\left((x - x_0)^4\right)$$

using $x_0 = 0$, with $f(x) = \ln(1 + x)$ and neglecting the error term $o\left((x - x_0)^4\right)$. So first not that $f(0) = \ln(1) = 0$ and then compute the derivatives

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1,$$

$$f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f'(0) = -1,$$

$$f'''(x) = 2\,\frac{1}{(1+x)^3} \Rightarrow f'(0) = 2,$$

whence

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{2}{3!}x^3 + o\left(x^4\right).$$

Neglecting the error term $o\left(x^4\right)$ and computing the formula above for $x = 1$ we get

$$\ln(2) \approx 1 - \frac{1}{2} + \frac{1}{3} = \frac{6-3+2}{6} = \frac{5}{6}.$$

6.5 Graphs of Functions

Solution of Exercise 13

- **Domain**. The function $x^2 e^{-x}$ is the product of the function x^2 with e^{-x} and they are both defined everywhere on the real line, so the domain D of the function is $D = \mathbb{R}$.
- **Asymptotes**. There are no vertical asymptotes since the function has no critical points. Now consider the limit

$$\lim_{x\to+\infty} x^2 e^{-x} = \infty \cdot 0 = \lim_{x\to+\infty} \frac{x^2}{\frac{1}{e^{-x}}} = \frac{\infty}{\infty} \overset{H}{=} \lim_{x\to+\infty} \frac{2\,x}{e^x} = \frac{\infty}{\infty} \overset{H}{=} \lim_{x\to+\infty} \frac{2}{e^x} = 0.$$

$$\lim_{x\to-\infty} x^2 e^{-x} = \lim_{x\to+\infty} x^2 e^{x} = (+\infty)\cdot(+\infty) = +\infty.$$

so $x = 0$ is an horizontal asymptote for $x \to +\infty$.

- **Intersection with** $x = 0$. If $x = 0$ we get $f(0) = 0$.

- **Intersection with** $y = 0$. The equation

$$x^2 e^{-x} = 0$$

is equivalent to

$$x^2 = 0$$

and this is because $e^{-x} > 0$ for all x. Hence the function intersects the axis $y = 0$ only in $x = 0$.

- **Sign**. Since, trivially, $x^2 \geq 0$ and $e^{-x} \geq 0$ we have that $f(x) \geq 0$ for all x.

- **Monotonicity**. Compute the first derivative

$$f'(x) = 2x e^{-x} - x^2 e^{-x} = e^{-x} x (2 - x).$$

Hence

$$f'(x) \geq 0 \Leftrightarrow x(2 - x) \geq 0$$

whence $f'(x) \geq 0$ if $x \in [0, 2]$, so the function is decreasing in $(-\infty, 0]$, increasing in $[0, 2]$ and decreasing in $[2, \infty)$

- **Maxima and minima**. Since $f'(0) = 0$ and $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $0 < x < 2$ we have that $x = 0$ is a minimum. Similarly since $f'(2) = 0$ and $f'(x) > 0$ for $0 < x < 2$ and $f'(x) < 0$ for $x > 2$ we have that $x = 2$ is a maximum.

- **Concavity and convexity**. Consider the second derivative

$$f''(x) = -e^{-x} x (2 - x) + e^{-x} (2 - x) - e^{-x} x = e^{-x}(x^2 - 4x + 2).$$

Consider the roots of the polynomial $x^2 - 4x + 2$

$$x_{1,2} = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

so $f''(x) > 0$, and hence f is convex, for $x \in \left(-\infty, 2 - \sqrt{2}\right)$ or $x \in \left(2 + \sqrt{2}, \infty\right)$. Viceversa $f''(x) < 0$, and hence f is concave, if $x \in \left(2 - \sqrt{2}, 2 + \sqrt{2}\right)$.

- **Graph.** See Figure 2.

Solution of Exercise 14

- **Domain**. Since the numerator is a polynomial and the denominator is $1 + x^2 > 0$ for all x then the domain D of the function is $D = \mathbb{R}$.

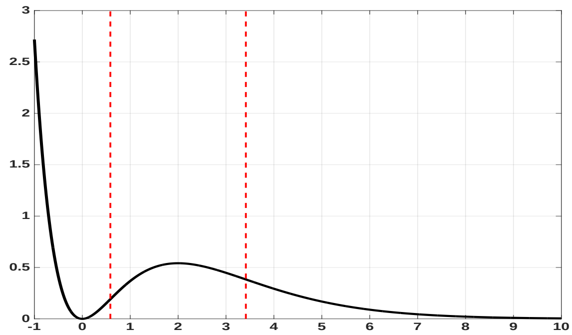

Fig. 2: The red dotted lines indicate the position of the points $2+\sqrt{2}$ and $2-\sqrt{2}$.

- **Asymptotes**. There are no vertical asymptotes since the function has no critical points. Now consider the limit

$$\lim_{x\to+\infty}\frac{x-x^3}{1+x^2}=\lim_{x\to+\infty}\frac{x^3\left(\frac{1}{x^2}-1\right)}{x^2\left(\frac{1}{x^4}+1\right)}=-\infty$$

$$\lim_{x\to-\infty}\frac{x-x^3}{1+x^2}=\lim_{x\to-\infty}\frac{x^3\left(\frac{1}{x^2}-1\right)}{x^2\left(\frac{1}{x^4}+1\right)}=+\infty$$

so there are no horizontal asymptotes.

- **Intersection with** $x=0$. If $x=0$ we get $f(0)=0$.

- **Intersection with** $y=0$. The equation

$$\frac{x-x^3}{1+x^2}=0$$

is equivalent to

$$x-x^3=x\left(1-x^2\right)=0$$

and this is because $1+x^2>0$ for all x. Hence the function intersects the axis $y=0$ in $x=0$ and $x=\pm1$.

- **Sign**. Since, trivially, $1+x^2\geq0$ we have that $f(x)\geq0$ for all x such that $x\left(1-x^2\right)\geq0$ hence for all x such that $x\in(-\infty,-1]$ or $x\in[0,1]$.

- **Monotonicity**. Compute the first derivative

$$f'(x)=-\frac{x^4+4x^2-1}{(x^2+1)^2}$$

Hence

$$f'(x)\geq0\Leftrightarrow x^4+4x^2-1\leq0.$$

Put $x^2 = t$ and find the solution of

$$t^2 + 4t - 1 = 0$$

which are

$$t_{1,2} = -2 \pm \sqrt{5}.$$

Since $t = x^2 \geq 0$ only the solution $-2+\sqrt{5}$ is acceptable. So the equation

$$x^4 + 4x^2 - 1 = 0$$

has the two real solutions $-\sqrt{-2+\sqrt{5}}$ and $+\sqrt{-2+\sqrt{5}}$. So the function is decreasing in $\left(-\infty, -\sqrt{-2+\sqrt{5}}\right]$, increasing in $\left[-\sqrt{-2+\sqrt{5}}, +\sqrt{-2+\sqrt{5}}\right]$ and decreasing in $\left[+\sqrt{-2+\sqrt{5}}, \infty\right)$

- **Maxima and minima**. Since $f'\left(-\sqrt{-2+\sqrt{5}}\right) = 0$ and $f'(x) < 0$ for $x < -\sqrt{-2+\sqrt{5}}$ and $f'(x) > 0$ for $-\sqrt{-2+\sqrt{5}} < x < \sqrt{-2+\sqrt{5}}$ we have that $x = -\sqrt{-2+\sqrt{5}}$ is a minimum. Similarly since $f'\left(\sqrt{-2+\sqrt{5}}\right) = 0$ and $f'(x) > 0$ for $-\sqrt{-2+\sqrt{5}} < x < \sqrt{-2+\sqrt{5}}$ and $f'(x) < 0$ for $x > \sqrt{-2+\sqrt{5}}$ we have that $x = \sqrt{-2+\sqrt{5}}$ is a maximum.

- **Concavity and convexity**. Consider the second derivative

$$f''(x) = \frac{4x\left(x^2-3\right)}{\left(x^2+1\right)^3}$$

so $f''(x) > 0$, and hence f is convex, for $x \in \left(-\sqrt{3}, 0\right)$ or $x \in \left(\sqrt{3}, \infty\right)$. Viceversa $f''(x) < 0$, and hence f is concave, if $x \in \left(0, \sqrt{3}\right)$.

- **Graph.** See Figure 3.

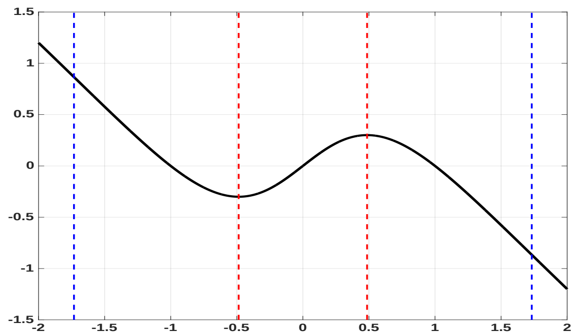

Fig. 3: The red dotted lines indicate the position of the points $-\sqrt{-2+\sqrt{5}}$ and $\sqrt{-2+\sqrt{5}}$. The blue dotted lines indicate the position of $-\sqrt{3}$ and $\sqrt{3}$.

Solution of Exercise 15

- **Domain**. The domain of
$$f(x) = 2x + \ln\left(\frac{1-x}{1+x}\right)$$
coincides with the domain of $\ln\left(\frac{1-x}{1+x}\right)$. The logarithmic function $\ln(y)$ is defined if and only if $y > 0$ so we need to impose that $\frac{1-x}{1+x} > 0$ which implies $x \in (-1,1)$ so the domain is $D = (-1,1)$.

- **Asymptotes**. There are two possible vertical asymptotes. Consider the limits in the critical points $x = -1$ and $x = 1$. If $x \to -1^+$ then $\frac{1-x}{1+x} \to +\infty$ and hence
$$\lim_{x\to -1^+}\left(2x + \ln\left(\frac{1-x}{1+x}\right)\right) = +\infty$$
so $x = -1$ is a vertical asymptote. If $x \to 1^-$ then $\frac{1-x}{1+x} \to 0^+$ and hence
$$\lim_{x\to 1^-}\left(2x + \ln\left(\frac{1-x}{1+x}\right)\right) = -\infty$$
so $x = 1$ is a vertical asymptote. We cannot look for horizontal asymptotes given that the domain of f is bounded.

- **Intersection with** $x = 0$. If $x = 0$ we get $f(0) = 0$.

- **Intersection with** $y = 0$. The equation
$$2x + \ln\left(\frac{1-x}{1+x}\right) = 0 \tag{6.1}$$
has at least the solution $x = 0$. From the sign of the derivative we can establish if this solution is unique or note, so let's move forward.

- **Sign**. We cannot say anything on the inequality
$$2x + \ln\left(\frac{1-x}{1+x}\right) \geq 0,$$
again we have to use the sign of the derivative to say more.

- **Monotonicity**. Compute the first derivative
$$f'(x) = -\frac{2x^2}{1-x^2}$$
so $f'(x) < 0$ for all x in the domain of the function, except for $x \neq 0$. This means that the function is **strictly** decreasing in its domain. Since $f(0) = 0$ this means that $x = 0$ is the unique solution of the equation (6.2) and, besides, that $f(x) > 0$ for $x < 0$ and $f(x) < 0$ for $x > 0$.

- **Maxima and minima**. Since f is **strictly** decreasing in its domain there are no maxima and no minima.

- **Concavity and convexity**. Consider the second derivative

$$f''(x) = -\frac{4x}{(x^2-1)^2}$$

so $f''(x) > 0$, and hence f is convex, for $x < 0$. Viceversa $f''(x) < 0$, and hence f is concave, if $x > 0$.

- **Graph.** See Figure 4.

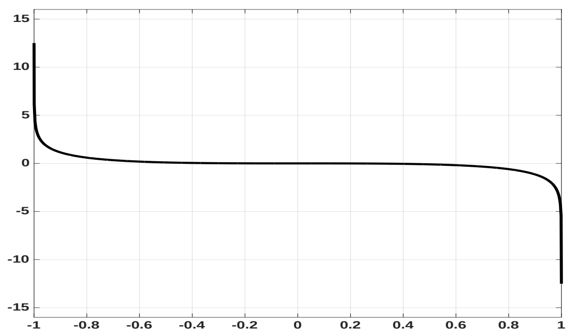

Fig. 4: The graph of $f(x) = 2\,x + \ln\left(\frac{1-x}{1+x}\right)$.

Solution of Exercise 15

- **Domain**. The domain of
$$f(x) = \frac{\sqrt{x}}{1+\ln(x)}$$
is determined by the conditions $x > 0$ (in order to have $\sqrt{x}$ and $\ln(x)$ defined) and $\ln(x) \neq -1$ (in order to have the denominator different from zero), which is equivalent to $x \neq e^{-1} = 1/e$. So the domain is $D = (0, 1/e) \cup (1/e, \infty)$.

- **Asymptotes**. There is one possible vertical asymptote at the critical point $x = 1/e$ and one at the critical point $x = 0$. If $x \to (1/e)^+$ then $1 + \ln(x) \to 0^+$ and hence
$$\lim_{x\to(1/e)^+} \frac{\sqrt{x}}{1+\ln(x)} = +\infty$$
while if $x \to (1/e)^-$ then $1 + \ln(x) \to 0^-$ and hence
$$\lim_{x\to(1/e)^-} \frac{\sqrt{x}}{1+\ln(x)} = -\infty.$$
Hence $x = 1/e$ is a vertical asymptote. Nevertheless since
$$\lim_{x\to0^+} \frac{\sqrt{x}}{1+\ln(x)} = \frac{0}{-\infty} = 0.$$
then $x = 0$ it is not a vertical asymptote. Now consider
$$\lim_{x\to\infty} \frac{\sqrt{x}}{1+\ln(x)} = \frac{\infty}{\infty} \overset{H}{=} \lim_{x\to\infty} \frac{x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1}{2}\sqrt{x} = +\infty,$$
so there are no horizontal asymptotes.

- **Intersection with** $x = 0$. The point $x = 0$ is outside of the domain.

- **Intersection with** $y = 0$. The equation
$$\frac{\sqrt{x}}{1+\ln(x)} = 0 \tag{6.2}$$
has no solution, since the numerator is zero at $x = 0$ but the denominator is not defined at $x = 0$. Nevertheless we already know that the function $\frac{\sqrt{x}}{1+\ln(x)}$ approaches zero as $x \to 0^+$.

- **Sign**. Since $\sqrt{x} \geq 0$ always, the sign of
$$\frac{\sqrt{x}}{1+\ln(x)},$$
is equivalent to the sign of $1+\ln(x)$. Hence $f(x) \geq 0$ if $x \in (1/e, \infty)$ and $f(x) \leq 0$ if $x \in (0, 1/e)$.

- **Monotonicity**. Compute the first derivative

$$f'(x) = \frac{\log(x) - 1}{2\sqrt{x}(\log(x) + 1)^2}$$

so $f'(x) < 0$ for all $x < e$ (f is decreasing) and $f'(x) > 0$ for all $x > e$ (f is increasing).

- **Maxima and minima**. By the considerations above $x = e$ is a minimum and there are no maxima.

- **Concavity and convexity**. Consider the second derivative

$$f''(x) = \frac{7 - \log(x)(\log(x) + 2)}{4x^{3/2}(\log(x) + 1)^3}.$$

In order to find the zeros of $f''(x)$ we have to solve the equation

$$7 - \log(x)(\log(x) + 2) = 0,$$

which, putting $y = \log(x)$, is equivalent to

$$7 - y\,(y + 2) = 0$$

whose solutions are

$$y_{1,2} = -1 \pm 2\sqrt{2}$$

hence the solutions of $7 - \log(x)(\log(x) + 2) = 0$ are

$$x_{1,2} = e^{y_{1,2}} = e^{-1\pm 2\sqrt{2}}.$$

Hence the numerator of $f''(x)$ is positive for $x \in \left(e^{-1-2\sqrt{2}}, e^{-1+2\sqrt{2}}\right)$ while the denominator is positive for $(\log(x) + 1)^3 > 0$ which is equivalent to $\log(x) + 1 > 0$, that is for $x > 1/e$. So combining the sign of the numerator and of the denominator we get $f''(x) > 0$ for $x \in \left(0, e^{-1-2\sqrt{2}}\right)$, $f''(x) < 0$ for $x \in \left(e^{-1-2\sqrt{2}}, 1/e\right)$, $f''(x) > 0$ for $x \in \left(1/e, e^{-1+2\sqrt{2}}\right)$ and $f''(x) < 0$ for $x \in \left(e^{-1+2\sqrt{2}}, \infty\right)$.

- **Graph.** See Figure 4.

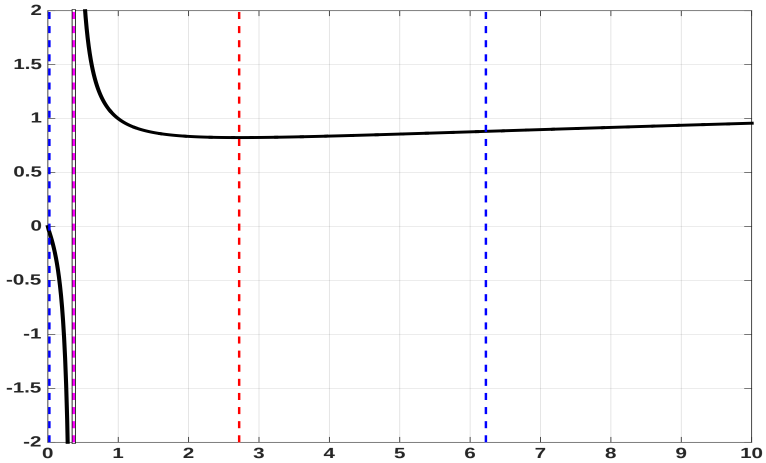

Fig. 5: Blue lines represents the position of the points $e^{-1\pm 2\sqrt{2}}$. The magenta line is the vertical asymptote $x = 1/e$ while the red line is the position of the minimum $x = e$.