Part IV

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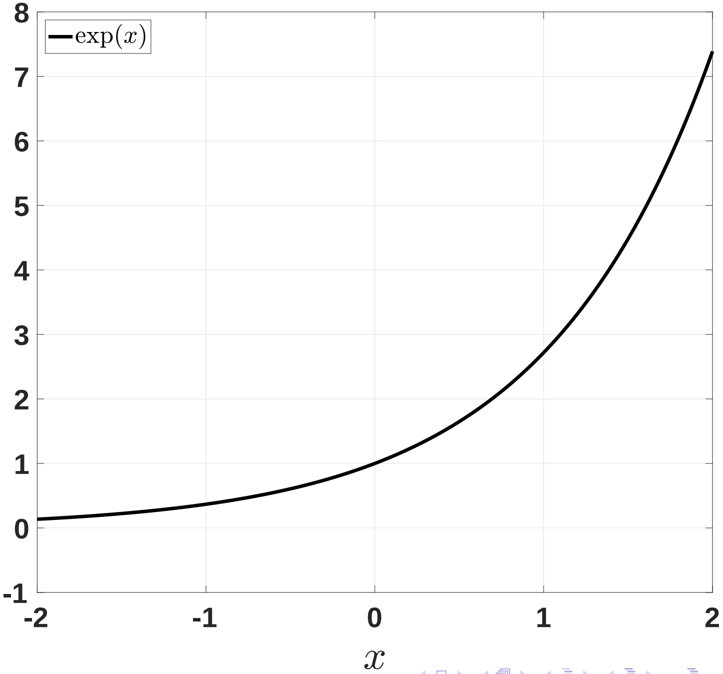
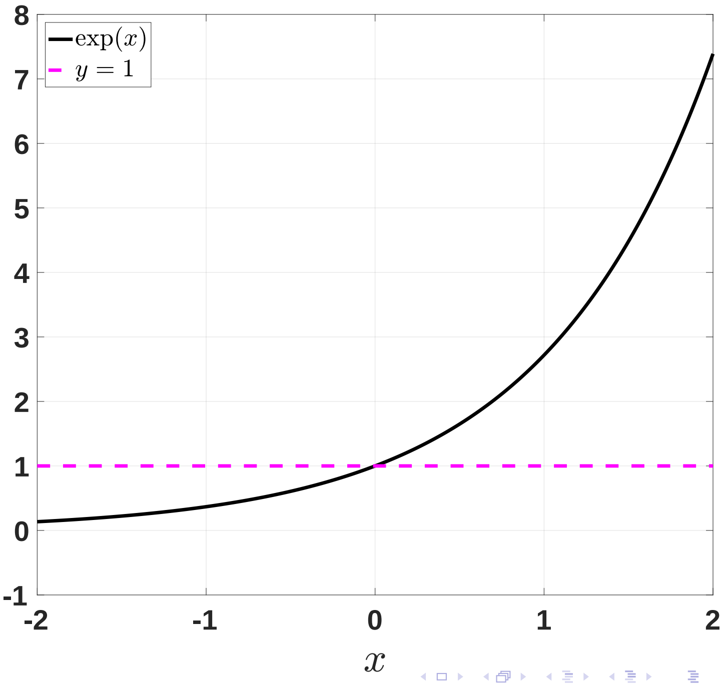
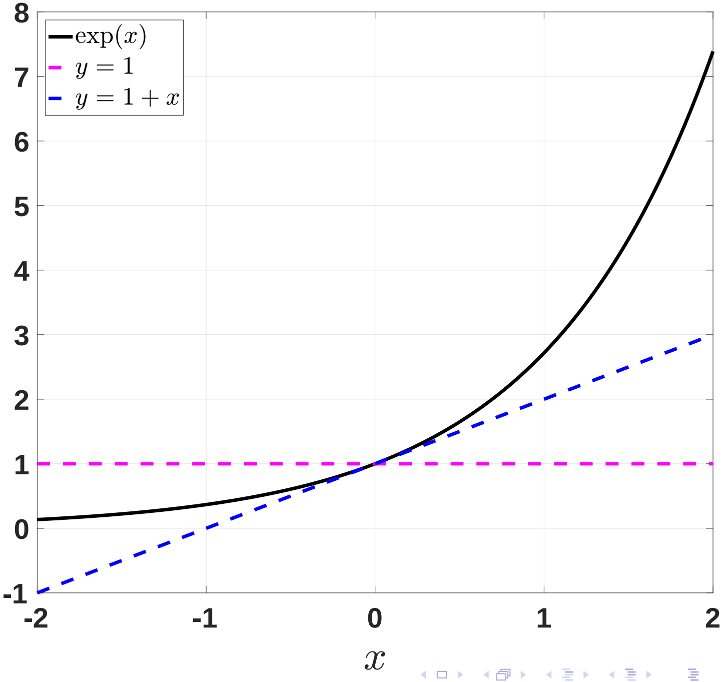
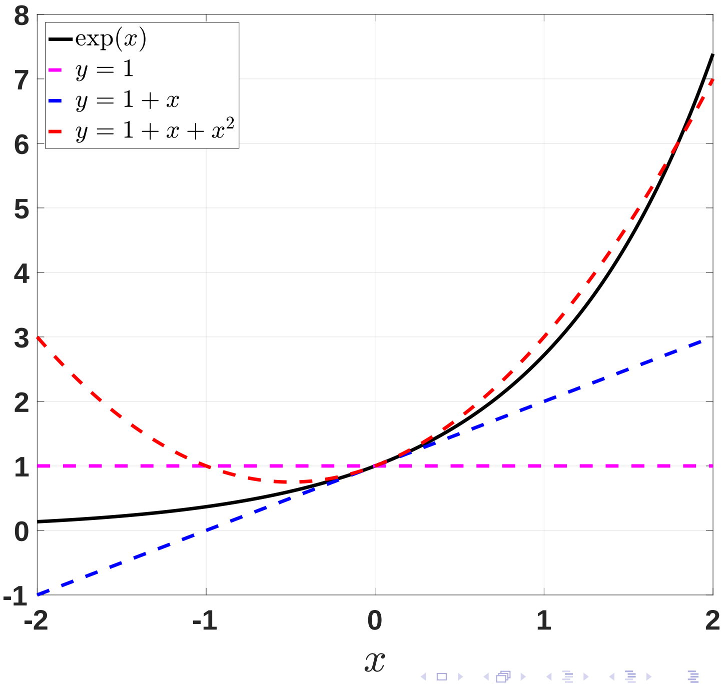
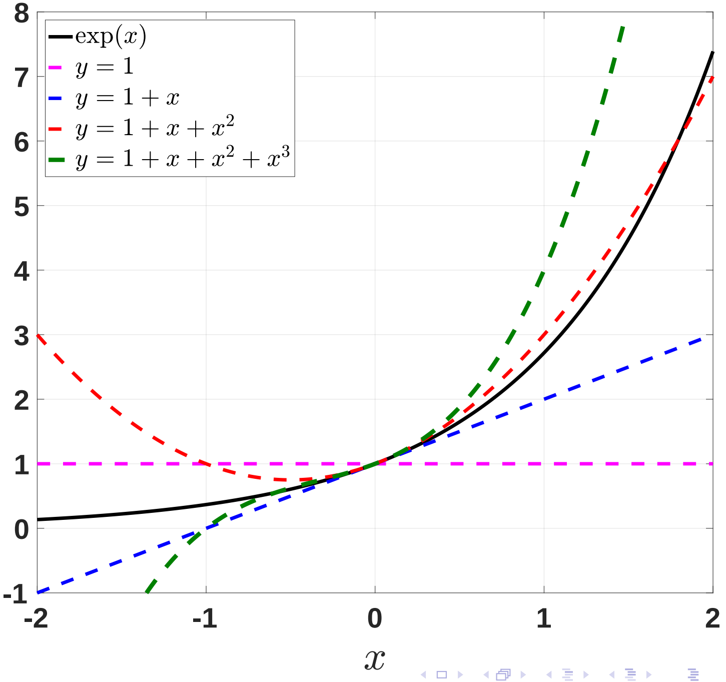

$\exp(x)$
8
7
6
5
4
3
2
1
0
-1
-2
-1
0
1
2
x

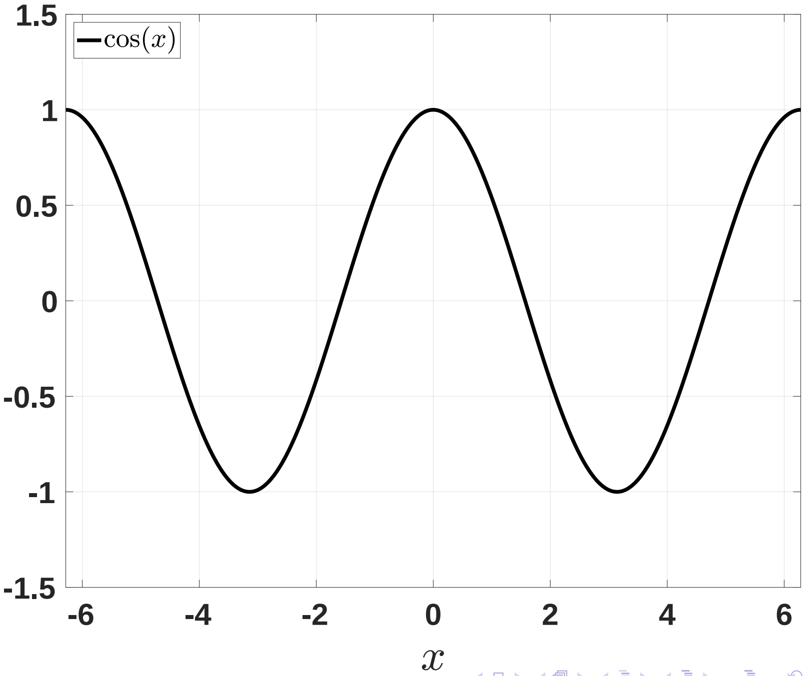

$\exp(x)$
$y = 1$
8
7
6
5
4
3
2
1
0
-1
-2
-1
0
1
2
x

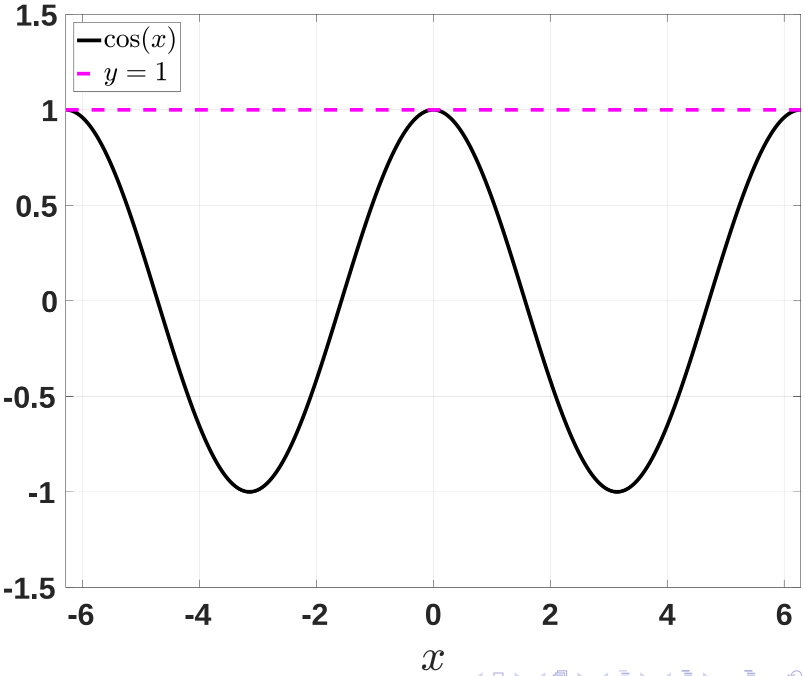

exp(x)
y = 1
y = 1 + x
8
7
6
5
4
3
2
1
0
-1
-2
-1
0
1
2
x


exp(x)
y = 1
y = 1 + x
y = 1 + x + x^2
8
7
6
5
4
3
2
1
0
-1
-2
-1
0
1
2
x

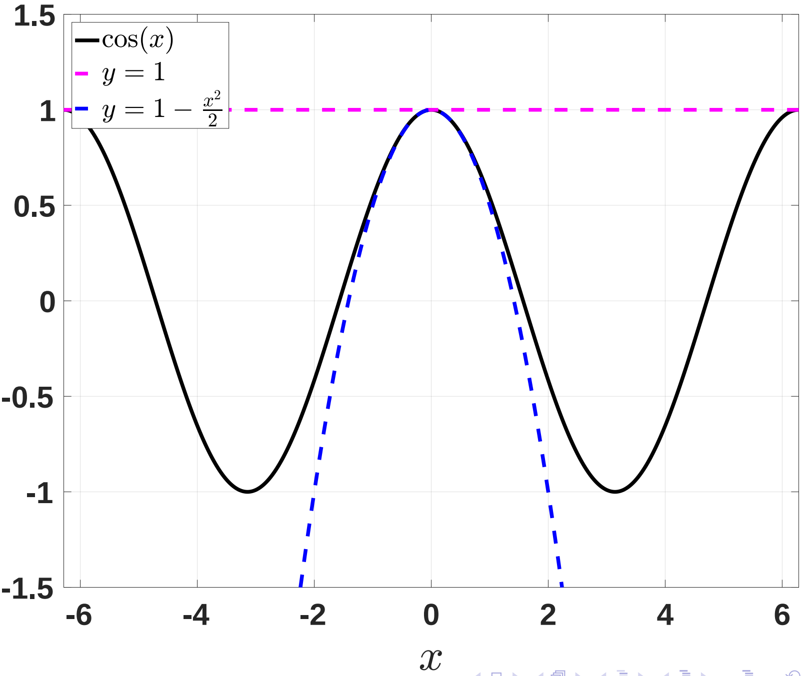


exp(x)
$y = 1$
$y = 1 + x$
$y = 1 + x + x^2$
$y = 1 + x + x^2 + x^3$
8
7
6
5
4
3
2
1
0
-1
-2
-1
0
1
2
x

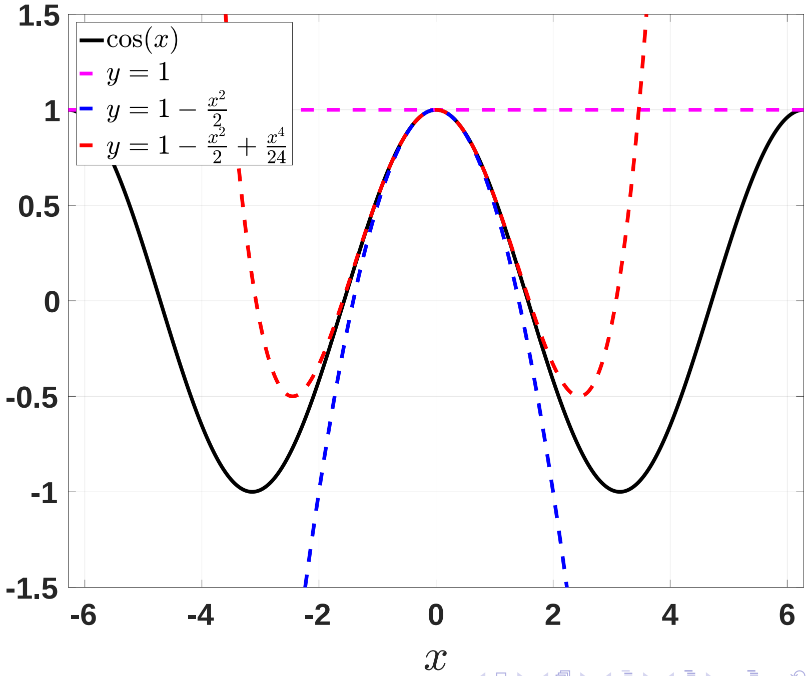

cos(x)
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6
x

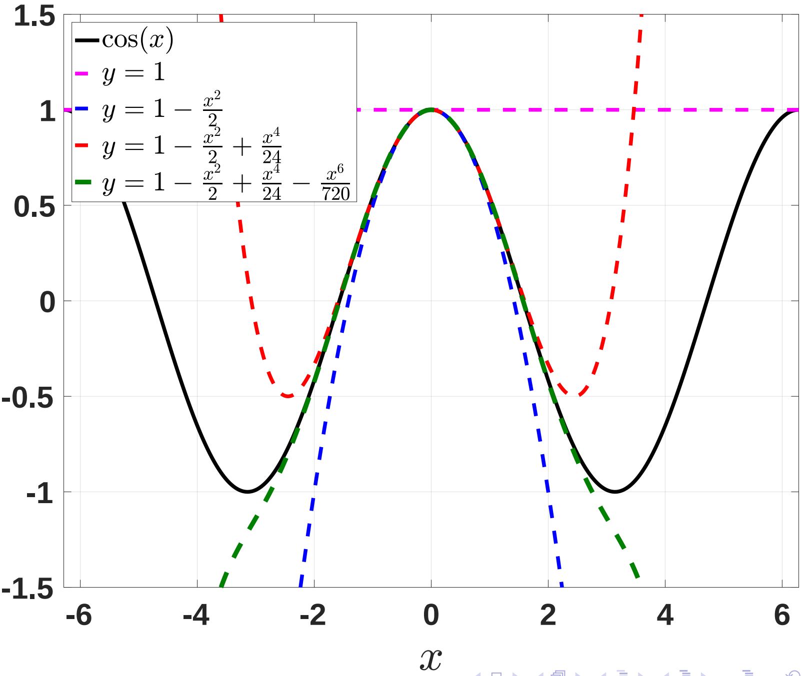


cos(x)
y = 1
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6
x



$\cos(x)$
$y = 1$
$y = 1 - \frac{x^2}{2}$
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6
x


$\cos(x)$
$y = 1$
$y = 1 - \frac{x^2}{2}$
$y = 1 - \frac{x^2}{2} + \frac{x^4}{24}$
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6
x


$\cos(x)$
$y = 1$
$y = 1 - \frac{x^2}{2}$
$y = 1 - \frac{x^2}{2} + \frac{x^4}{24}$
$y = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6
x

The Taylor's formula

Before proceeding...let's talk about "little-o" functions.

Definition

Let $\alpha > 0$. We indicate with $o\left((x - x_0)^\alpha\right)$ (we say "little oh of $(x - x_0)^\alpha$") every function that, as $x \to x_0$, goes to zero faster than $(x - x_0)^\alpha$. In formula

$$\lim_{x \to x_0} \frac{o\left((x - x_0)^\alpha\right)}{(x - x_0)^\alpha} = 0.$$

Example

$$\lim_{x \to 0} \frac{\ln\left(1 + x^2\right)}{x} = \lim_{x \to 0} x \underbrace{\frac{\ln\left(1 + x^2\right)}{x^2}}_{\to 1} = 0 \Rightarrow \ln\left(1 + x^2\right) = o\left(x\right).$$

$$\lim_{x \to 0} \frac{x^2}{x} = 0 \Rightarrow x^2 = o\left(x\right).$$

$$\lim_{x \to 0} \frac{\sin\left(x\right)}{\sqrt{x}} = \lim_{x \to 0} \sqrt{x} \underbrace{\frac{\sin\left(x\right)}{x}}_{\to 1} = 0 \Rightarrow \sin\left(x\right) = o\left(\sqrt{x}\right).$$

Algebra of little-oh

Strange things happen with little-oh:

$$o\left(x\right) + o\left(x^2\right) = o\left(x\right).$$

This is because

$$\begin{aligned}
\lim_{x\to 0} \frac{o\left(x\right) + o\left(x^2\right)}{x} &= \lim_{x\to 0} \left(\frac{o\left(x\right)}{x} + \frac{o\left(x^2\right)}{x} \right) \\
&= \lim_{x\to 0} \left(\frac{o\left(x\right)}{x} + \frac{o\left(x^2\right)}{x^2} x \right) = 0.
\end{aligned}$$

More generally

$$o\left(x^m\right) + o\left(x^n\right) = o\left(x^{\min(m,n)}\right).$$

Besides, trivially:

$$o\left(x^m\right)\, o\left(x^n\right) = o\left(x^{m+n}\right).$$

The Taylor's formula

Theorem

Let $f : [a, b] \to \mathbb{R}$ be a function n-times differentiable in a neighborhood of $x_0 \in (a, b)$ and assume that the n-th derivative is continuous in x_0. Then $f(x)$ can be written as:

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n \\ &\quad + o\left((x - x_0)^n\right). \end{aligned}$$

In other words

$$\begin{aligned} f(x) &= \underbrace{f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n}_{\text{Approximating polynomial}} + \\ &\quad + \underbrace{o\left((x - x_0)^n\right)}_{\text{Reminder}} \end{aligned}$$

we can approximate the function as polynomial plus a reminder that goes to zero faster than $(x - x_0)^n$.

The Taylor's formula: proof

Case $n = 1 \Rightarrow \exists f'(x)$ and it is continuous in x_0. Let $g(x)$ be defined by

$$g(x - x_0) = f(x) - f(x_0) - f'(x_0)(x - x_0)$$

that is

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{Approximating polynomial}} + \underbrace{g(x - x_0)}_{\text{Reminder}}$$

f is differentiable $x_0 \Rightarrow f$ is continuous in $x_0 \Rightarrow \lim_{x \to x_0} g(x - x_0) = 0$.
Moreover $g(x - x_0)$ is differentiable as f. Now compute:

$$\lim_{x \to x_0} \frac{g(x - x_0)}{(x - x_0)} = \frac{0}{0} \overset{\mathrm{H}}{=} \lim_{x \to x_0} \frac{g'(x - x_0)}{1} = \lim_{x \to x_0} \left[f'(x) - f'(x_0) \right] = 0,$$

therefore we have proved that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o((x - x_0)). \quad \square$$

The Taylor's formula: proof

Case $n = 2 \Rightarrow \exists f''(x)$ and it is continuous in x_0. Let $g(x)$ be defined by

$$g(x - x_0) = f(x) - f(x_0) - f'(x_0)(x - x_0) - \frac{1}{2!} f''(x_0)(x - x_0)^2$$

that is

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2}_{\text{Approximating polynomial}} + \underbrace{g(x - x_0)}_{\text{Reminder}}$$

f is differentiable $x_0 \Rightarrow f$ is continuous in $x_0 \Rightarrow \lim_{x \to x_0} g(x - x_0) = 0$. Moreover $g(x - x_0)$ is differentiable as f with

$$\begin{aligned} g'(x - x_0) &= f'(x) - f'(x_0) - f''(x_0)(x - x_0) \Rightarrow \lim_{x \to x_0} g'(x - x_0) = 0 \\ g''(x - x_0) &= f''(x) - f''(x_0) \Rightarrow \lim_{x \to x_0} g''(x - x_0) = 0 \end{aligned}$$

Now compute:

$$\lim_{x \to x_0} \frac{g(x - x_0)}{(x - x_0)^2} = \frac{0}{0} \overset{\text{H}}{=} \lim_{x \to x_0} \frac{g'(x - x_0)}{2(x - x_0)} = \frac{0}{0} \overset{\text{H}}{=} \lim_{x \to x_0} \frac{g''(x - x_0)}{2} = 0$$

therefore we have proved that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + o\left((x - x_0)^2\right). \quad \square$$

The Taylor's formula of $\cos(x)$ around $x_0 = 0$.

Compute the Taylor polynomial expansion

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)\,(x - x_0)^2 + \ldots + \frac{1}{n!} f^{(n)}(x_0)\,(x - x_0)^n + o\left((x - x_0)^n\right).$$

of $f(x) = \cos(x)$ around $x_0 = 0$.

$$\begin{aligned}
f(x) &= \cos(x) \Rightarrow f(0) = 1 \\
f'(x) &= -\sin(x) \Rightarrow f'(0) = 0 \\
f''(x) &= -\cos(x) \Rightarrow f''(0) = -1 \\
f'''(x) &= \sin(x) \Rightarrow f'''(0) = 0 \\
f''''(x) &= \cos(x) \Rightarrow f''''(0) = 1 \\
\vdots &\quad \vdots
\end{aligned}$$

whence

$$\cos(x) = 1 - \frac{x}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + o\left(x^{2n}\right).$$

The Taylor's formula of sin (x) around $x_0 = 0$.

Compute the Taylor polynomial expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + o((x - x_0)^n).$$

of $f(x) = \sin(x)$ around $x_0 = 0$.

$$\begin{aligned}
f(x) &= \sin(x) \Rightarrow f(0) = 0 \\
f'(x) &= \cos(x) \Rightarrow f'(0) = 1 \\
f''(x) &= -\sin(x) \Rightarrow f''(0) = 0 \\
f'''(x) &= -\cos(x) \Rightarrow f'''(0) = -1 \\
f''''(x) &= \sin(x) \Rightarrow f''''(0) = 0 \\
\vdots & \quad \vdots
\end{aligned}$$

whence

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+1}\right).$$

The Taylor's formula of $\ln 1 + x$ around $x_0 = 0$.

Compute the Taylor polynomial expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \tfrac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \ldots + \tfrac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + o((x - x_0)^n).$$

of $f(x) = \ln(1 + x)$ around $x_0 = 0$.

$$\begin{aligned}
f(x) &= \ln(1+x) \Rightarrow f(0) = 0 \\
f'(x) &= \frac{1}{1+x} \Rightarrow f'(0) = 1 \\
f''(x) &= -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1 \\
f'''(x) &= 2\frac{1}{(1+x)^3} \Rightarrow f'''(0) = 2 \\
f''''(x) &= -6\frac{1}{(1+x)^4} \Rightarrow f''''(0) = -6.
\end{aligned}$$

whence

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{2}{3 \cdot 2}x^3 - \frac{6}{4 \cdot 3 \cdot 2}x^4 + \ldots + = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^n \frac{x^n}{n} + o(x^n).$$

The Taylor's formula of e^x around $x_0 = 0$.

Compute the Taylor polynomial expansion

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n + o((x - x_0)^n).$$

of $f(x) = e^x$ around $x_0 = 0$.

$$\begin{aligned} f(x) &= e^x \Rightarrow f(0) = 1 \\ f'(x) &= e^x \Rightarrow f'(0) = 1 \\ f''(x) &= e^x \Rightarrow f''(0) = 1 \\ f'''(x) &= e^x \Rightarrow f'''(0) = 1 \\ f''''(x) &= e^x \Rightarrow f''''(0) = 1. \end{aligned}$$

whence

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + o(x^n).$$

Exercize

Write the Taylor's polynomial of degree 3 *of* $\sqrt{1+x}$ *in* $x_0 = 0$ *and use it to find an approximation of* $\sqrt{2}$.

How to proceed

- Write the generic formula with the requested degree

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \frac{1}{3!} f^{(3)}(x_0)(x - x_0)^3 + o\left((x - x_0)^3\right).$$

- Plug in the formula the value for x_0

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!} f^{(2)}(0)\,x^2 + \frac{1}{3!} f^{(3)}(0)\,x^3 + o\left(x^3\right).$$

- From the last formula isolate the quantities that must be computed

$$f(0), \quad f'(0), \quad f^{(2)}(0), \quad f^{(3)}(0).$$

Exercize

Write the Taylor's polynomial of degree 3 *of* $\sqrt{1+x}$ *in* $x_0 = 0$ *and use it to find an approximation of* $\sqrt{2}$.

How to proceed

- Write the generic formula with the requested degree

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f^{(2)}(x_0)(x - x_0)^2 + \frac{1}{3!} f^{(3)}(x_0)(x - x_0)^3 + o\left((x - x_0)^3\right).$$

- Plug in the formula the value for x_0

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!} f^{(2)}(0)\,x^2 + \frac{1}{3!} f^{(3)}(0)\,x^3 + o\left(x^3\right).$$

- From the last formula isolate the quantities that must be computed

$$f(0), \quad f'(0), \quad f^{(2)}(0), \quad f^{(3)}(0).$$

- Compute the quantities and conclude with some algebra.

The Taylor's formula for $\sqrt{2} = 1.41421...$

Exercize

Write the Taylor's polynomial of degree 3 of $\sqrt{1+x}$ in $x_0 = 0$ and use it to find an approximation of $\sqrt{2}$.

$$\begin{aligned}
f(x) &= \sqrt{1+x} \Rightarrow f(0) = 1\\
f'(x) &= \frac{1}{2\sqrt{1+x}} \Rightarrow f'(0) = \frac{1}{2}\\
f''(x) &= -\frac{1}{4(1+x)^{3/2}} \Rightarrow f''(0) = -\frac{1}{4}\\
f'''(x) &= \frac{3}{8(1+x)^{5/2}} \Rightarrow f'''(0) = \frac{3}{8}.
\end{aligned}$$

By plugging the above quantities into the formula, we get

$$\sqrt{1+x} = f(0)+f'(0)\,x+\frac{1}{2!}f^{(2)}(0)\,x^2+\frac{1}{3!}f^{(3)}(0)\,x^3+o\left(x^3\right) = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+o\left(x^3\right).$$

Write the **approximated formula**:

$$\sqrt{1+x} \approx 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16} \Rightarrow \sqrt{2} = \sqrt{1+1} \approx 1+\frac{1}{2}-\frac{1}{8}+\frac{1}{16} = \frac{23}{16} = 1.4375$$

The Taylor's formula for $\ln(2) = 0.69314...$

Exercize

Write the Taylor's polynomial of degree 3 *of* $\ln(1+x)$ *in* $x_0 = 0$ *and use it to find an approximation of* $\ln(2)$.

$$\begin{aligned} f(x) &= \ln(1+x) \Rightarrow f(0) = 0 \\ f'(x) &= \frac{1}{1+x} \Rightarrow f'(0) = 1 \\ f''(x) &= -\frac{1}{(1+x)^2} \Rightarrow f''(0) = -1 \\ f'''(x) &= 2\frac{1}{(1+x)^3} \Rightarrow f'''(0) = 2. \end{aligned}$$

By plugging the above quantities into the formula, we get

$$\ln(1+x) = f(0) + f'(0)\,x + \frac{1}{2!}f^{(2)}(0)\,x^2 + \frac{1}{3!}f^{(3)}(0)\,x^3 + o\left(x^3\right) = x - \frac{x^2}{2} + \frac{x^3}{3} + o\left(x^3\right).$$

Write the **approximated formula**:

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} \Rightarrow \ln(2) = \ln(1+1) \approx 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = 0.8333$$

The Taylor's formula for $\sin(1) = 0.84147...$

Exercize

Write the Taylor's polynomial of degree 5 *of* $\sin(x)$ *in* $x_0 = 0$ *and use it to find an approximation of* $\sin(1)$.

$$\begin{aligned} f(x) &= \sin(x) \Rightarrow f(0) = 0 \\ f^{(1)}(x) &= \cos(x) \Rightarrow f'(0) = 1 \\ f^{(2)}(x) &= -\sin(x) \Rightarrow f''(0) = 0 \\ f^{(3)}(x) &= -\cos(x) \Rightarrow f'''(0) = -1 \\ f^{(4)}(x) &= \sin(x) \Rightarrow f''''(0) = 0 \\ f^{(5)}(x) &= \cos(x) \Rightarrow f'''''(0) = 1. \end{aligned}$$

By plugging the above quantities into the formula, we get

$$\sin(x) = f(0) + f'(0)\,x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + o\left(x^5\right)$$

$$= x - \frac{x^3}{6} + \frac{x^5}{120} + o\left(x^5\right). \tag{0.1}$$

$$\sin(x) \approx x - \frac{x^3}{6} + \frac{x^5}{120} \Rightarrow \sin(1) \approx 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120} = 0.8416666...$$