DERIVATA DELLA FUNZIONE INVERSA: SIA $f: D=[a,b] \to \mathbb{R}$

$x_0 \in (a,b)$ E SIA f DERIVABILE IN x_0.

f È INVERTIBILE

$y_0 = f(x_0)$

$$\lim_{y \to y_0} \frac{f^{(-1)}(y) - f^{(-1)}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{1}{\frac{f(x)-f(x_0)}{x - x_0}}$$

$$\boxed{\begin{array}{l} x = f^{(-1)}(y) \Rightarrow y = f(x) \\ x_0 = f^{(-1)}(y_0) \Rightarrow y_0 = f(x_0) \end{array}}$$

$$= \frac{1}{f'(x_0)} = \frac{1}{f'(f^{(-1)}(y_0))}$$

$$= f^{(-1)'}(y_0)$$

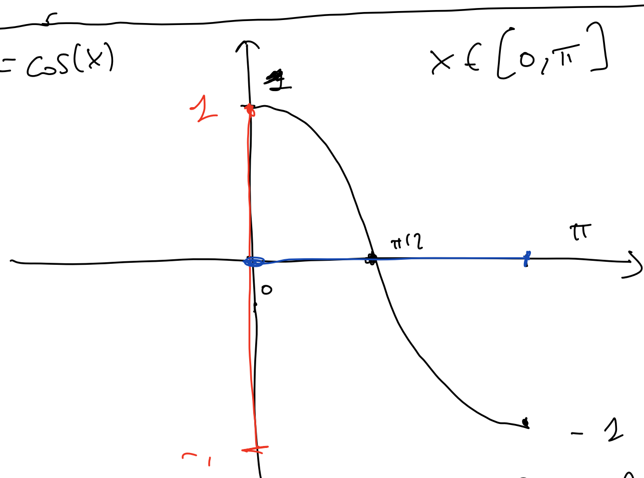
$f^{(-1)}$ È DERIVABILE IN y_0 E SI HA CHE

$$\boxed{f^{(-1)'}(y_0) = \frac{1}{f'(f^{(-1)}(y_0))}}$$

$$(\text{ARCSIN}(x))' = \frac{1}{\sqrt{1-x^2}}$$

$$x \in (-1, 1)$$

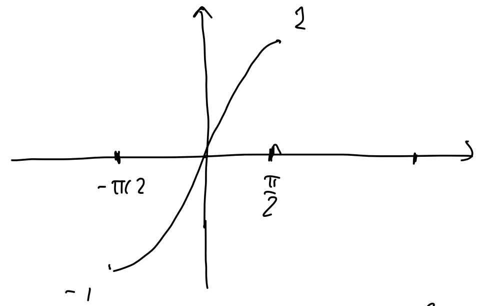
$f(x) = \cos(x)$ $\quad x \in [0, \pi]$



$f^{(-1)}(x) = \text{ARCCOS}(x) : [-1, 1] \to [0, \pi]$

$$(\text{ARCCOS}(x))' = -\frac{1}{\sqrt{1-x^2}}$$

$f(x) = \sin(x)$



$f^{(-1)}(x) = \text{ARCSIN}(x) : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin(0) = 0 \Rightarrow \text{ARCSIN}(0) = 0$

$\sin\left(\frac{\pi}{2}\right) = 1 \Rightarrow \text{ARCSIN}(1) = \pi/2$

$\sin\left(-\frac{\pi}{2}\right) = -1 \Rightarrow \text{ARCSIN}(-1) = -\pi/2$

$$(\text{ARCSIN}(x))' = \frac{1}{\sin'(\text{ARCSIN}(x))}$$

$$= \frac{1}{\cos(\text{ARCSIN}(x))}$$

$\cos(x) = \pm\sqrt{1 - \sin^2(x)}$

$\cos^2(x) + \sin^2(x) = 1$

SE $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \cos(y) = +\sqrt{1-\sin^2(y)}$

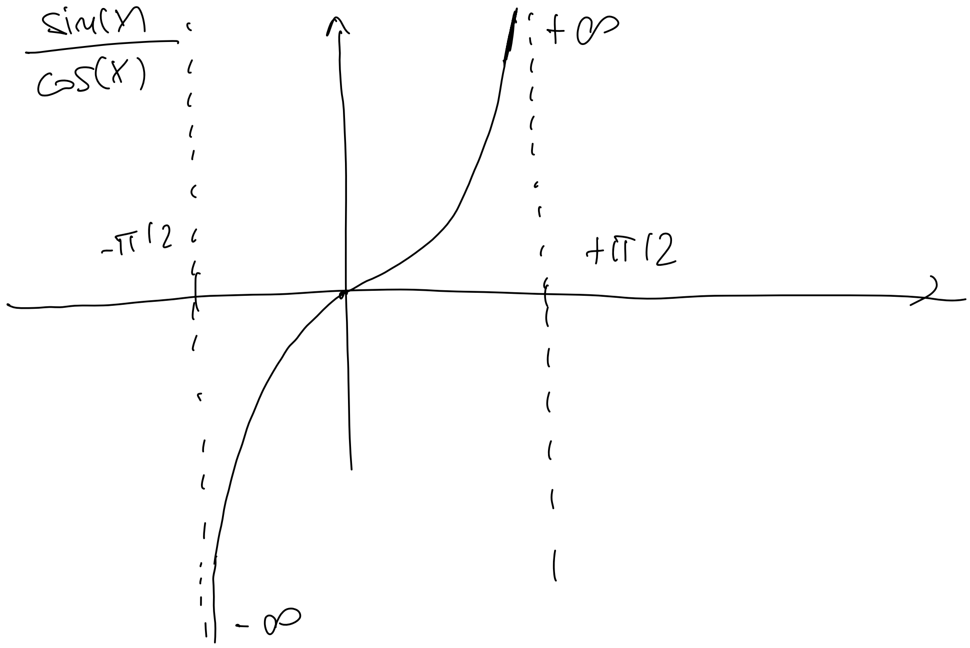
$x \in [-1, 1] \Rightarrow \text{ARCSIN}(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow$

$$(\text{ARCSIN}(x))' = \frac{1}{\cos(\text{ARCSIN}(x))} =$$

$$= \frac{1}{+\sqrt{1 - (\sin(\text{ARCSIN}(x)))^2}}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

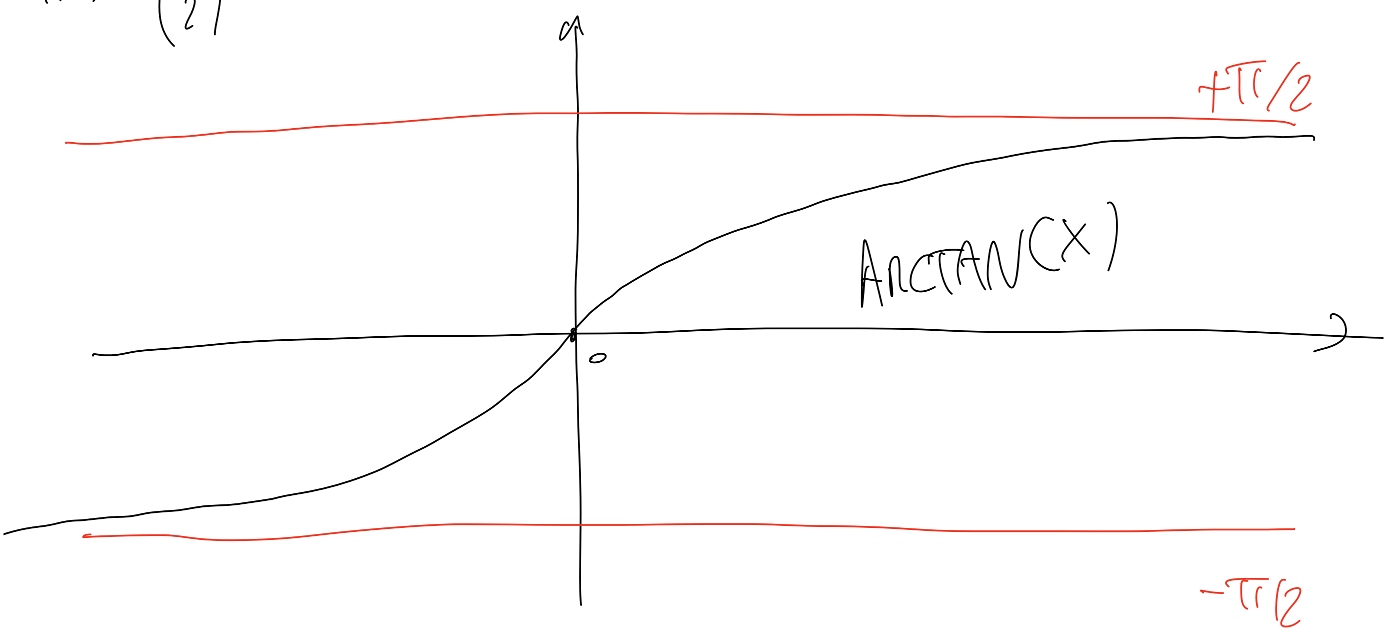
$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$



$$f^{(-1)}(x) = \mathrm{ARCTAN}(x) : (-\infty, +\infty) \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\lim_{x \to +\left(\frac{\pi}{2}\right)^-} \tan(x) = +\infty \Rightarrow \lim_{x \to +\infty} \mathrm{ARCTAN}(x) = \frac{\pi}{2}$$

$$\lim_{x \to -\left(\frac{\pi}{2}\right)^+} \tan(x) = -\infty \Rightarrow \lim_{x \to -\infty} \mathrm{ARCTAN}(x) = -\frac{\pi}{2}$$



$$(\tan(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{1}{\cos^2(x)} \Rightarrow$$

$$(\arctan(x))' = \frac{1}{\dfrac{1}{\cos^2(\arctan(x))}} = \cos^2(\arctan(x)) =$$

$$\boxed{\cos^2(y) = \frac{\cos^2(y)}{1} = \frac{\cos^2(y)}{\cos^2(y)+\sin^2(y)} = \frac{1}{1+\left(\dfrac{\sin(y)}{\cos(y)}\right)^2} = \frac{1}{1+(\tan(y))^2}}$$

$$\Rightarrow (\arctan(x))' = \cos^2(\arctan(x)) = \frac{1}{1+(\tan(\arctan(x)))^2} = \frac{1}{1+x^2}$$

$$(\arctan(x))' = \frac{1}{1+x^2} \qquad x \in (-\infty, +\infty)$$

$$(\ln(x))' = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

REGOLA DI DE L'HOPITAL: SIANO f E g DUE FUNZIONI DEFINITE IN $[a,b]$ E DERIVABILI IN (a,b) E SIA $x_0 \in (a,b)$. SE:

1) $\displaystyle\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$ E ESISTE $\displaystyle\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L$

OPPURE SE

2) $\displaystyle\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ E ESISTE $\displaystyle\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L$

ALLORA $\exists \lim_{x \to x_0} \frac{f(x)}{g(x)} = L$

IL TEOREMA VALE ANCHE QUANDO $x_0 = +\infty$ o $x_0 = -\infty$

ESEMPIO: $0 = \lim_{x \to 0^+} x \log(x) = 0 \cdot (-\infty) = \lim_{x \to 0^+} \frac{\log(x)}{\frac{1}{x}} = \frac{-\infty}{+\infty}$

$$\overset{H}{=} \lim_{x \to 0^+} \frac{(\log(x))'}{\left(\frac{1}{x}\right)'} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -\frac{x^2}{x} = \lim_{x \to 0^+} -x = 0$$

$$\lim_{x \to +\infty} \frac{\log(x)}{\sqrt{x}} = \frac{+\infty}{+\infty} \overset{H}{=} \lim_{x \to +\infty} \frac{(\log(x))'}{(\sqrt{x})'} = \lim_{x \to +\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x \to +\infty} 2\frac{\sqrt{x}}{x} =$$

$$= \lim_{x \to +\infty} 2\frac{x^{1/2}}{x} = \lim_{x \to +\infty} 2\frac{1}{x^{1-1/2}} = \lim_{x \to +\infty} \frac{2}{x^{1/2}} = 0$$

$$\lim_{x \to +\infty} \frac{x + \sin(x)}{x} = \frac{\infty}{\infty} \overset{H}{=} \lim_{x \to +\infty} \frac{(x+\sin(x))'}{1} = \lim_{x \to +\infty} \frac{1+\cos(x)}{1} \ \cancel{\nexists}$$

$$\| $$

$$\lim_{x \to +\infty} \left(1 + \frac{\sin(x)}{x}\right) = 1$$

HOPITAL NON SI PUÓ APPLICARE IN QUESTO CASO.

ESEMPIO: $\lim_{x \to +\infty} \underbrace{\left(\arctan(x) - \frac{\pi}{2}\right)}_{\to 0} \cdot \underbrace{e^x}_{\to +\infty} = 0 \cdot (+\infty) = -$

$$= \lim_{x \to +\infty} \frac{\left(\arctan(x) - \pi/2\right)}{e^{-x}} = \frac{0}{0} \overset{H}{=} \lim_{x \to +\infty} \frac{\left(\arctan(x) - \pi/2\right)'}{\left(e^{-x}\right)'} =$$

$$(e^{-x})' = -e^{-x}$$

$$= \lim_{x\to+\infty} \frac{\frac{1}{1+x^2}}{-e^{-x}} = \lim_{x\to+\infty} -\frac{e^x}{1+x^2} = \frac{\infty}{\infty} \overset{H}{=} \lim_{x\to+\infty} -\frac{e^x}{2x}$$

$$= \frac{\infty}{\infty} \overset{H}{=} \lim_{x\to+\infty} -\frac{e^x}{2} = -\infty$$

$$\frac{\frac{1}{1+x^2}}{e^{-x}} = \frac{1}{(1+x)^2 e^{-x}} = \frac{e^x}{1+x^2}$$

$$\lim_{x\to\infty} \frac{5x+\sin(x)+\log(\sqrt{x})}{3x} = \frac{\infty}{\infty}$$

$$\overset{H}{=} \lim_{x\to+\infty} \frac{5+\cos(x)+(\log(\sqrt{x}))'}{3} = \lim_{x\to+\infty} \frac{5+\cos(x)+\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}}{3}$$

$$= \lim_{x\to+\infty} \frac{5+\cos(x)+\frac{1}{2x}}{3} = \not\exists$$

HOPITAL NON FUNZIONA

$$= \lim_{x\to\infty} \left[\frac{5}{3} + \frac{\sin(x)}{3x} + \frac{\log(\sqrt{x})}{3x}\right] = \frac{5}{3}$$

$$\frac{\sin(x)}{3x} \to 0$$

$$-\frac{1}{x} \le \frac{\sin(x)}{x} \le +\frac{1}{x} \qquad x>0$$

$$-\frac{1}{x}\to 0 \qquad \frac{\sin(x)}{x}\to 0 \qquad \frac{1}{x}\to 0$$

$$\lim_{x\to\infty} \frac{\log(\sqrt{x})}{3x} = \frac{\infty}{\infty} \overset{H}{=} \lim_{x\to\infty} \frac{(\log(\sqrt{x}))'}{(3x)'} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}\,\frac{1}{2\sqrt{x}}}{3} =$$

$$= 0$$

$$\lim_{x\to0^+} \frac{\arctan(x)}{\log(1+x)} = \frac{0}{0} \overset{H}{=} \lim_{x\to0^+} \frac{(\arctan(x))'}{(\log(1+x))'} = \lim_{x\to0^+} \frac{\frac{1}{1+x^2}}{\frac{1}{1+x}}$$

$$= \lim_{x \to 0^+} \frac{1+x}{1+x^2} = 1$$

DEF. SIA $x_0 \in \mathbb{R}$. SI CHIAMA "o PICCOLO DI $x - x_0$" UNA QUALSIASI FUNZIONE CHE VA A ZERO PIÙ VELOCEMENTE DI $x - x_0$ QUANDO $x \to x_0$ OVVERO

$$\lim_{x \to x_0} \frac{o(x-x_0)}{x-x_0} = 0$$

O PIÙ GENERALMENTE SI CHIAMA O PICCOLO DI $(x-x_0)^n$ UNA QUALSIASI FUNZIONE CHE VA A ZERO PIÙ VELOCEMENTE DI $(x-x_0)^n$ QUANDO $x \to x_0$.

$$\lim_{x \to x_0} \frac{o((x-x_0)^n)}{(x-x_0)^n} = 0$$

ESEMPIO: $x^3 = o(x^2)$ $\qquad x^4 = o(x^3) = o(x^2)$

$$o(x^2) + o(x^3) = o(x^2)$$

$$\lim_{x \to 0} \frac{o(x^2)+o(x^3)}{x^2} = \lim_{x \to 0} \frac{o(x^2)}{x^2} + \frac{o(x^3)}{x^2} =$$

$$= \lim_{x \to 0} \underbrace{\frac{o(x^2)}{x^2}}_{\to 0} + x \underbrace{\frac{o(x^3)}{x^3}}_{\to 0} = 0$$

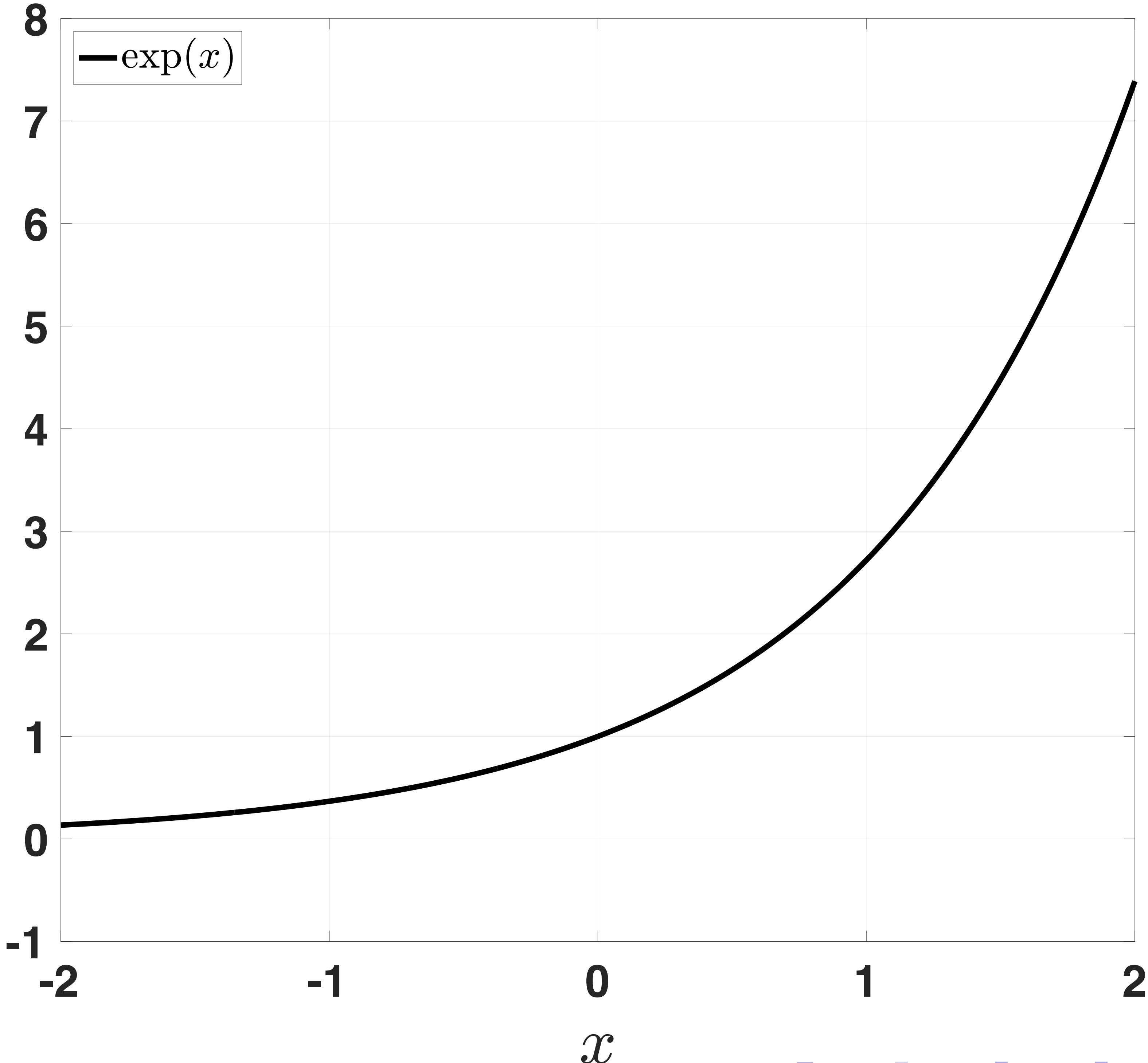
$$o(x^n) + o(x^m) = o(x^{\min(n,m)})$$

$$o(x^n) \cdot o(x^m) = o(x^{n+m})$$

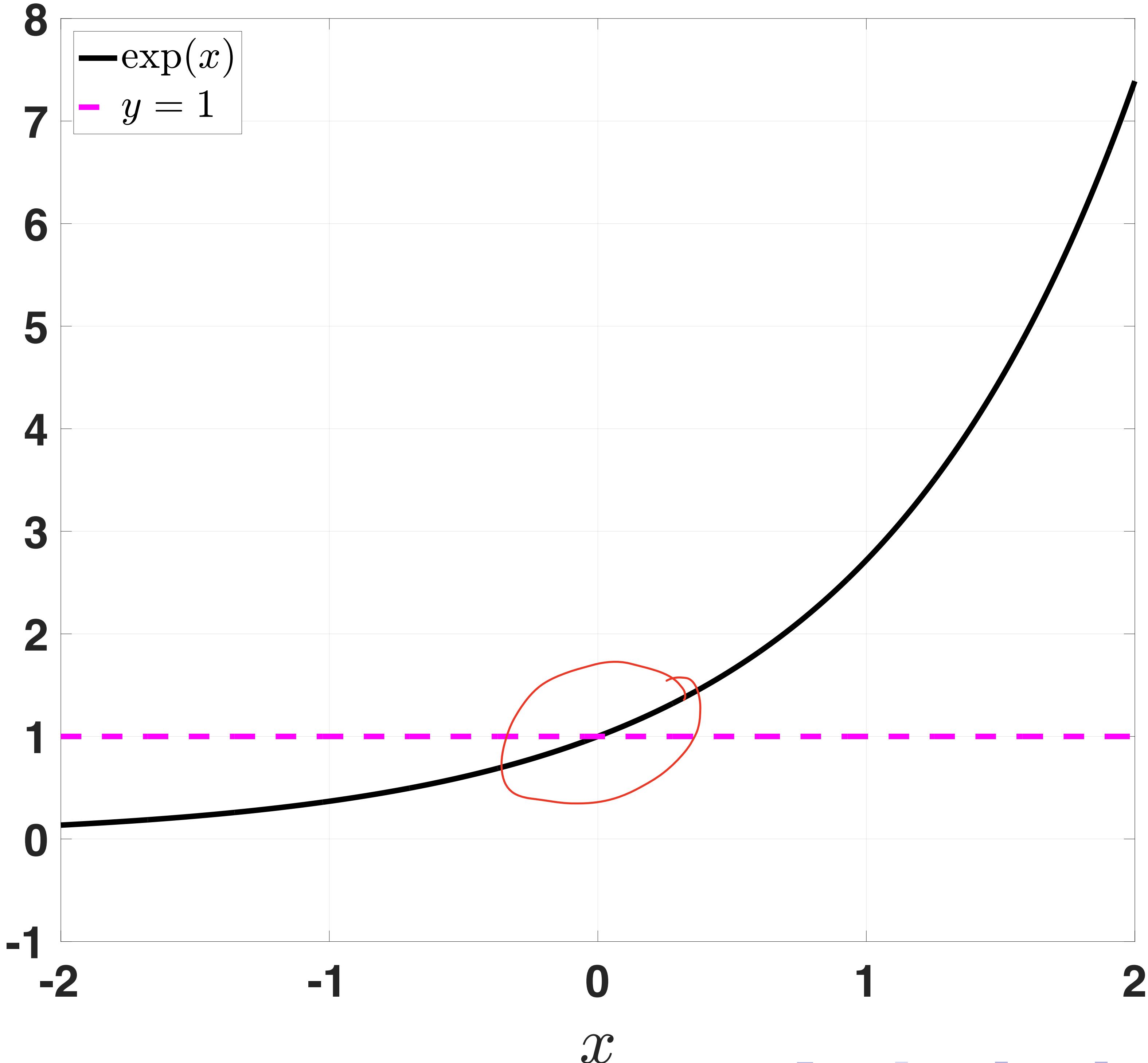
TEO: SIA f UNA FUNZIONE $f: [a,b] \to \mathbb{R}$, $x_0 \in (a,b)$.
SIA f DERIVABILE n-VOLTE IN x_0 CON DERIVATA n-ESIMA CONTINUA IN x_0. ALLORA:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \ldots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n + o\left((x-x_0)^n\right)$$

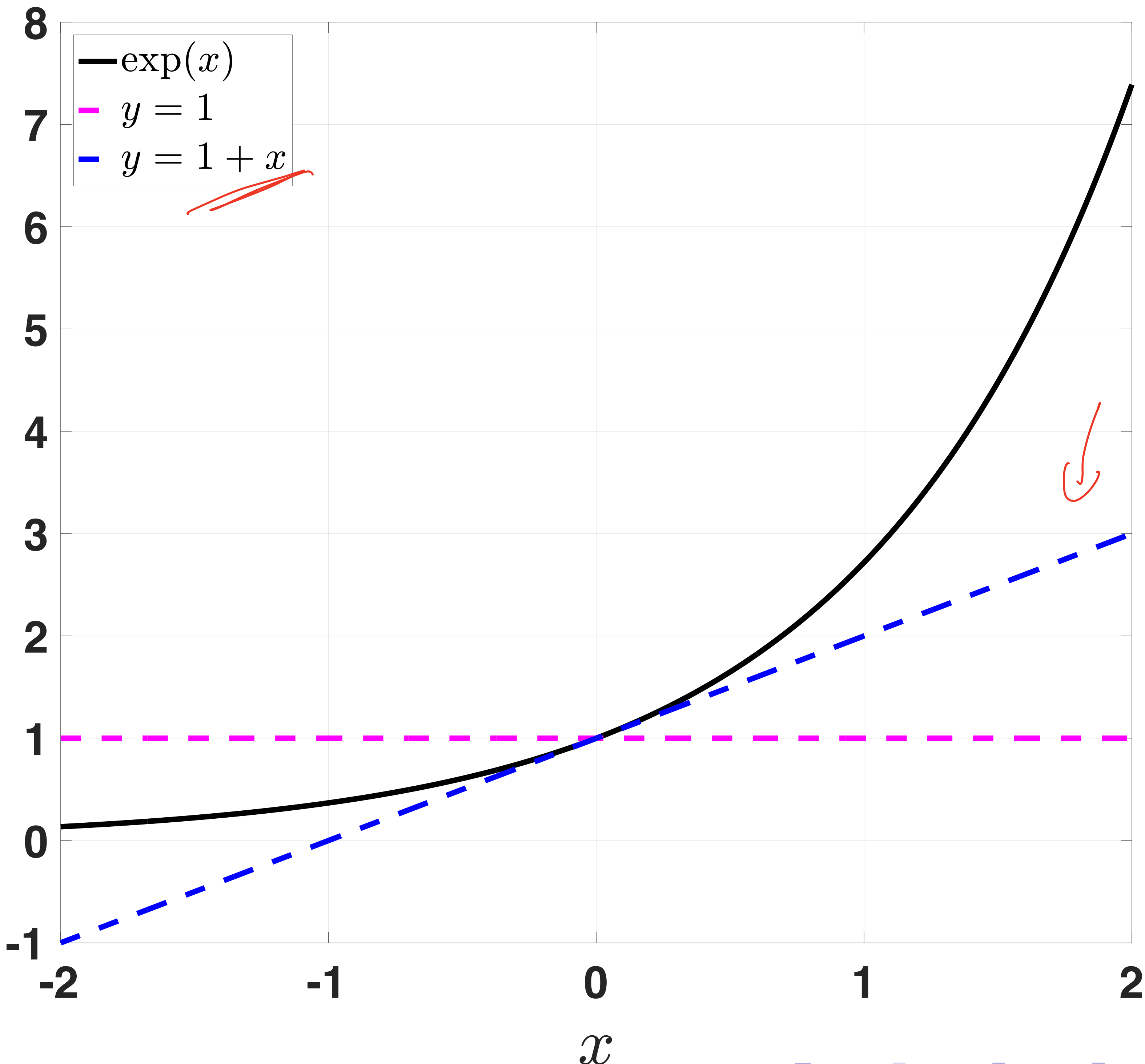
November 2, 2023



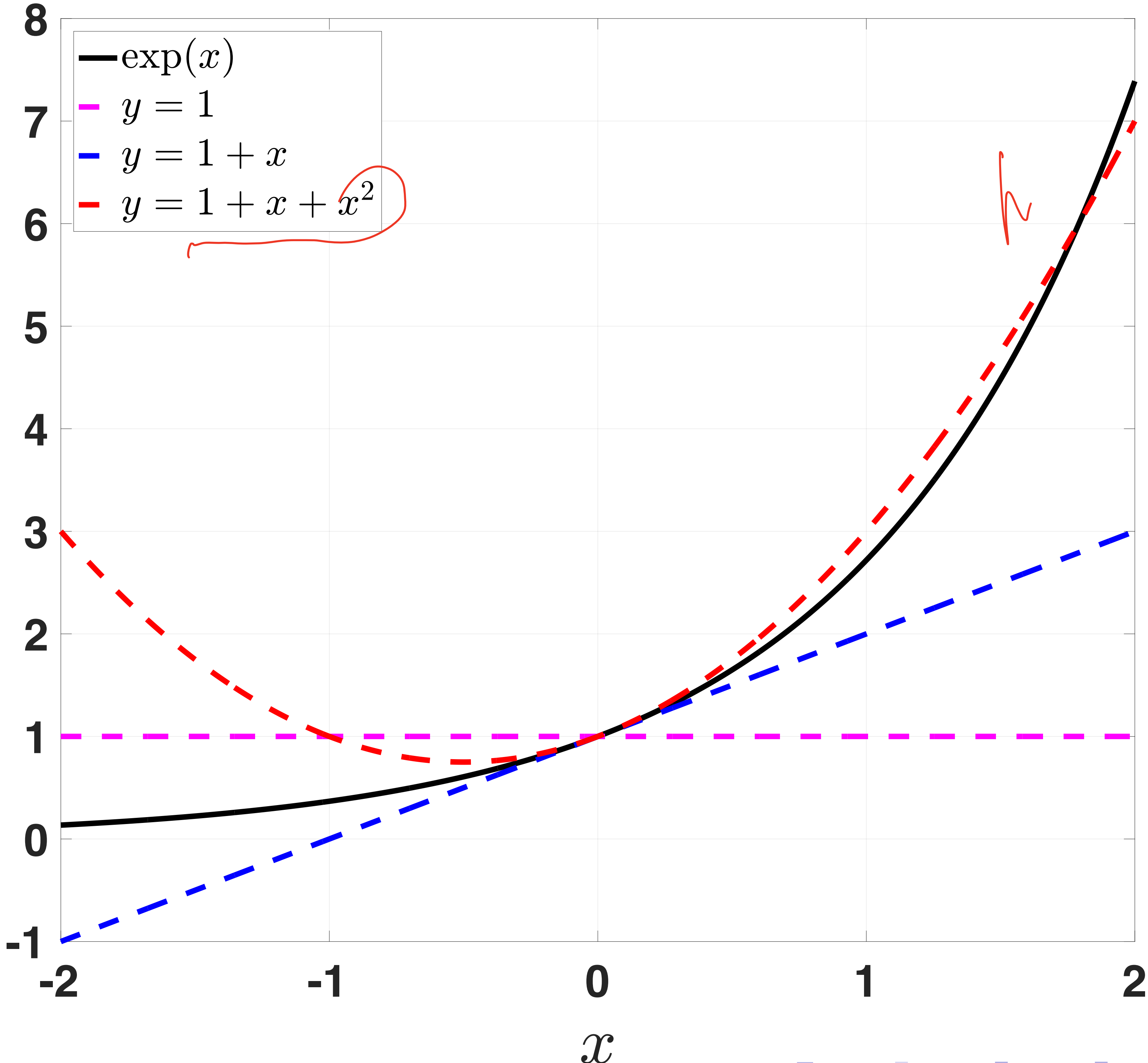
8
7
6
5
4
3
2
1
0
-1
-2
-1
0
1
2
x
$\exp(x)$

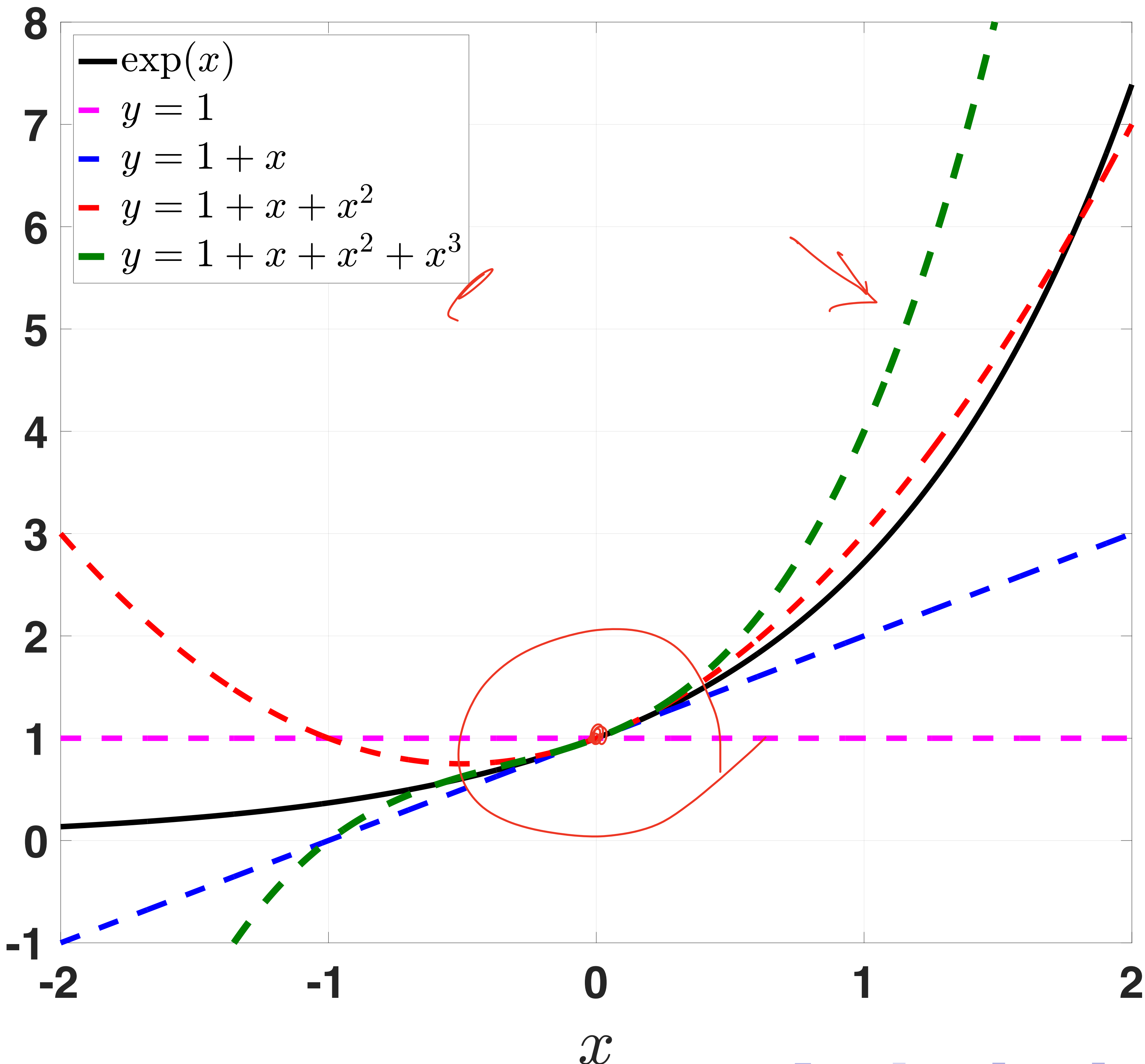


$\exp(x)$
$y = 1$
8
7
6
5
4
3
2
1
0
-1
-2
-1
0
1
2
x

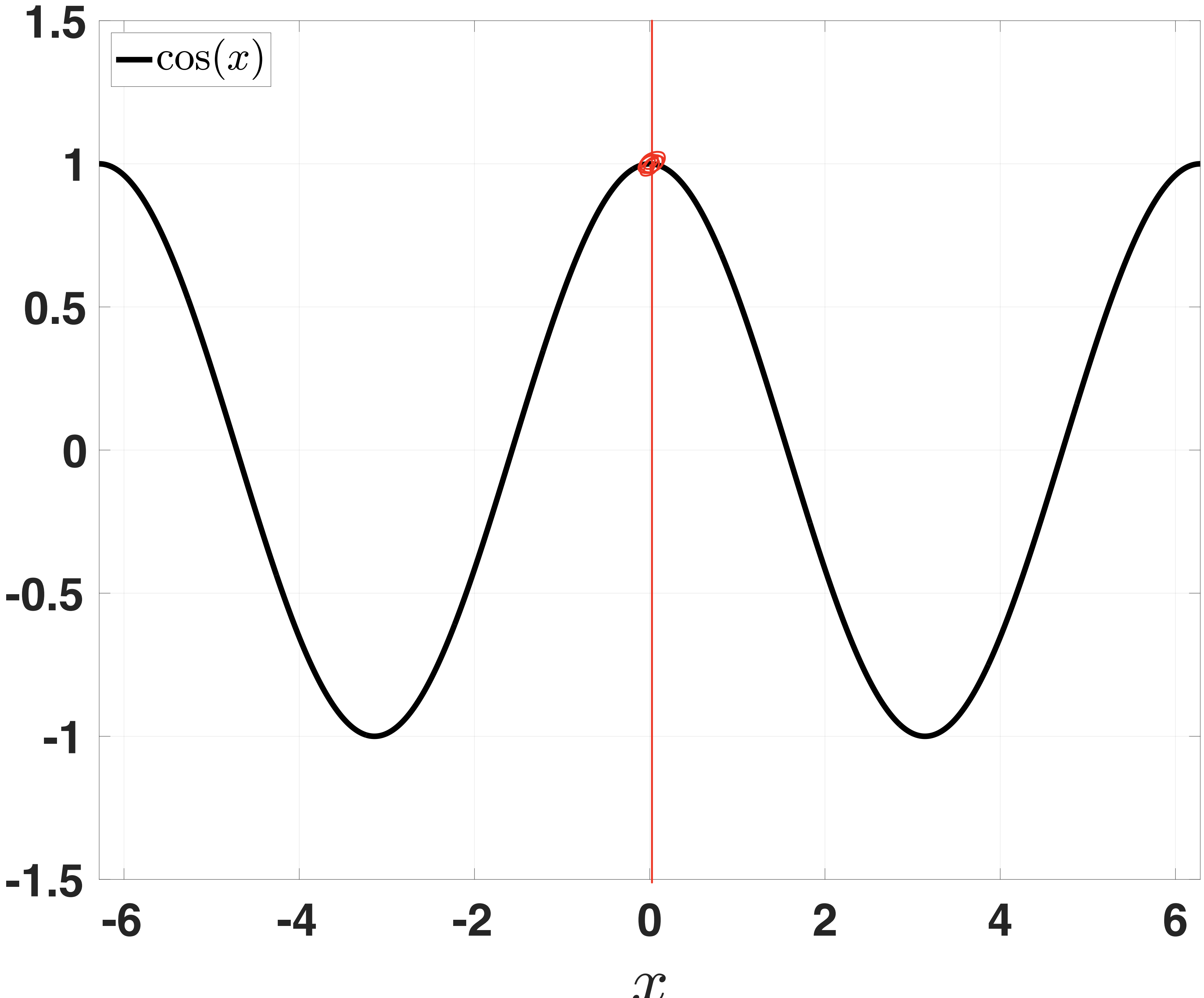


$\exp(x)$
$y = 1$
$y = 1 + x$
8
7
6
5
4
3
2
1
0
-1
-2
-1
0
1
2
x

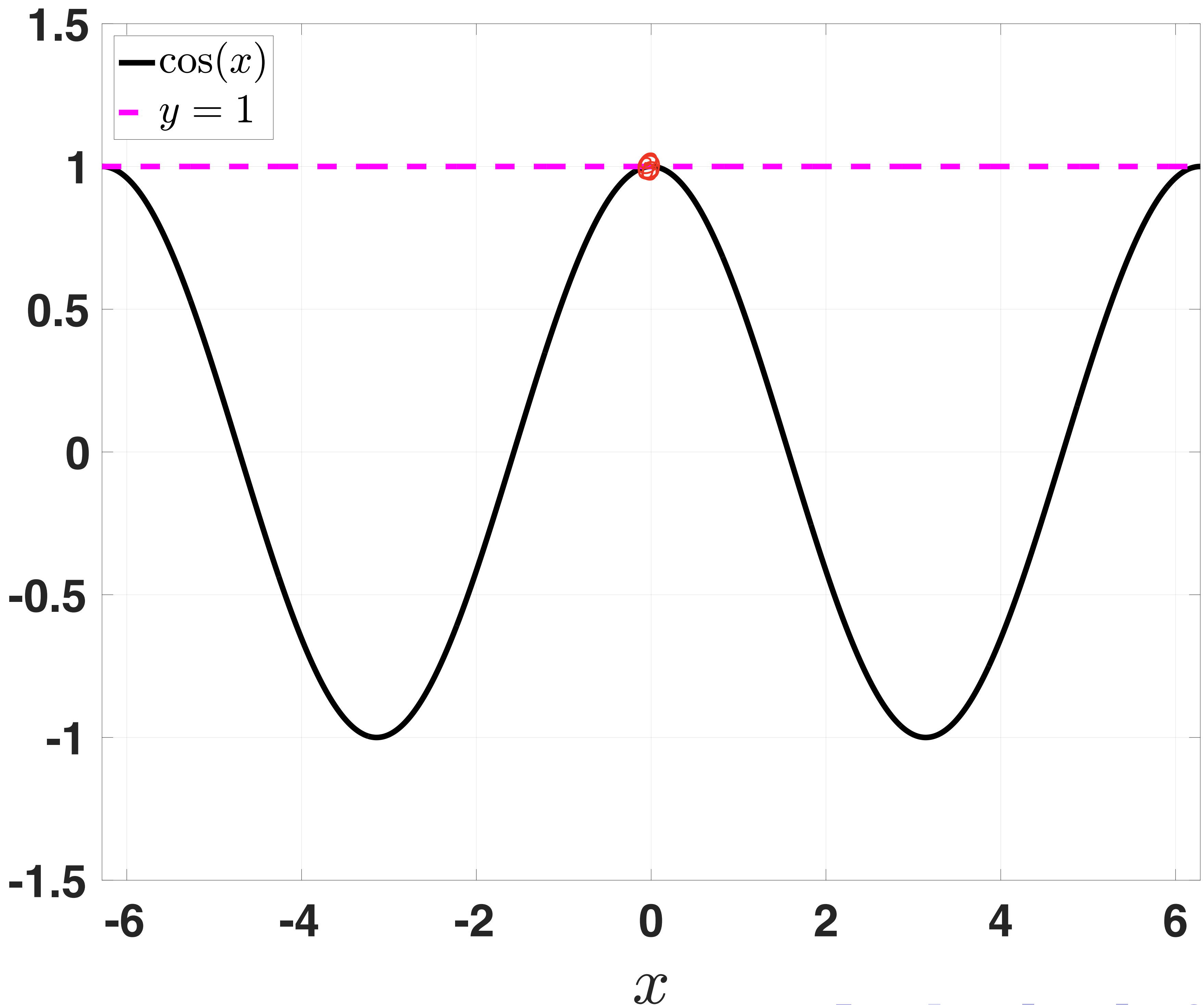

exp(x)
$y = 1$
$y = 1 + x$
$y = 1 + x + x^2$
h
8
7
6
5
4
3
2
1
0
-1
-2
-1
0
1
2
x



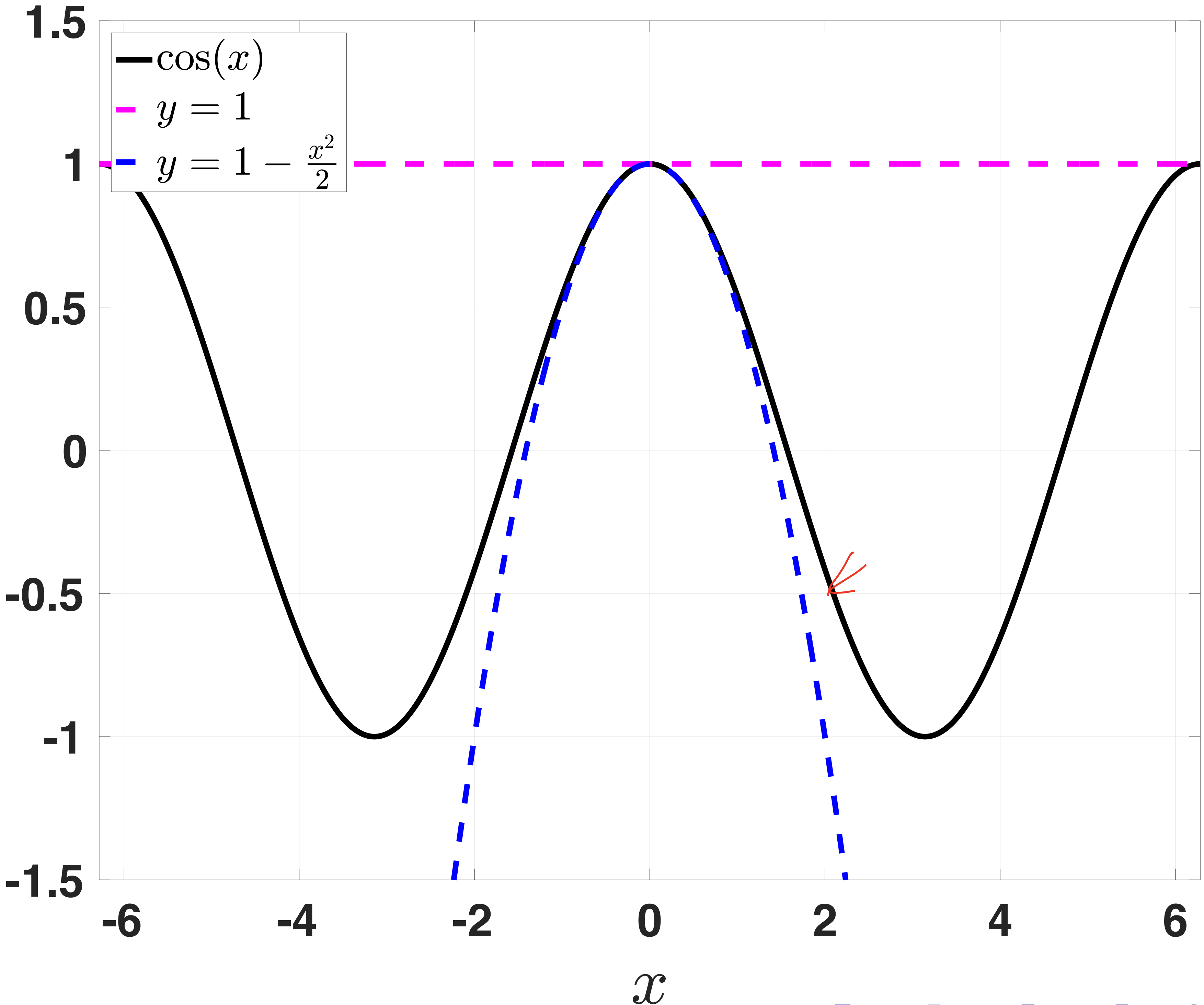
$\exp(x)$
$y = 1$
$y = 1 + x$
$y = 1 + x + x^2$
$y = 1 + x + x^2 + x^3$
8
7
6
5
4
3
2
1
0
-1
-2
-1
0
1
2
x

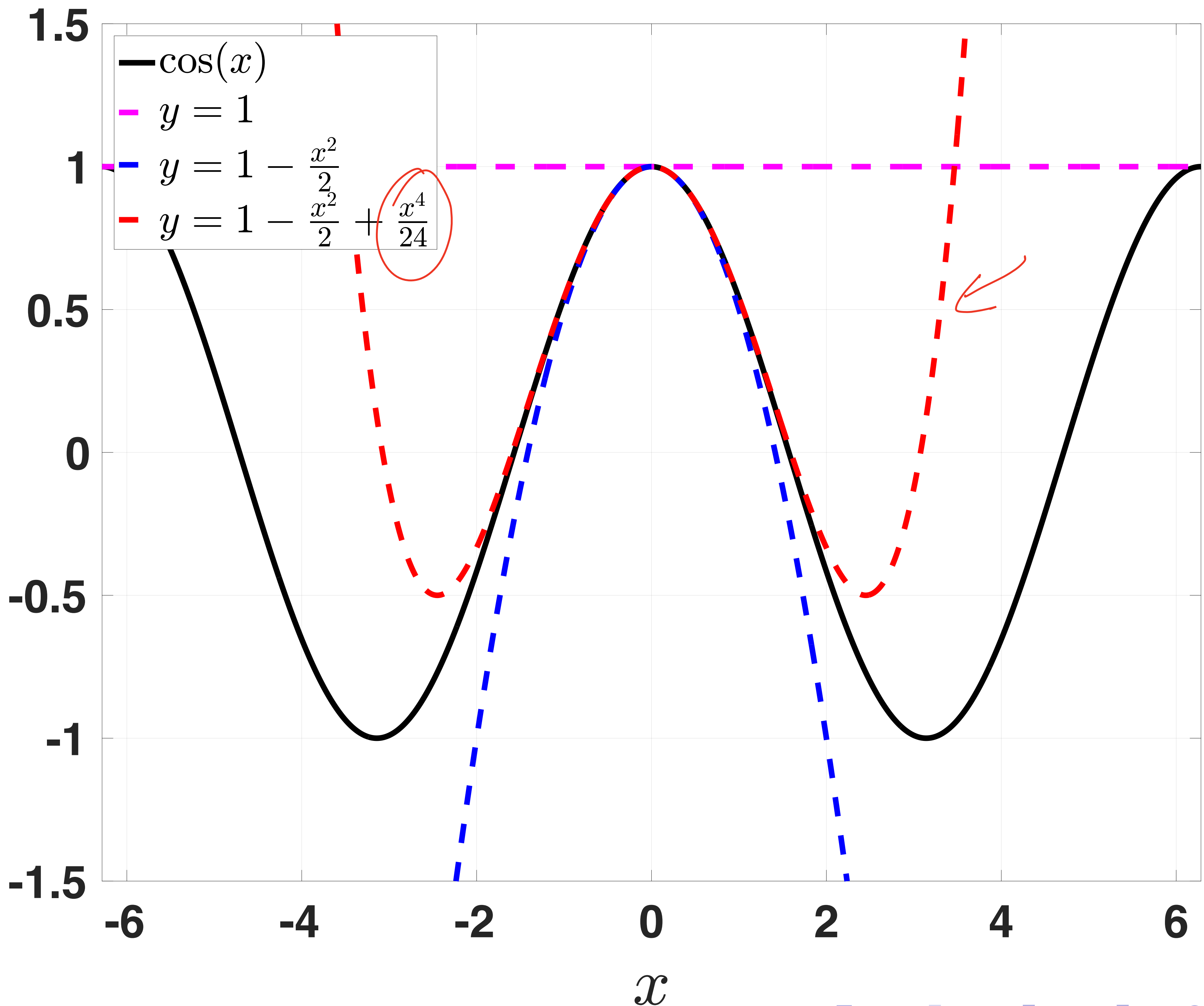


$\cos(x)$
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6
x

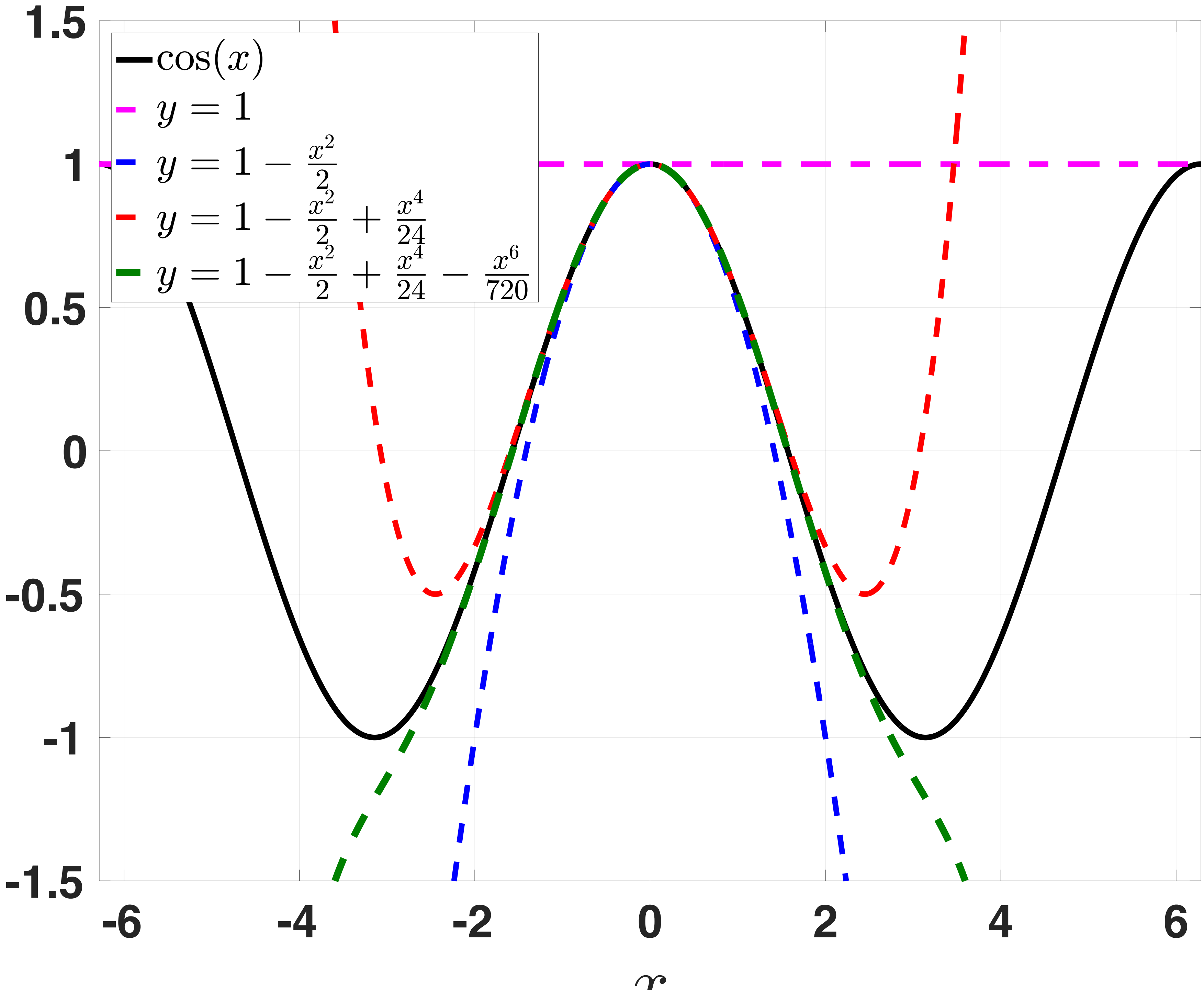


1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6
x
$\cos(x)$
$y = 1$


1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6
x
$\cos(x)$
$y = 1$
$y = 1 - \frac{x^2}{2}$



$\cos(x)$
$y = 1$
$y = 1 - \frac{x^2}{2}$
$y = 1 - \frac{x^2}{2} + \frac{x^4}{24}$
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6
x


$\cos(x)$
$y = 1$
$y = 1 - \frac{x^2}{2}$
$y = 1 - \frac{x^2}{2} + \frac{x^4}{24}$
$y = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$
1.5
1
0.5
0
-0.5
-1
-1.5
-6
-4
-2
0
2
4
6
x