1 The Riemann and the Riemann-Stieltjes integrals

Definition 1. *Let $[a, b]$ be a given interval. A partition $\mathcal{P}$ of $[a, b]$ is a collection of points $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ such that $a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$, i.e. $\exists M \geq 0$ such that $\left|f(x)\right| \leq M$ for all $x \in [a, b]$. For a given partition $\mathcal{P}$ of $[a, b]$ we set*

$$M_j = \sup\left\{f(x) \,\middle|\, x \in \left[x_{j-1}, x_j\right]\right\},\; m_j = \inf\left\{f(x) \,\middle|\, x \in \left[x_{j-1}, x_j\right]\right\}.$$

We define the upper and lower sums of f on $\mathcal{P}$ respectively as

$$U(\mathcal{P}, f) \doteq \sum_{j=1}^{n} M_j \, \Delta_j x, \;\; L(\mathcal{P}, f) \doteq \sum_{j=1}^{n} m_j \, \Delta_j x,$$

where $\Delta_j x = x_j - x_{j-1}$. In addition we define the upper and the lower Riemann integrals respectively as:

$$\overline{\int_a^b} f(x)\; dx = \inf_{\mathcal{P}} U(\mathcal{P}, f), \;\; \underline{\int_a^b} f(x)\; dx = \sup_{\mathcal{P}} L(\mathcal{P}, f).$$

If $\overline{\int_a^b} f\, dx = \underline{\int_a^b} f\, dx$, we say that f is Riemann-integrable on $[a, b]$ and we call the Riemann integral of f on $[a, b]$ the quantity

$$\int_a^b f(x)\; dx \doteq \overline{\int_a^b} f(x)\; dx = \underline{\int_a^b} f(x)\; dx.$$

Remark. The upper and lower integrals are well-defined for a bounded function. In fact if f is bounded in $[a, b]$ then there exist two constants m and M such that

$$m \leq f(x) \leq M, \; \forall x \in [a, b].$$

Hence for every partition $\mathcal{P}$ we have

$$m\;(b - a) \leq L(\mathcal{P}, f) \leq U(\mathcal{P}, f) \leq M\;(b - a),$$

so the quantities $\overline{\int_a^b} f(x)\; dx$ and $\underline{\int_a^b} f(x)\; dx$ are finite (but they might be different).

For the purpose of statistical applications is quite useful to use a slightly different version of the Riemann integral.

Definition 2. *Let α be a monotonically increasing function on $[a, b]$. For each partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ of $[a, b]$ we define $\Delta_j \alpha \doteq \alpha\left(x_j\right) - \alpha\left(x_{j-1}\right)$, $j = 1, \ldots, n$ and, accordingly, for every bounded function f on $[a, b]$ we define*

$$U(\mathcal{P}, f, \alpha) = \sum_{j=1}^{M} M_j \, \Delta_j \alpha, \;\; L(\mathcal{P}, f, \alpha) = \sum_{j=1}^{M} m_j \, \Delta_j \alpha.$$

We define the upper and the lower Riemann-Stieltjes integrals on the interval $[a, b]$ of f w.r.t. the measure α respectively as:

$$\overline{\int_a^b} f(x)\; \alpha(dx) = \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha), \;\; \underline{\int_a^b} f(x)\; \alpha(dx) = \sup_{\mathcal{P}} L(\mathcal{P}, f, \alpha),$$

and we say that the function f is Reimann-Stieltjes integrable on the interval $[a, b]$ w.r.t. the measure α, and we write

$f \in \mathcal{R}(\alpha)$ on $[a,b]$, if:

$$\overline{\int_a^b} f(x)\ \alpha(dx) = \underline{\int_a^b} f\,\alpha(dx) \equiv \int_a^b f(x)\ \alpha(dx).$$

Remark. The Riemann integral is a special case of the Riemann-Stieltjes integral for $\alpha(x) = x$. Nevertheless note that $\alpha(x)$ is not required to be continuous.

Definition 3. *We say that the partition $\mathcal{P}^\star$ is a refinement of the partition $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{P}^\star$. Given two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ we say that the partition $\mathcal{P}_1 \cup \mathcal{P}_2$ is their common refinement.*

Theorem 1.1. *If $\mathcal{P}^\star$ is a refinement of the partition $\mathcal{P}$ then:*

$$L(\mathcal{P}, f, \alpha) \le L(\mathcal{P}^\star, f, \alpha),\ \ U(\mathcal{P}, f, \alpha) \ge U(\mathcal{P}^\star, f, \alpha). \tag{1.1}$$

Moreover:

$$\underline{\int_a^b} f(x)\ \alpha(dx) \le \overline{\int_a^b} f(x)\ \alpha(dx).$$

Proof. Suppose that $\mathcal{P}^\star$ contains one point more than $\mathcal{P}$. Let $x^\star$ be this point. Suppose, without loss of generality, that $x_{j-1} < x^\star < x_j$ where x_{j-1} and x_j are point of the partition $\mathcal{P}$. Put

$$w_1 = \inf_{x_{j-1} \le x \le x^\star} f(x), \quad w_2 = \inf_{x^\star \le x \le x_j} f(x)$$

Since both $\left[x_{j-1}, x^\star\right]$ and $\left[x^\star, x_j\right]$ are subsets of $\left[x_{j-1}, x_j\right]$ we get that $m_j \le w_1$ and $m_j \le w_2$. Whence

$$\begin{aligned} L(\mathcal{P}^\star, f, \alpha) - L(\mathcal{P}, f, \alpha) &= \underbrace{w_1\left(\alpha(x^\star) - \alpha\left(x_{j-1}\right)\right) + w_2\left(\alpha\left(x_j\right) - \alpha(x^\star)\right)}_{\text{Appear in } L(\mathcal{P}^\star, f, \alpha) \text{ but not in } L(\mathcal{P}, f, \alpha)} - \underbrace{m_j\left(\alpha(x_i) - \alpha\left(x_{j-1}\right)\right)}_{\text{vice versa}} \\ &= \left(w_1 - m_j\right)\left(\alpha(x^\star) - \alpha\left(x_{j-1}\right)\right) + \left(w_2 - m_j\right)\left(\alpha\left(x_j\right) - \alpha(x^\star)\right) \ge 0, \end{aligned}$$

If $\mathcal{P}^\star$ contains k more points we repeat this procedure k times. With an identical reasoning we prove $U(\mathcal{P}, f, \alpha) \ge U(\mathcal{P}^\star, f, \alpha)$. Now consider two arbitrary partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ and consider $\mathcal{P}^\star \doteq \mathcal{P}_1 \bigcup \mathcal{P}_2$ their common refinement. Hence

$$L(\mathcal{P}_1, f, \alpha) \le L(\mathcal{P}^\star, f, \alpha) \le U(\mathcal{P}^\star, f, \alpha) \le U(\mathcal{P}_2, f, \alpha),$$

whence for arbitrary partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ it holds that

$$L(\mathcal{P}_1, f, \alpha) \le U(\mathcal{P}_2, f, \alpha).$$

Now fix $\mathcal{P}_2$ and take the supremum on the left over all $\mathcal{P}_1$, obtaining

$$\underline{\int_a^b} f(x)\ \alpha(dx) \le U(\mathcal{P}_2, f, \alpha).$$

Now take the infimum on the right over all $\mathcal{P}_2$, obtaining

$$\underline{\int_a^b} f(x)\,\alpha(dx) \le \overline{\int_a^b} f(x)\ \alpha(x).$$

□

Theorem 1.2. *$f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if $\forall \varepsilon > 0$ there exists a partition $\mathcal{P}$ of $[a,b]$ such that:*

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon. \tag{1.2}$$

Proof. Suppose first that $\forall \varepsilon > 0$ there exists a partition $\mathcal{P}$ of $[a,b]$ such that:

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon.$$

For every partition $\mathcal{P}$ we have

$$L(\mathcal{P}, f, \alpha) \leq \underline{\int_a^b} f(x)\ \alpha(dx) \leq \overline{\int_a^b} f(x)\ \alpha(dx) \leq U(\mathcal{P}, f, \alpha).$$

Hence

$$0 \leq \overline{\int_a^b} f(x)\ \alpha(dx) - \underline{\int_a^b} f(x)\ \alpha(dx) \leq U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon.$$

Since ε is arbitrarily small we get

$$\overline{\int_a^b} f(x)\ \alpha(dx) = \underline{\int_a^b} f(x)\ \alpha(dx)$$

whence $f \in \mathcal{R}(\alpha)$ on $[a,b]$. To prove the other implication, assume that $f \in \mathcal{R}(\alpha)$ on $[a,b]$. Let $\varepsilon > 0$ be given. By definition of supremum and infimum we can say that there exists partitions such that

$$U(\mathcal{P}_2, f, \alpha) - \int_a^b f(x)\ \alpha(dx) < \varepsilon/2, \text{ and } \int_a^b f(x)\ \alpha(dx) - L(\mathcal{P}_1, f, \alpha) < \varepsilon/2.$$

If $\mathcal{P}_1 = \mathcal{P}_2$ we have finished, if $\mathcal{P}_1 \neq \mathcal{P}_2$ then the common refinement $\mathcal{P}^\star = \mathcal{P}_1 \bigcup \mathcal{P}_2$ contains more points than both $\mathcal{P}_1$ and $\mathcal{P}_2$ and then inequalities (0.0.1) are strict. Then

$$U(\mathcal{P}^\star, f, \alpha) < U(\mathcal{P}_2, f, \alpha) < \frac{\varepsilon}{2} + \int_a^b f(x)\ \alpha(dx) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + L(\mathcal{P}_1, f, \alpha) = \varepsilon + L(\mathcal{P}_1, f, \alpha) < \varepsilon + L(\mathcal{P}^\star, f, \alpha),$$

whence

$$U(\mathcal{P}^\star, f, \alpha) - L(\mathcal{P}^\star, f, \alpha) < \varepsilon.$$

□

Theorem 1.3. *If f is continuous in $[a,b]$ then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.*

Proof. For the Heine-Borel Theorem the interval $[a,b]$ is compact and for the Heine-Cantor Theorem a continuous function on a compact set is uniformly continuous, hence no matter how small we take a $\eta > 0$ we can always find a $\delta > 0$ such that if $|x - t| < \delta$ then $\left|f(x) - f(t)\right| < \eta$. Now for all $\varepsilon > 0$ we look for a partition $\mathcal{P} = \{x_0, \ldots, x_n\}$ such that equation (0.0.2) holds. Take a partion $\mathcal{P}$ such that $\Delta x_j = x_j - x_{j-1} < \delta$ for all $j = 1, \ldots, n$. Therefore

$$M_j - m_j = \sup_{x \in [x_{j-1}, x_j]} f(x) - \inf_{x \in [x_{j-1}, x_j]} f(x) \leq \eta.$$

As a consequence

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) = \sum_{j=1}^{n} \left(M_j - m_j\right) \Delta_j \alpha \leq \eta \sum_{j=1}^{n} \Delta_j \alpha = \eta\ [\alpha(b) - \alpha(a)].$$

So it is enough to take an η such that $\eta\ [\alpha(b) - \alpha(a)] < \varepsilon$. $\square$

Theorem 1.4. ***(Integral as a linear operator).*** *If $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ then $f_1 + f_2 \in \mathcal{R}(\alpha)$. Moreover if $f \in \mathcal{R}(\alpha)$ on $[a,b]$ then $c\,f \in \mathcal{R}(\alpha)$ for every constant c. We also have that*

$$\int_a^b (f_1 + f_2)(x)\ \alpha(dx) = \int_a^b f_1(x)\ \alpha(dx) + \int_a^b f_2(x)\ \alpha(dx),$$

and that

$$\int_a^b c\,f(x)\ \alpha(dx) = c\int_a^b f(x)\ \alpha(dx).$$

Proof. Consider $f = f_1 + f_2$ and let P be a partition of $[a,b]$. We have

$$L(P, f_1, \alpha) + L(P, f_2, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P, f_1, \alpha) + U(P, f_2, \alpha).$$

The first inequality follows from the identity $\inf(f_1) + \inf(f_2) \le \inf(f_1 + f_2)$, while the second is a consequence of the trivial inequality $\inf(f) \le \sup(f)$. Finally, the last one follows from $\sup(f_1 + f_2) \le \sup(f_1) + \sup(f_2)$. Since $f_j \in \mathcal{R}(\alpha)$, $j = 1, 2$, for all $\varepsilon > 0$ there exists a partition $\mathcal{P}_j$ such that

$$U\left(\mathcal{P}_j, f_j, \alpha\right) - L\left(\mathcal{P}_j, f_j, \alpha\right) < \varepsilon,\ j = 1, 2. \tag{1.3}$$

Consider the common refinement $\mathcal{P} \doteq \mathcal{P}_1 \cup \mathcal{P}_2$. We know from the properties of the common refinement that

$$L\left(\mathcal{P}_j, f_j, \alpha\right) \le L\left(\mathcal{P}, f_j, \alpha\right) \text{ and } U\left(\mathcal{P}_j, f_j, \alpha\right) \ge U\left(\mathcal{P}, f_j, \alpha\right),\ j = 1, 2.$$

Using these properties with (0.0.3) we get

$$U\left(\mathcal{P}, f_j, \alpha\right) - L\left(\mathcal{P}, f_j, \alpha\right) < \varepsilon,\ j = 1, 2.$$

Now we have that

$$\begin{aligned} U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) &\le U(\mathcal{P}, f, \alpha) - [L(\mathcal{P}, f_1, \alpha) + L(\mathcal{P}, f_2, \alpha)] \\ &\le U(\mathcal{P}, f_1, \alpha) + U(\mathcal{P}, f_2, \alpha) - L(\mathcal{P}, f_1, \alpha) - L(\mathcal{P}, f_2, \alpha) \\ &= [U(\mathcal{P}, f_1, \alpha) - L(\mathcal{P}, f_1, \alpha)] + [U(\mathcal{P}, f_2, \alpha) - L(\mathcal{P}, f_2, \alpha)] < 2\,\varepsilon. \end{aligned}$$

This proves that $f \in \mathcal{R}(\alpha)$ on $[a,b]$. With the same partition $\mathcal{P}$ we have that

$$U\left(\mathcal{P}, f_j, \alpha\right) - \int_a^b f_j(x)\ \alpha(dx) < \varepsilon,\ j = 1, 2.$$

Therefore we can write

$$\int_a^b f(x)\ \alpha(dx) \le U(\mathcal{P}, f, \alpha) \le U(\mathcal{P}, f_1, \alpha) + U(\mathcal{P}, f_2, \alpha) < \int_a^b f_1(x)\ \alpha(dx) + \int_a^b f_2(x)\ \alpha(dx) + 2\,\varepsilon.$$

Now let $\varepsilon \to 0$, obtaining

$$\int_a^b f(x)\ \alpha(dx) \le \int_a^b f_1(x)\ \alpha(dx) + \int_a^b f_2(x)\ \alpha(dx).$$

On the other side we also have

$$\int_a^b f_j\,d\alpha - L\left(\mathcal{P}, f_j, \alpha\right) < \varepsilon,\ j = 1, 2.$$

and of course

$$\int_a^b f(x)\ \alpha(dx) \geq L(\mathcal{P}, f, \alpha) \geq L(\mathcal{P}, f_1, \alpha) + L(\mathcal{P}, f_2, \alpha).$$

Putting together the last two inequalities gives

$$\int_a^b f(x)\ \alpha(dx) > \int_a^b f_1(x)\ \alpha(dx) + \int_a^b f_1(x)\ \alpha(dx) - 2\,\varepsilon$$

Now let $\varepsilon \to 0$

$$\int_a^b f(x)\ \alpha(dx) \geq \int_a^b f_1(x)\ \alpha(dx) + \int_a^b f_2(x)\ \alpha(dx).$$

Concluding

$$\int_a^b f(x)\ \alpha(dx) = \int_a^b f_1(x)\ \alpha(dx) + \int_a^b f_2(x)\ \alpha(dx).$$

□

Theorem 1.5. *Let f be bounded function in $\mathcal{R}(\alpha)$ on $[a,b]$, i.e. $m \leq f \leq M$. Let g be a function continuous in $[m, M]$. Therefore the composit function $G \doteq g \circ f$ belongs to $\mathcal{R}(\alpha)$ on $[a,b]$.*

Proof. We have to show that $\forall \varepsilon^\star > 0$ there exists a partition $\mathcal{P}$ of $[a,b]$ such that $U(\mathcal{P}, G, \alpha) - L(\mathcal{P}, G, \alpha) < \varepsilon^\star$. As a first observation, note that the continuity of g on the compact set $[m, M]$ implies its uniform continuity, which means

$$\forall \varepsilon > 0,\ \exists \delta < \varepsilon : |t - s| < \delta \Rightarrow \left|g(t) - g(s)\right| < \varepsilon.$$

Notice that, we have added the condition $\delta < \varepsilon$, which is fully compatible with uniform integrability and will be useful at the end of the proof. Being f integrable, we can find a partition $\mathcal{P} = \{x_0, ..., x_n\}$ of $[a,b]$ such that $U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \delta^2$. Define, for $j = 1, \ldots, n$, the following quantities

$$\begin{aligned}
M_j &\doteq \sup\left\{f(x)\ \middle|\ x \in \left[x_{j-1}, x_j\right]\right\}, \\
m_j &\doteq \inf\left\{f(x)\ \middle|\ x \in \left[x_{j-1}, x_j\right]\right\}, \\
M_j^\star &\doteq \sup\left\{g(f(x))\ \middle|\ x \in \left[x_{j-1}, x_j\right]\right\}, \\
m_j^\star &\doteq \inf\left\{g(f(x))\ \middle|\ x \in \left[x_{j-1}, x_j\right]\right\}.
\end{aligned}$$

Divide the integers $\{1, ..., n\}$ into two classes A and B defined in this way

$$j \in A \Leftrightarrow M_j - m_j < \delta \text{ and } j \in B \Leftrightarrow M_j - m_j \geq \delta.$$

For the uniform continuity of g we have that

$$j \in A \Rightarrow M_j^\star - m_j^\star \leq \varepsilon.$$

Now let K be defined as $K = \sup_{x \in [m,M]} \left|g(f(x))\right|$. It is clear that

$$M_j^\star = \sup_{x \in \left[x_{j-1}, x_j\right]} g(f(x)) \leq \sup_{x \in \left[x_{j-1}, x_j\right]} \left|g(f(x))\right| \leq \sup_{x \in [a,b]} \left|g(f(x))\right| = K,$$

and that

$$m_j^\star = \inf_{x\in\left[x_{j-1},x_j\right]} g\left(f\left(x\right)\right) \geq \inf_{x\in[a,b]} g\left(f\left(x\right)\right) \geq \inf_{x\in[a,b]} -\left|g\left(f\left(x\right)\right)\right| = -\sup_{x\in[a,b]} \left|g\left(f\left(x\right)\right)\right| = -K,$$

where the second in equality follows from $g \geq -\left|g\right|$. Thus

$$M_j^\star - m_j^\star \leq K - m_j^\star \leq K + K = 2\,K.$$

By definition of the class B we have that

$$\sum_{j\in B} \left(M_j - m_j\right)\,\Delta_j\alpha \geq \delta \sum_{j\in B} \Delta_j\alpha. \tag{1.4}$$

Nevertheless $U\left(\mathcal{P}, f, \alpha\right) - L\left(\mathcal{P}, f, \alpha\right) < \delta^2$ is equivalent to

$$\sum_{j\in B} \left(M_j - m_j\right)\,\Delta_j\alpha < \delta^2. \tag{1.5}$$

Putting together inequalities (0.0.4) and (0.0.5) gives

$$\delta \sum_{j\in B} \Delta_j\alpha \leq \sum_{j\in B} \left(M_j - m_j\right)\,\Delta_j\alpha < \delta^2,$$

which, in turn, implies $\sum_{j\in B} \Delta_j\alpha < \delta$. Using the definition of the class A we get

$$\sum_{j\in A} \left(M_j^\star - m_j^\star\right)\,\Delta_j\alpha \leq \varepsilon \sum_{j\in A} \Delta_j\alpha \leq \varepsilon \sum_{j=1}^{n} \Delta_j\alpha = \varepsilon\,\left[\alpha\left(b\right) - \alpha\left(a\right)\right].$$

Finally, we compute the difference between the upper and lower sums of $G = g \circ f$ on $\mathcal{P}$, obtaining

$$\begin{aligned}
U\left(\mathcal{P}, G, \alpha\right) - L\left(\mathcal{P}, G, \alpha\right) &= \sum_{j=1}^{n} \left(M_j^\star - m_j^\star\right)\,\Delta_j\alpha = \sum_{j\in A} \left(M_j^\star - m_j^\star\right)\,\Delta_j\alpha + \sum_{j\in B} \left(M_j^\star - m_j^\star\right)\,\Delta_j\alpha \\
&\leq \varepsilon\,\left[\alpha\left(b\right) - \alpha\left(a\right)\right] + 2\,K\,\delta \\
&< \varepsilon\,\left[\alpha\left(b\right) - \alpha\left(a\right)\right] + 2\,K\,\varepsilon = \varepsilon\,\left[\alpha\left(b\right) - \alpha\left(a\right) + 2\,K\right].
\end{aligned}$$

Since ε is arbitrary the statement follows. □

Theorem 1.6. *Let f_1 and f_2 be bounded functions such that $f_j \in \mathcal{R}\left(\alpha\right)$ on $[a,b]$, $j = 1,2$. Then $f_1 \cdot f_2 \in \mathcal{R}\left(\alpha\right)$ on $[a,b]$, $\left|f_1\right| \in \mathcal{R}\left(\alpha\right)$ and*

$$\left|\int_a^b f_1\left(x\right)\,\alpha\left(dx\right)\right| \leq \int_a^b \left|f_1\left(x\right)\right|\,\alpha\left(dx\right).$$

Proof. Since $x \to x^2$ is a continuous function, from Theorem 0.0.5 we obtain that $f_j^2 \in \mathcal{R}\left(\alpha\right)$ on $[a,b]$ for $j = 1,2$. Using the algebraic identity

$$f_1\,f_2 = \frac{\left(f_1 + f_2\right)^2 - \left(f_1 - f_2\right)^2}{4},$$

we get that $f_1\,f_2 \in \mathcal{R}\left(\alpha\right)$ on $[a,b]$. Similarly, being $x \to |x|$ continuous we obtain from Theorem 0.0.5 that $\left|f\right| \in \mathcal{R}\left(\alpha\right)$ on $[a,b]$. Furthermore, given that

$$-\left|f\left(x\right)\right| \leq f\left(x\right) \leq \left|f\left(x\right)\right|,$$

by integration on both sides we get

$$-\int_a^b \left|f(x)\right| \alpha(dx) \leq \int_a^b f(x)\ \alpha(dx) \leq \int_a^b \left|f(x)\right| \alpha(dx),$$

which is equivalent to

$$\left|\int_a^b f(x)\ \alpha(dx)\right| \leq \int_a^b \left|f(x)\right| \alpha(dx).$$

□

Observation 1. *Note that if the absolute value of a function belongs to $\mathcal{R}(\alpha)$ on $[a,b]$ we cannot say that f belongs to $\mathcal{R}(\alpha)$ on $[a,b]$. Consider the function defined as*

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{R}/\mathbb{Q} \end{cases}.$$

Therefore $\left|f\right| = 1$ is integrable while f it is clearly not integrable.

The following theorem provides the correspondence between Riemann and Riemann-Stieltjes integrals:

Theorem 1.7. *Assume that α is monotonic and that α' is Riemann integrable on $[a,b]$. Let f be a bounded real function on $[a,b]$. Then $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if $g(x) = f(x)\ \alpha'(x)$ is Riemann integrable on $[a,b]$. In that case:*

$$\int_a^b f(x)\ \alpha(dx) = \int_a^b f(x)\ \alpha'(x)\ dx,$$

which is written unformally as:

$$\alpha(dx) = \alpha'(x)\ dx.$$

Theorem 1.8. *Suppose that φ is a strictly increasing function from $[A,B]$ into $[a,b]$ with $\varphi(A) = a$ and $\varphi(B) = b$. Suppose that α is monotonically increasing on $[a,b]$ and $f \in \mathcal{R}(\alpha)$ on (a,b). Define the new function*

$$g(y) \doteq f(\varphi(y)) : [A,B] \to \mathbb{R},$$

and the new measure

$$\beta(y) = \alpha(\varphi(y)).$$

Then $g \in \mathcal{R}(\beta)$ on $[A,B]$ and

$$\int_A^B g(y)\ \beta(dy) = \int_a^b f(x)\ \alpha(dx). \tag{1.6}$$

In particular if $\alpha(x) = x$ then $\beta = \varphi$ and applying Theorem 0.0.7 to the left side of equation (0.0.6) gives

$$\int_a^b f(x)\ dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(y))\ \varphi'(y)\ dy. \tag{1.7}$$

Remark 1. Let f be in $\mathcal{R}(\alpha)$ on $[a,b]$. We know that for all $\varepsilon > 0$ there exists a partition $\mathcal{P} = \{x_0, ..., x_n\}$ such that

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon. \tag{1.8}$$

By simply adding new points to the partition $\mathcal{P}$ we can assume that the partition $\mathcal{P}$ is the uniformly spaced partition

$$x_j = a + j\,\frac{b-a}{n}, \quad j = 0, ..., n.$$

Having added more points and given the monotone behaviour of the upper and lower sums with respect to a refinement of the partition, the relationship (0.0.8) still holds. Now note that

$$L(\mathcal{P}, f, \alpha) \le \sum_{j=1}^{n} f\left(x_j\right) \Delta_j \alpha \le U(\mathcal{P}, f, \alpha),$$

where, as usual, $\Delta_j \alpha = \alpha\left(x_j\right) - \alpha\left(x_{j-1}\right)$. It should be also obvious that

$$L(\mathcal{P}, f, \alpha) \le \int_a^b f(x)\ \alpha(dx) \le U(\mathcal{P}, f, \alpha).$$

Therefore

$$\int_a^b f(x)\ \alpha(dx) - \sum_{j=1}^{n} f\left(x_j\right) \Delta_j \alpha \le U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon,$$

an, simultaneously,

$$\int_a^b f(x)\ \alpha(dx) - \sum_{j=1}^{n} f\left(x_j\right) \Delta_j \alpha \ge L(\mathcal{P}, f, \alpha) - U(\mathcal{P}, f, \alpha) > -\varepsilon.$$

In summary

$$\left| \int_a^b f(x)\ \alpha(dx) - \sum_{j=1}^{n} f\left(x_j\right) \Delta_j \alpha \right| < \varepsilon,$$

which means

$$\lim_{n \to 0} \sum_{j=1}^{n} f\left(x_j\right) \Delta_j \alpha = \int_a^b f(x)\ \alpha(dx).$$

It is straightforward to verify that the same result holds if in the sum $\sum_{j=1}^{n} f\left(x_j\right) \Delta_j \alpha$ we replace each of the $f\left(x_j\right)$ with a new $f\left(t_j\right)$, but with $t_j \in \left[x_j, x_{j-1}\right]$

In practical applications, integrals are computed using the fundamental theorem of calculus, stated below. Here, we propose a list of solved exercises that exploit the results in Remark 1 to obtain explicit expression for some simple integrals.

Exercize 1. *Compute $\int_a^b K\, dx$, where K is a real constant.*

Solution. The constant function is integrable, so using Remark 1 we can write

$$\int_a^b K\, dx = \lim_{n \to \infty} \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) K = \lim_{n \to \infty} K \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) = \lim_{n \to \infty} K \sum_{j=1}^{n} \frac{b-a}{n} = \lim_{n \to \infty} K\, n\, \frac{b-a}{n} = K\,(b-a).$$

Exercize 2. *Compute $\int_a^b x\, dx$.*

Solution. The function $x \to x$ is integrable, so using Remark 1 we can write

$$\begin{aligned}\int_a^b x\,dx &= \lim_{n\to\infty}\sum_{j=1}^{n}\left(x_{j+1}-x_j\right)\left(a+\frac{b-a}{n}\,j\right)=\lim_{n\to\infty}\frac{b-a}{n}\sum_{j=1}^{n}\left(a+\frac{b-a}{n}\,j\right)=\lim_{n\to\infty}\frac{b-a}{n}\left(a\,n+\frac{b-a}{n}\sum_{j=1}^{n}j\right)\\ &= \lim_{n\to\infty}\frac{b-a}{n}\left(a\,n+\frac{b-a}{n}\,\frac{(n+1)\,n}{2}\right)=\lim_{n\to\infty}\left[a\,(b-a)+\frac{(b-a)^2}{n}\,\frac{n+1}{2}\right]=a\,(b-a)+\frac{(b-a)^2}{2}\\ &= a\,b-a^2+\frac{1}{2}\,b^2+\frac{1}{2}\,a^2-a\,b=\frac{1}{2}\left(b^2-a^2\right).\end{aligned}$$

Note that using the identity

$$\sum_{j=1}^{n} j^2=\frac{n\,(n+1)\,(2\,n+1)}{6},$$

and proceeding in the exact same way it is possible to show that

$$\int_a^b x^2\,dx=\frac{1}{3}\left(b^3-a^3\right)$$

Theorem 1.9. ***(First fundamental theorem of calculus).*** *Let f be Riemann-integrable in $[a,b]$ and F be defined as:*

$$F(x)=\int_0^x f(t)\;dt,$$

then $F(x)$ is uniformly continuous (and thus continuous) on $[a,b]$. Moreover if f is continuous in $x_0 \in [a,b]$ then F is differentiable and

$$F'(x_0)=f(x_0)\,.$$

Proof. Boundedness of f implies $\left|f(t)\right| \le M$ for all $t \in [a,b]$ for some constant $M > 0$. Take two points x and y such that $a \le x \le y \le b$ to have

$$\left|F(y)-F(x)\right|=\left|\int_x^y f(t)\;dt\right|\le\int_x^y\left|f(t)\right|\,dt\le M\int_x^y 1\,dt=M\,(y-x)\,.$$

Now for all $\varepsilon > 0$ it is enpugh to take x and y such that $\left|y-x\right| < \frac{\varepsilon}{M}$ to have uniform continuity of F . Suppose that f is continuous in x_0. Therefore $\forall\varepsilon > 0$ there exists $\delta > 0$ such that $\left|f(t)-f(x_0)\right| < \varepsilon$ whenever $|t-x_0| < \delta$ or, equivalently, for all $t \in (x_0-\delta, x_0+\delta)$. For h positive, but sufficiently small, we get $x_0+h \in (x_0-\delta, x_0+\delta)$ and, accordingly, that

$$\left|\frac{F(x_0+h)-F(x_0)}{h}-f(x_0)\right|=\left|\frac{\int_{x_0}^{x_0+h}f(t)\;dt}{h}-f(x_0)\right|=\left|\frac{\int_{x_0}^{x_0+h}\left[f(t)-f(x_0)\right]\,dt}{h}\right|\le\frac{\int_{x_0}^{x_0+h}\left|f(t)-f(x_0)\right|\,dt}{h}<\varepsilon.$$

Taking the limit $h \to 0^+$ we obtain $F'\left(x_0^+\right)=f(x_0)$ and, with a specular argument, $F'\left(x_0^-\right)=f(x_0)$. In summary $F'(x_0)=f(x_0)$. □

Theorem 1.10. ***(Second fundamental theorem of calculus).*** *Let f be a function in $\mathcal{R}(\alpha)$ on $[a,b]$. If there exists a differentiable function F on $[a,b]$ such that $f=F'$ then*

$$\int_a^b f(t)\;dt=F(b)-F(a)\,.$$

The function F is called a primitive of f.

Proof. Let $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. The mean value theorem and the first fundamental theorem of calculus applied to F assert that, for all $j = 1, .., n$, there exists a point $t_j \in \left[x_{j-1}, x_j\right]$ such that:

$$F\left(x_j\right) - F\left(x_{j-1}\right) = F'\left(t_j\right) \Delta_j x = f\left(t_j\right) \Delta_j x.$$

Summing across all the indexes gives

$$\sum_{j=1}^{n} f\left(t_j\right) \Delta_j x = \sum \left[F\left(x_j\right) - F\left(x_{j-1}\right)\right] = F(b) - F(a).$$

Note that the left-hand side depends on n while the right-hand does not. Taking the limit for $n \to \infty$ and using Remark 1 we find that

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{j=1}^{n} f\left(t_j\right) \Delta_j x = \int_a^b f(x)\, dx.$$

$\square$

In what follows we shall use the notation $\int f(x)\, dx$ to indicate all the primitives of the function f. Sometimes the integral $\int f(x)\, dx$ is called the *indefinite integral* of the function f. Accordingly, computing the indefinite integral of f amounts to find all of its primitives.

We now proceed to illustrate implications of the two fundamental theorems of calculus with a collection of solved exercises.

Exercize 3. *Find a function such that f is integrable, but $\nexists g$ such that $f = g'$.*

Solution. If an integrable function is changed in one point, or in a finite number of points, the function remains integrable and, moreover, the integral does not change value. Now take a function f continuous in a closed interval $[a, b]$ and modify it as follows:

$$\widetilde{f}(x) = \begin{cases} L \neq f(x_0) & x = x_0 \\ f(x) & x \neq x_0. \end{cases}$$

The function $\widetilde{f}$ is integrable but has a simple discontinuity and for Theorem **??** a function that is the derivative of another function cannot have simple discontinuities. Therefore $\nexists g$ such that $f = g'$.

Exercize 4. *Compute the derivative of* $F(x) = \int_a^{x^3} (\sin(t))^3\, dt$.

Solution. Note that $F(x) = h(g(x))$, where $h(g) = \int_a^g \sin^3(t)\, dt$ and $g(x) = x^3$. Therefore:

$$F'(x) = h'(g(x))\, g'(x) = \sin^3\left(x^3\right) 3\, x^2.$$

Exercize 5. *Compute the derivative of*

$$F(x) = \int_x^b \frac{1}{1 + t^2 + \sin(t)}\, dt.$$

Solution. Write $F(x) = -\int_b^x dt/\left(1+t^2+\sin(t)\right)$. Then

$$F'(x) = -\frac{1}{1+x^2+\sin(x)}.$$

Exercize 6. *Find all the primitives of the function* $f(x) = 1/x$.

Solution. Since, for $x > 0$, we have $(\log(x))' = 1/x$ and, for all $x \neq 0$, we have $|x|' = |x|/x$, we get that $F(x) = \log(|x|) + c$ is a primitive for all $c \in \mathbb{R}$.

Exercize 7. *Find all the primitives of the function* $f(x) = \sqrt{e^{3x}}$.

Solution. Observing that $(e^x)' = e^x$ it is immediate to verify that $F(x) = \frac{3}{2} e^{\frac{3}{2}x} + c$ is a primitive for all $c \in \mathbb{R}$.

Exercize 8. *Compute* $\int_0^{\frac{\pi}{2}} x \sin(x)\, dx$.

Solution. We introduce the so-called integration by part method. Let $a < b$ be two real numbers. If f and g are two integrable functions such that f' and g' exist and are integrable on $[a, b]$, from the derivative of the product

$$(f\,g)' = f'\,g + f\,g'$$

we derive

$$f'\,g = (f\,g)' - f\,g'$$

whence

$$\int_a^b f'(x)\, g(x)\, dx = f(b)\, g(b) - f(a)\, g(a) - \int_a^b f(x)\, g'(x)\, dx. \tag{1.9}$$

To solve the exercise, we use the integration-by-part formula (0.0.9) with $f'(x) = \sin(x)$ and $g(x) = x$. Hence $g'(x) = 1$ and $f(x) = -\cos(x)$, whence

$$\int_0^{\frac{\pi}{2}} x \sin(x)\, dx = (-x\cos(x))|_{\frac{\pi}{2}} - (-x\cos(x))|_0 - \int_0^{\frac{\pi}{2}} (-\cos(x))\, dx = \sin\left(\frac{\pi}{2}\right) = 1,$$

where we used the notation $f(x)\big|_{x_0}$ to indicate that the function f must be computed in x_0. Sometimes we use the short-hand notation (quite common) $[f(x)]_{x_0}^{x_1} = f(x)\big|_{x_1} - f(x)\big|_{x_0} = f(x_1) - f(x_0)$.

Exercize 9. *Compute* $\int_{-1}^1 x \arcsin(x)\, dx$.

Solution. It is convenient to use the change-of-variable formula (0.0.7). Call $x = \sin(\vartheta)$, from which $dx = \cos(\vartheta)\, d\vartheta$ and so

$$\int_{-1}^1 x \arcsin x\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \vartheta \sin(\vartheta) \cos(\vartheta)\, d\vartheta.$$

Using the trigonometric identity $\cos(\vartheta)\sin(\vartheta) = \frac{1}{2}\sin(2\vartheta)$ we get

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \vartheta \sin(\vartheta) \cos(\vartheta)\, d\vartheta = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \vartheta \sin(2\vartheta)\, d\vartheta.$$

Using formula (0.0.9) with $g(\vartheta) = \frac{1}{2}\,\vartheta$ and $f'(\vartheta) = \sin(2\,\vartheta)$, that is $f(\vartheta) = -\frac{1}{2}\,\cos(2\,\vartheta)$, we get

$$\begin{aligned}
\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \vartheta\,\sin(2\,\vartheta)\;d\vartheta &= \left[-\frac{1}{4}\,\vartheta\,\cos(2\,\vartheta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-)\,\cos(2\,\vartheta)\;d\vartheta = \left[-\frac{1}{4}\,\vartheta\,\cos(2\,\vartheta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left[\frac{1}{8}\,\sin(2\,\vartheta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.
\end{aligned}$$

Exercize 10. *Compute* $\int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$.

Solution. Substitute $x = \sin(\vartheta) \Rightarrow dx = \cos(\vartheta)\;d\vartheta$, obtaining

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{\sin^2(\vartheta)}{\cos(\vartheta)}\cos(\vartheta)\;d\vartheta = \int_0^{\frac{\pi}{4}} \sin^2(\vartheta)\;d\vartheta.$$

Use $\sin^2(\vartheta) = \frac{1-\cos(2\,\vartheta)}{2}$ to have

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{1-\cos(2\,\vartheta)}{2}\,d\vartheta = \frac{1}{2}\left[\vartheta - \frac{1}{2}\,\sin(2\,\vartheta)\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{\pi-2}{8}.$$

Exercize 11. *Compute all the primitives of the function* $f(x) = \frac{x-1}{x\,(x+1)^2}$.

Solution. We look for constants A, B and C such that

$$\frac{x-1}{x\,(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Straightforward computations show that $A = -1$, $B = 1$ and $C = 2$. Whence

$$F(x) = -\log|x| + \log|1-x| - \frac{2}{(x+1)} + c,$$

is a primitive for all $c \in \mathbb{R}$.

Exercize 12. *Compute all the primitives of the function* $f(x) = \frac{2\,x-1}{2\,x^2-2\,x+3}$.

Solution. Put $g(x) = 2\,x^2 - 2\,x + 3$ and note that

$$\frac{2\,x-1}{2\,x^2-2\,x+3} = \frac{1}{2}\,\frac{g'(x)}{g(x)},$$

whence $F(x) = \frac{1}{2}\,\log\left|2\,x^2 - 2\,x + 3\right| + c$ is a primitive for all $c \in \mathbb{R}$.

Exercize 13. *Compute all the primitives of the function* $f(x) = \frac{x^3-1}{x^2-3\,x+2}$.

Solution. The function f is a fraction in which the degree of the numerator is greater than the degree of the denominator. We can thus proceed using the Ruffini's decomposition algorithm:

$x^3 - 1$	$x^2 - 3\,x + 2$
$(+x)\cdot\left(x^2 - 3\,x + 2\right) = x^3 - 3\,x^2 + 2\,x$	$\frac{x^3}{x^2} = x$
$x^3 - 1 - x^3 + 3\,x^2 - 2\,x = 3\,x^2 - 2\,x - 1$	

and we continue till division is not more allowed, obtaining

x^3-1	$x^2-3\,x+2$
$x^3-3\,x^2+2\,x$	$x+\frac{3\,x^2}{x^2}=x+3\equiv Q\,(x)$
$3\,x^2-2\,x-1$ $3\cdot\left(x^2-3\,x+2\right)=3\,x^2-9\,x+6$	
$3\,x^2-2\,x-1-\left(3\,x^2-9\,x+6\right)=7\,x-7\equiv R\,(x)$	

In summary

$$\frac{x^3-1}{x^2-3\;x+2}=(x+3)+\frac{7\,x-7}{x^2-3\,x+2}=(x+3)+7\,\frac{x-1}{(x-1)\;(x-2)}=(x+3)+7\,\frac{1}{x-2},$$

whence

$$F\,(x)=\frac{1}{2}\,x^2+3\,x+7\;\log|x-2|+c,$$

is a primitive for all $c\in\mathbb{R}$.

Exercize 14. *Compute the indefinite integral* $\int\sqrt{2\,x-x^2}\,dx$.

Solution. Note that $\sqrt{2\,x-x^2}=\sqrt{1-(x-1)^2}$. Change variable $u=x-1$, obtaining $\sqrt{2\,x-x^2}\,dx=\sqrt{1-u^2}\,du$. Now change again with $u=\sin(\vartheta)$, which gives $du=\cos(\vartheta)\;d\vartheta$ and so

$$\sqrt{2\,x-x^2}\,dx=\sqrt{1-u^2}\,du=\cos^2(\vartheta)\;d\vartheta.$$

Remember that $\cos^2(\vartheta)=\frac{1+\cos(2\,\vartheta)}{2}$. Therefore

$$\begin{aligned}\int\sqrt{2\,x-x^2}\,dx &= \int\sqrt{1-u^2}\,du=\int\cos^2(\vartheta)\;d\vartheta=\int\frac{1+\cos(2\,\vartheta)}{2}\,d\vartheta=\frac{\vartheta}{2}+\frac{1}{4}\;\sin(2\,\vartheta)+c\\ &= \frac{\vartheta}{2}+\frac{1}{2}\;\sin(\vartheta)\;\cos(\vartheta)+c=\frac{\arcsin(u)}{2}+\frac{1}{2}\;\sin(\arcsin(u))\;\cos(\arcsin u)+c\\ &= \frac{\arcsin(u)}{2}+\frac{1}{2}\,u\;\cos(\arcsin(u))+c,\end{aligned}$$

where c is any real constant. Now note that $\cos(\arcsin(u))=\sqrt{1-\sin^2(\arcsin(u))}=\sqrt{1-u^2}=\sqrt{1-(x-1)^2}$. Therefore

$$\int\sqrt{2\,x-x^2}\,dx=\frac{\arcsin(x-1)}{2}+\frac{1}{2}\;(x-1)\;\sqrt{1-(x-1)^2}+c.$$