# Exercises $5^{th}$ Week: Solution

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## Exercise 2

Let $X$ be a random variable with p.d.f.:

$$f(x|\theta) = \frac{exp(-(x-\theta))}{(1+exp(-(x-\theta)))^2} \qquad -\infty < x < \infty \quad -\infty < \theta < \infty$$

Use the pivotal method to verify that if $0 < \alpha_1 < 0.5$ and $0 < \alpha_2 < 0.5$, then

$$\left(X - log\left(\frac{1-\alpha_2}{\alpha_2}\right), X - log\left(\frac{\alpha_1}{1-\alpha_1}\right)\right)$$

is a confidence interval for $\theta$ with coverage probability $1-(\alpha_1+\alpha_2)$.

### Solution Exercise 2

- Pivotal Variable : $Y = X - \theta$
- Distribution of $Y$

$$\begin{aligned} f_Y(y) &= \frac{exp(-y))}{(1+exp(-y))^2} \qquad -\infty < y < \infty \\ F_Y(y) = P(Y \le y) &= \int_{-\infty}^{y} \frac{exp(-t))}{(1+exp(-t))^2} dt = \frac{1}{1+exp(-y)} \end{aligned}$$

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$$\begin{aligned} & P\left(X - log\left(\frac{1-\alpha_2}{\alpha_2}\right) < \theta \le X - log\left(\frac{\alpha_1}{1-\alpha_1}\right)\right) \\ = & P\left(log\left(\frac{\alpha_1}{1-\alpha_1}\right) \le Y < log\left(\frac{1-\alpha_2}{\alpha_2}\right)\right) \\ = & P\left(Y < log\left(\frac{1-\alpha_2}{\alpha_2}\right)\right) - P\left(Y < log\left(\frac{\alpha_1}{1-\alpha_1}\right)\right) \\ = & \frac{1}{1+exp\left(-log\left(\frac{1-\alpha_2}{\alpha_2}\right)\right)} - \frac{1}{1+exp\left(-log\left(\frac{\alpha_1}{1-\alpha_1}\right)\right)} \\ = & \frac{1}{1+\frac{\alpha_2}{1-\alpha_2}} - \frac{1}{1+\frac{1-\alpha_1}{\alpha_1}} = 1 - \alpha_1 - \alpha_2 \end{aligned}$$

## Exercise 3

Compare the asymptotic efficiency of method of moments estimator of parameter $\alpha$ of the Pareto distribution with maximum likelihood estimator (assume $x_m$ known). Remind that a r.v. X with a Pareto distribution has the following density:

$$f(x;\alpha,x_{\mathrm{m}}) = \alpha\, x_{\mathrm{m}}^{\alpha} x^{-(\alpha+1)} \quad \text{for } x \geq x_m$$

### Solution: Exercise 3

Observe that the joint pdf of $X = (X_1, \ldots, X_n)$

$$\begin{aligned} f(x;\alpha,x_{\mathrm{m}}) &= \prod_{i=1}^{n} \frac{\alpha\, x_{\mathrm{m}}^{\alpha}}{x_i^{\alpha+1}} \\ &= \frac{\alpha^n\, x_{\mathrm{m}}^{n\alpha}}{\prod_{i=1}^{n} x_i^{\alpha+1}} \\ &= g(t,\alpha)h(x) \end{aligned}$$

where $t = \prod_{i=1}^{n} x_i$ $g(t,\alpha) = c\alpha^n\, x_{\mathrm{m}}^{n\alpha} t^{-(\alpha+1)}$ and $h(x) = 1$.

By the factorization theorem: $T(X) = \prod_{i=1}^{n} X_i$ is a sufficient statistics for $\alpha$

The log-likelihood function is

$$logL(\alpha,x_{\mathrm{m}}) = nlog\alpha + n\alpha log(x_{\mathrm{m}}) - (\alpha+1)\sum_{i=1}^{n} logx_i$$

$$\begin{aligned} \frac{\delta logL(\alpha,x_{\mathrm{m}})}{\delta\alpha} &= \frac{n}{\alpha} + nlog(x_{\mathrm{m}}) - \sum_{i=1}^{n} logx_i \\ \frac{\delta^2 logL(\alpha,x_{\mathrm{m}})}{\delta\alpha^2} &= -\frac{n}{\alpha^2} \end{aligned}$$

Solving for $\frac{\delta logL(\alpha,x_{\mathrm{m}})}{\delta\alpha} = 0$, since $\frac{\delta^2 logL(\alpha,x_{\mathrm{m}})}{\delta\alpha^2} < 0$ the mle of $\alpha$ is given by

$$\begin{aligned} \hat{\alpha}_{MLE} &= \frac{n}{\sum_{i=1}^{n} logx_i - nlog(x_{\mathrm{m}})} \\ Var(\hat{\alpha}_{MLE}) &= (nI(\alpha))^{-1} = \frac{\alpha^2}{n} \end{aligned}$$

Compute by yourself $E(X)$ and $Var(X)$

$$\begin{aligned} E(X) &= \frac{\alpha x_m}{\alpha-1} \qquad \alpha > 1 \\ Var(X) &= \frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)} \qquad \alpha > 2 \end{aligned}$$

Equating first population moment with first sample moment, we find $\alpha_{MOM}$

$$\begin{aligned} E(X) &= \bar{X} \\ \frac{\alpha x_m}{\alpha - 1} &= \bar{X} \\ \hat{\alpha}_{MOM} &= \frac{\bar{X}}{\bar{X} - x_m} \end{aligned}$$

Distribution of $\hat{\alpha}_{MOM}$ can be found by delta method, REMIND:
Let $Y_n, \quad n = 1, 2, \ldots,$ be a sequence of random variables such that

$$\sqrt{n}\,(Y_n - \theta) \longrightarrow_d N(0, \sigma^2)$$

Furthermore let $g()$ be a twice differentiable real function defined on $R$ such that $g'(\theta) \neq 0$. Then

$$\sqrt{n}\,(g(Y_n) - g(\theta)) \longrightarrow_d N(0, \left(g'(\theta)\right)^2 \sigma^2)$$

- $Y_n = \bar{X}$
- $\theta = E(\bar{X}) = \frac{\alpha x_m}{\alpha - 1}$
- $\sigma^2 = Var(\bar{X}) = \frac{1}{n} \frac{\alpha x_m^2}{(\alpha-1)^2(\alpha-2)}$
- $$\begin{aligned} g(t) &= \frac{t}{t - x_m} \\ g\prime(t) &= \frac{-x_m}{(t - x_m)^2} \\ (g\prime(t))^2 &= \frac{x_m^2}{(t - x_m)^4} \\ (g\prime(\theta))^2 &= \frac{(\alpha - 1)^4}{x_m^2} \\ (g\prime(\theta))^2 \sigma^2 &= \frac{1}{n} \frac{\alpha(\alpha - 1)^2}{\alpha - 2} \end{aligned}$$

$$Var(\hat{\alpha}_{MOM}) = \frac{1}{n} \frac{\alpha(\alpha - 1)^2}{\alpha - 2}$$

Comparison of the asymptotic efficiency of the two estimators:

$$\frac{Var(\hat{\alpha}_{MOM})}{Var(\hat{\alpha}_{MLE})} = \frac{\frac{1}{n}\frac{\alpha(\alpha-1)^2}{\alpha-2}}{\frac{\alpha^2}{n}} = \frac{(\alpha - 1)^2}{\alpha(\alpha - 2)}$$

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\begin{aligned}
\frac{(\alpha-1)^2}{\alpha(\alpha-2)} &> 1 \\
\alpha-1)^2 &> \alpha(\alpha-2)) \\
\alpha^2-2\alpha+1 &> \alpha^2-2\alpha \\
1 &> 0 \\
always &
\end{aligned}
$$

## Exercise 4

The life times of the neon lamps in the Uni Mail rooms can be modelled by an exponential distribution, i.e. we suppose that $(X_1, \ldots, X_n)$ are from the exponential distribution $Exp(\lambda)$ with probability distribution function:

$$f(x|\lambda) = \lambda exp(-\lambda x) I_{(0,\infty)}(x)$$

We would like to construct point and interval estimates of the median survival time:

1. Compute the median $Me(\lambda)$ of the model,
2. Find $\hat{Me}(\lambda)_{MLE}$: the maximum likelihood estimator for $Me(\lambda)$, the median of this distribution,
3. Give an approximate (asymptotic) confidence interval for $Me(\lambda)$ based on $\hat{Me}(\lambda)_{MLE}$,
4. Show that $2\lambda \sum_i X_i$ is a pivotal statistic and give the exact confidence interval for $Me(\lambda)$.
5. Suppose that for a random sample of $n = 100$ neon lamps you observe $\bar{x} = 5$, compare the exact and approximate confidence interval and discuss results found.

### Solution: Exercise 4

1.

2. Compute the median $Me(\lambda)$ of the model,

$$\begin{aligned}
\int_0^{Me(\lambda)} f(x|\lambda)dx &= 0.5 \\
\int_0^{Me(\lambda)} \lambda exp(-\lambda x)dx &= |_0^{Me(\lambda)} - exp(-\lambda x) \\
1 - exp(-\lambda Me(\lambda)) &= 0.5 \\
exp(-\lambda Me(\lambda)) &= 0.5 \\
Me(\lambda) &= \frac{1}{\lambda} log(2)
\end{aligned}$$

Median of an exponential distribution: $Median = \frac{log(2)}{\lambda}$.

3. For invariance property of MLE: $\hat{Median}_{MLE} = log(2)\bar{X}$ Find $\hat{M}e(\lambda)_{MLE}$: the maximum likelihood estimator for $Me(\lambda)$, the median of this distribution,

$$\begin{aligned}
\hat{M}e_{MLE}(\lambda) &= \frac{1}{\hat{\lambda}} log(2) \\
\hat{\lambda}_{MLE} &= \frac{1}{\bar{X}} \\
\hat{M}e(\lambda)_{MLE} &= \bar{X} log(2)
\end{aligned}$$

$$\left[ log(2)\bar{X} - z_{1-\alpha/2}\frac{(log(2)\bar{X})^2}{n}; \, log(2)\bar{X} + z_{1-\alpha/2}\frac{(log(2)\bar{X})^2}{n} \right]$$

## Exercise 5

To study the mean of a population variable $X$, $\mu = E(X)$, a simple random sample of size $n$ is considered. Imagine that we do not trust the first and the last data, so we consider the following statistics:

$$\tilde{X} = \frac{1}{n-2}\sum_{j=2}^{n-1} X_j = \frac{X_2 + \ldots + X_{n-1}}{n-2}$$

Calculate the expectation and the variance of this statistic. Calculate the mean square error (MSE) and its limit when $n$ tends to infinite. Study the consistency. Compare the previous error with that of the ordinary sample mean.