

Likelihood ratio test: Example

Suppose that the distribution of lifetimes of TV tubes can be adequately modelled by an exponential distribution with mean λ so

$$f(x|\lambda) = \frac{1}{\lambda} \exp\left(-\frac{x}{\lambda}\right)$$

for $x \geq 0$. Under usual production conditions, the mean lifetime is 50 hours but if a fault occurs in the process, the mean lifetime drops to 40 hours. A random sample of 20 tube lifetimes is taken in order to test the hypotheses

$$H_0 : \lambda = 50 \quad H_1 : \lambda = 30.$$

And the following statistics is observed $\sum_i x_i = 680$

Use the Neyman-Pearson lemma to find the most powerful test with significance level $\alpha = 0.05$.

Likelihood ratio test: Example

$$R = \left\{ x : \sum_i x_i \leq K \right\}$$

$$R = \left\{ x : \sum_i x_i \leq \text{InvGamma}(\alpha, a = 20, b = 50) \right\}$$

$$R = \left\{ x : \sum_i x_i \leq 662.73 \right\}$$

$$R = \{x : \bar{x} \leq 33.14\}$$

ACCEPT NULL HYPOTHESIS

Let (X_1, \dots, X_n) be a random sample of i.i.d. random variables distributed as a Pareto distribution with parameters α and x_m

$$f(x; \alpha, x_m) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}} \quad \text{for } x \geq x_m.$$

The Pareto distribution is used to describe the distribution of income in the top portion of the income distribution. Therefore, if x is income, the pdf is defined for incomes in excess of x_m . It is usually used to describe the income in higher income groups.

$$E(X) = \frac{\alpha x_m}{\alpha - 1}$$

Consider a sample of people of 100 family with incomes in excess of 50,000\$ and suppose we want to test $H_0, \alpha = 5$ versus an alternative $H_0, \alpha \neq 5$. Suppose (x_1, \dots, x_{100}) has an observed sample mean of 60,000\$ and observed mean of logarithm equal to 11.0698.

Observe that the joint pdf of $X = (X_1, \dots, X_n)$

$$\begin{aligned} f(x; \alpha, x_m) &= \prod_{i=1}^n \frac{\alpha x_m^\alpha}{x_i^{\alpha+1}} \\ &= \frac{\alpha^n x_m^{n\alpha}}{\prod_{i=1}^n x_i^{\alpha+1}} \\ &= g(t, \alpha) h(x) \end{aligned}$$

where $t = \prod_{i=1}^n x_i$ $g(t, \alpha) = c \alpha^n x_m^{n\alpha} t^{-(\alpha+1)}$ and $h(x) = 1$. By the factorization theorem:

$T(X) = \prod_{i=1}^n X_i$ is a sufficient statistics for α

The log-likelihood function is

$$\log L(\alpha, x_m) = n \log \alpha + n \alpha \log(x_m) - (\alpha + 1) \sum_{i=1}^n \log x_i$$

Thus

$$\frac{\delta \log L(\alpha, x_m)}{\delta \alpha} = \frac{n}{\alpha} + n \log(x_m) - \sum_{i=1}^n \log x_i$$

Solving for $\frac{\delta \log L(\alpha, x_m)}{\delta \alpha} = 0$, the mle of α is given by

$$\begin{aligned}\hat{\alpha} &= \frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)} = \frac{1}{\overline{\log x_i} - \log(x_m)} \\ \hat{\alpha} &= 4\end{aligned}$$

MLE Estimator

$$\frac{n}{\sum_{i=1}^n \log X_i - n \log(x_m)} = \frac{1}{\log \bar{X} - \log(x_m)}$$

MLE Estimate

$$\alpha = 4$$

MLE Estimator

$$\frac{n}{\sum_{i=1}^n \log X_i - n \log(x_m)} = \frac{1}{\log \bar{X} - \log(x_m)}$$

MLE Estimate

$$\alpha = 4$$

$$\begin{aligned}\frac{\delta \log L(\alpha, x_m)}{\delta \alpha} &= \frac{n}{\alpha} + n \log(x_m) - \sum_{i=1}^n \log x_i \\ \frac{\delta^2 \log L(\alpha, x_m)}{\delta \alpha^2} &= -\frac{n}{\alpha^2}\end{aligned}$$

$$\frac{\delta \log L(\alpha, x_m)}{\delta \alpha} = \frac{n}{\alpha} + n \log(x_m) - \sum_{i=1}^n \log x_i$$

$$\frac{\delta^2 \log L(\alpha, x_m)}{\delta \alpha^2} = -\frac{n}{\alpha^2}$$

$$\text{Var}(\hat{\alpha}) = (nI(\alpha))^{-1} = \frac{\alpha^2}{n}$$

$$\frac{I(\alpha_0)}{n} = 0.25$$

$$\frac{I(\hat{\alpha})}{n} = 0.16$$

Score Function

$$\frac{\delta \log L(\alpha, x_m)}{\delta \alpha} = \frac{n}{\alpha} + n \log(x_m) - \sum_{i=1}^n \log x_i$$

Fisher Information

$$I_n(\alpha) = \frac{n}{\alpha^2}$$

Score Function

$$\frac{\delta \log L(\alpha, x_m)}{\delta \alpha} = \frac{n}{\alpha} + n \log(x_m) - \sum_{i=1}^n \log x_i$$

Fisher Information

$$I_n(\alpha) = \frac{n}{\alpha^2}$$

Pareto: Large Sample Test

Find the Wald test statistic for testing

$$H_0 : \alpha_0 = 5 \quad \text{versus} \quad H_1 : \alpha_0 \neq 5$$

Specify the distribution and Verify the null hypothesis, with $\alpha = 0.05$

Find the Score test statistic for testing

$$H_0 : \alpha_0 = 5 \quad \text{versus} \quad H_1 : \alpha_0 \neq 5$$

Specify the distribution and Verify the null hypothesis, with $\alpha = 0.05$

Pareto: Large Sample Test

Find the Wald test statistic for testing

$$H_0 : \alpha_0 = 5 \quad \text{versus} \quad H_1 : \alpha_0 \neq 5$$

Specify the distribution and Verify the null hypothesis, with $\alpha = 0.05$

Find the Score test statistic for testing

$$H_0 : \alpha_0 = 5 \quad \text{versus} \quad H_1 : \alpha_0 \neq 5$$

Specify the distribution and Verify the null hypothesis, with $\alpha = 0.05$

Find the Likelihood Ratio test statistic for testing

$$H_0 : \alpha_0 = 5 \quad \text{versus} \quad H_1 : \alpha_0 \neq 5$$

Specify the distribution and Verify the null hypothesis, with $\alpha = 0.05$

Pareto: Large Sample Test

Find the Wald test statistic for testing

$$H_0 : \alpha_0 = 5 \quad \text{versus} \quad H_1 : \alpha_0 \neq 5$$

Specify the distribution and Verify the null hypothesis, with $\alpha = 0.05$

Find the Score test statistic for testing

$$H_0 : \alpha_0 = 5 \quad \text{versus} \quad H_1 : \alpha_0 \neq 5$$

Specify the distribution and Verify the null hypothesis, with $\alpha = 0.05$

Find the Likelihood Ratio test statistic for testing

$$H_0 : \alpha_0 = 5 \quad \text{versus} \quad H_1 : \alpha_0 \neq 5$$

Specify the distribution and Verify the null hypothesis, with $\alpha = 0.05$

$$\hat{\alpha} \sim N\left(\alpha_0, \frac{1}{nl(\alpha_0)}\right)$$
$$(\hat{\alpha} - \alpha_0)^2 \times nl(\alpha_0) \sim \chi_1$$

Wald test Statistics

$$(\hat{\alpha} - \alpha_0)^2 nl(\hat{\alpha})$$

Wald test Statistics

$$(\hat{\alpha} - \alpha_0) \times \sqrt{nl(\hat{\theta})}$$

$$\hat{\alpha} \sim N\left(\alpha_0, \frac{1}{nl(\alpha_0)}\right)$$
$$(\hat{\alpha} - \alpha_0)^2 \times nl(\alpha_0) \sim \chi_1$$

Wald test Statistics

$$(\hat{\alpha} - \alpha_0)^2 nl(\hat{\alpha})$$

Wald test Statistics

$$(\hat{\alpha} - \alpha_0) \times \sqrt{nl(\hat{\theta})}$$

Wald test Statistics

$$(\hat{\alpha} - \alpha_0) \times \sqrt{nl(\hat{\alpha})}$$

Wald test Statistics

$$\frac{\frac{\sum_{i=1}^n \log x_i - n \log(x_m)}{n} - \alpha_0}{\frac{\sum_{i=1}^n \log x_i - n \log(x_m)}{n \sqrt{n}}}$$

Wald test Statistics

$$(\hat{\alpha} - \alpha_0) \times \sqrt{nl(\hat{\alpha})}$$

Wald test Statistics

$$\frac{\frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)} - \alpha_0}{\frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)} \sqrt{n}}$$

Distribution of test statistics

$$N(0, 1)$$

Wald test Statistics

$$(\hat{\alpha} - \alpha_0) \times \sqrt{nl(\hat{\alpha})}$$

Wald test Statistics

$$\frac{\frac{\sum_{i=1}^n \log x_i - n \log(x_m)}{\sum_{i=1}^n \log x_i - n \log(x_m)} - \alpha_0}{\frac{n}{\sqrt{n}}}$$

Distribution of test statistics

$$N(0, 1)$$

Pareto- Wald Test

Wald test Statistics

$$\frac{\frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)} - \alpha_0}{\frac{\frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)}}{\sqrt{n}}}$$

Distribution of test statistics

$$N(0, 1)$$

Pareto- Wald Test

Wald test Statistics

$$\frac{\frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)} - \alpha_0}{\frac{\frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)}}{\sqrt{n}}}$$

Distribution of test statistics

$$N(0, 1)$$

Observed value of test statistics

$$\frac{4 - 5}{\sqrt{0.16}} = -2.5 \quad p\text{-value} = 0.0124$$

Pareto- Wald Test

Wald test Statistics

$$\frac{\frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)} - \alpha_0}{\frac{\frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)}}{\sqrt{n}}}$$

Distribution of test statistics

$$N(0, 1)$$

Observed value of test statistics

$$\frac{4 - 5}{\sqrt{0.16}} = -2.5 \quad p\text{-value} = 0.0124$$

Decision: REJECT

Pareto- Wald Test

Wald test Statistics

$$\frac{\frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)} - \alpha_0}{\frac{\frac{n}{\sum_{i=1}^n \log x_i - n \log(x_m)}}{\sqrt{n}}}$$

Distribution of test statistics

$$N(0, 1)$$

Observed value of test statistics

$$\frac{4 - 5}{\sqrt{0.16}} = -2.5 \quad p\text{-value} = 0.0124$$

Decision: REJECT

Wald Test -Alternative (variance under H_0)

Wald test Statistics

$$\frac{\frac{\sum_{i=1}^n \log x_i - n \log(x_m)}{n} - \alpha_0}{\frac{\alpha_0}{\sqrt{n}}}$$

Distribution of test statistics

$$N(0, 1)$$

Wald Test -Alternative (variance under H_0)

Wald test Statistics

$$\frac{\frac{\sum_{i=1}^n \log x_i - n \log(x_m)}{n} - \alpha_0}{\frac{\alpha_0}{\sqrt{n}}}$$

Distribution of test statistics

$$N(0, 1)$$

Observed value of test statistics

$$\frac{4 - 5}{\sqrt{0.25}} = -2 \quad p - value = 0.0455$$

Wald Test -Alternative (variance under H_0)

Wald test Statistics

$$\frac{\frac{\sum_{i=1}^n \log x_i - n \log(x_m)}{n} - \alpha_0}{\frac{\alpha_0}{\sqrt{n}}}$$

Distribution of test statistics

$$N(0, 1)$$

Observed value of test statistics

$$\frac{4 - 5}{\sqrt{0.25}} = -2 \qquad p - value = 0.0455$$

Decision: REJECT

Wald Test -Alternative (variance under H_0)

Wald test Statistics

$$\frac{\frac{\sum_{i=1}^n \log x_i - n \log(x_m)}{n} - \alpha_0}{\frac{\alpha_0}{\sqrt{n}}}$$

Distribution of test statistics

$$N(0, 1)$$

Observed value of test statistics

$$\frac{4 - 5}{\sqrt{0.25}} = -2 \quad p\text{-value} = 0.0455$$

Decision: REJECT

$$\frac{\left(\frac{\partial \log L(\alpha_0)}{\partial \alpha}\right)^2}{nI(\alpha_0)} \sim \chi_1^2$$

Score Test Statistics

$$\frac{\left(\frac{n}{\alpha_0} + n \log(x_m) - \sum_{i=1}^n \log(x_i)\right)^2}{\frac{n}{\alpha_0^2}}$$

$$\frac{\left(\frac{\partial \log L(\alpha_0)}{\partial \alpha}\right)^2}{nI(\alpha_0)} \sim \chi_1^2$$

Score Test Statistics

$$\frac{\left(\frac{n}{\alpha_0} + n \log(x_m) - \sum_{i=1}^n \log(x_i)\right)^2}{\frac{n}{\alpha_0^2}}$$

Distribution of Score Test Statistics

$$\chi_1^2$$

Score Test

$$\frac{\left(\frac{\partial \log L(\alpha_0)}{\partial \alpha}\right)^2}{nI(\alpha_0)} \sim \chi_1^2$$

Score Test Statistics

$$\frac{\left(\frac{n}{\alpha_0} + n \log(x_m) - \sum_{i=1}^n \log(x_i)\right)^2}{\frac{n}{\alpha_0^2}}$$

Distribution of Score Test Statistics

$$\chi_1^2$$

Score Test

Score Test Statistics

$$\frac{\frac{n}{\alpha_0} + n \log(x_m) - \sum_{i=1}^n \log(x_i)}{\sqrt{\frac{n}{\alpha_0^2}}}$$

Distribution of the Score Test Statistics

$$N(0, 1)$$

Observed value of test statistics

$$\begin{aligned} \frac{\delta \log L(\alpha_0, x_m)}{\delta \alpha} &= \frac{100}{5} + 100 \log(50000) - 100 \times 11.069 = -4.922 \\ \frac{\log L(\alpha_0)}{\sqrt{nl(\alpha_0)}} &= \frac{-4.922}{2} = -2.461 \quad pvalue = 0.014 \end{aligned}$$

Score Test

Score Test Statistics

$$\frac{\frac{n}{\alpha_0} + n \log(x_m) - \sum_{i=1}^n \log(x_i)}{\sqrt{\frac{n}{\alpha_0^2}}}$$

Distribution of the Score Test Statistics

$$N(0, 1)$$

Observed value of test statistics

$$\begin{aligned} \frac{\delta \log L(\alpha_0, x_m)}{\delta \alpha} &= \frac{100}{5} + 100 \log(50000) - 100 \times 11.069 = -4.922 \\ \frac{\log L(\alpha_0)}{\sqrt{nl(\alpha_0)}} &= \frac{-4.922}{2} = -2.461 \quad pvalue = 0.014 \end{aligned}$$

Score Test

Score Test Statistics

$$\frac{\frac{n}{\alpha_0} + n \log(x_m) - \sum_{i=1}^n \log(x_i)}{\sqrt{\frac{n}{\alpha_0^2}}}$$

Distribution of the Score Test Statistics

$$N(0, 1)$$

Observed value of test statistics

$$\begin{aligned} \frac{\delta \log L(\alpha_0, x_m)}{\delta \alpha} &= \frac{100}{5} + 100 \log(50000) - 100 \times 11.069 = -4.922 \\ \frac{\log L(\alpha_0)}{\sqrt{nl(\alpha_0)}} &= \frac{-4.922}{2} = -2.461 \quad pvalue = 0.014 \end{aligned}$$

Score Test Statistics

$$\frac{\frac{n}{\alpha_0} + n \log(x_m) - \sum_{i=1}^n \log(x_i)}{\sqrt{\frac{n}{\alpha_0^2}}}$$

Distribution of the Score Test Statistics

$$N(0, 1)$$

Score Test

Score Test Statistics

$$\frac{\frac{n}{\alpha_0} + n \log(x_m) - \sum_{i=1}^n \log(x_i)}{\sqrt{\frac{n}{\alpha_0^2}}}$$

Distribution of the Score Test Statistics

$$N(0, 1)$$

Observed value of test statistics

score=-2.461, p-value=0.014

Score Test

Score Test Statistics

$$\frac{\frac{n}{\alpha_0} + n \log(x_m) - \sum_{i=1}^n \log(x_i)}{\sqrt{\frac{n}{\alpha_0^2}}}$$

Distribution of the Score Test Statistics

$$N(0, 1)$$

Observed value of test statistics

score=-2.461, p-value=0.014

Decision: REJECT

Score Test

Score Test Statistics

$$\frac{\frac{n}{\alpha_0} + n \log(x_m) - \sum_{i=1}^n \log(x_i)}{\sqrt{\frac{n}{\alpha_0^2}}}$$

Distribution of the Score Test Statistics

$$N(0, 1)$$

Observed value of test statistics

score=-2.461, p-value=0.014

Decision: REJECT

Likelihood ratio Test

Log Likelihood

$$\log L(\alpha) = n \log \alpha + n \alpha \log(x_m) - (\alpha + 1) \sum_{i=1}^n \log x_i$$

Likelihood ratio Test

$$2 \times (\log L(\hat{\alpha}) - \log L(\alpha_0)) \sim \chi_1^2$$

Likelihood ratio Test

Log Likelihood

$$\log L(\alpha) = n \log \alpha + n \alpha \log(x_m) - (\alpha + 1) \sum_{i=1}^n \log x_i$$

Likelihood ratio Test

$$2 \times (\log L(\hat{\alpha}) - \log L(\alpha_0)) \sim \chi_1^2$$

Likelihood ratio Test Statistics

$$2 \times (\log L(\hat{\alpha}) - \log L(\alpha_0))$$
$$2 \times \left(n \log \frac{\hat{\alpha}}{\alpha_0} + (\hat{\alpha} - \alpha_0) \left(n \log(x_m) - \sum_{i=1}^n \log(x_i) \right) \right)$$

Likelihood ratio Test

Log Likelihood

$$\log L(\alpha) = n \log \alpha + n \alpha \log(x_m) - (\alpha + 1) \sum_{i=1}^n \log x_i$$

Likelihood ratio Test

$$2 \times (\log L(\hat{\alpha}) - \log L(\alpha_0)) \sim \chi_1^2$$

Likelihood ratio Test Statistics

$$2 \times (\log L(\hat{\alpha}) - \log L(\alpha_0))$$
$$2 \times \left(n \log \frac{\hat{\alpha}}{\alpha_0} + (\hat{\alpha} - \alpha_0) \left(n \log(x_m) - \sum_{i=1}^n \log(x_i) \right) \right)$$

Likelihood ratio Test

Likelihood ratio Test Statistics

$$2 \times \left(n \log \frac{\hat{\alpha}}{\alpha_0} + (\hat{\alpha} - \alpha_0) \left(n \log(x_m) - \sum_{i=1}^n \log(x_i) \right) \right)$$

Distribution of the Test Statistics

$$\chi_1^2$$

Likelihood ratio Test

Likelihood ratio Test Statistics

$$2 \times \left(n \log \frac{\hat{\alpha}}{\alpha_0} + (\hat{\alpha} - \alpha_0) \left(n \log(x_m) - \sum_{i=1}^n \log(x_i) \right) \right)$$

Distribution of the Test Statistics

$$\chi_1^2$$

Observed Test Statistics

$$l(5) = -1071.0$$

$$l(4) = -1068.4$$

$$2 \times (\log L(\hat{\alpha}) - \log L(\alpha_0)) = 5.3756 \quad p\text{-value} = 0.0204$$

Likelihood ratio Test

Likelihood ratio Test Statistics

$$2 \times \left(n \log \frac{\hat{\alpha}}{\alpha_0} + (\hat{\alpha} - \alpha_0) \left(n \log(x_m) - \sum_{i=1}^n \log(x_i) \right) \right)$$

Distribution of the Test Statistics

$$\chi_1^2$$

Observed Test Statistics

$$l(5) = -1071.0$$

$$l(4) = -1068.4$$

$$2 \times (\log L(\hat{\alpha}) - \log L(\alpha_0)) = 5.3756 \quad p\text{-value} = 0.0204$$

Likelihood ratio Test

Likelihood ratio Test Statistics

$$2 \times \left(n \log \frac{\hat{\alpha}}{\alpha_0} + (\hat{\alpha} - \alpha_0) \left(n \log(x_m) - \sum_{i=1}^n \log(x_i) \right) \right)$$

Distribution of the Test Statistics

$$\chi_1^2$$

Likelihood ratio Test

Likelihood ratio Test Statistics

$$2 \times \left(n \log \frac{\hat{\alpha}}{\alpha_0} + (\hat{\alpha} - \alpha_0) \left(n \log(x_m) - \sum_{i=1}^n \log(x_i) \right) \right)$$

Distribution of the Test Statistics

$$\chi_1^2$$

Observed Test Statistics

$$2 \times (\log L(\hat{\alpha}) - \log L(\alpha_0)) = 5.3756 \quad p\text{-value} = 0.0204$$

Likelihood ratio Test

Likelihood ratio Test Statistics

$$2 \times \left(n \log \frac{\hat{\alpha}}{\alpha_0} + (\hat{\alpha} - \alpha_0) \left(n \log(x_m) - \sum_{i=1}^n \log(x_i) \right) \right)$$

Distribution of the Test Statistics

$$\chi_1^2$$

Observed Test Statistics

$$2 \times (\log L(\hat{\alpha}) - \log L(\alpha_0)) = 5.3756 \quad p\text{-value} = 0.0204$$

Decision: REJECT

Likelihood ratio Test

Likelihood ratio Test Statistics

$$2 \times \left(n \log \frac{\hat{\alpha}}{\alpha_0} + (\hat{\alpha} - \alpha_0) \left(n \log(x_m) - \sum_{i=1}^n \log(x_i) \right) \right)$$

Distribution of the Test Statistics

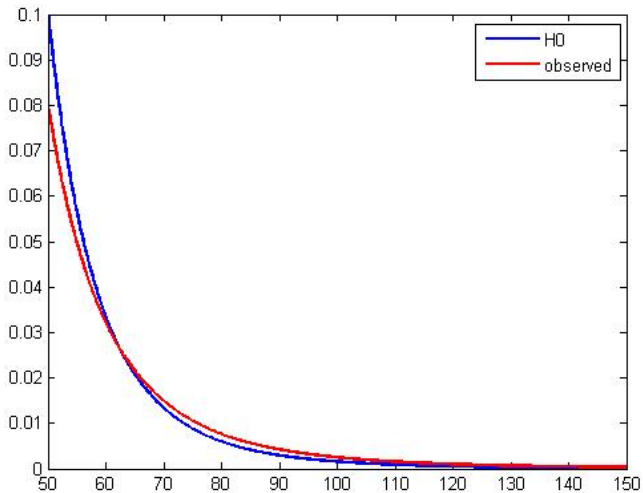
$$\chi_1^2$$

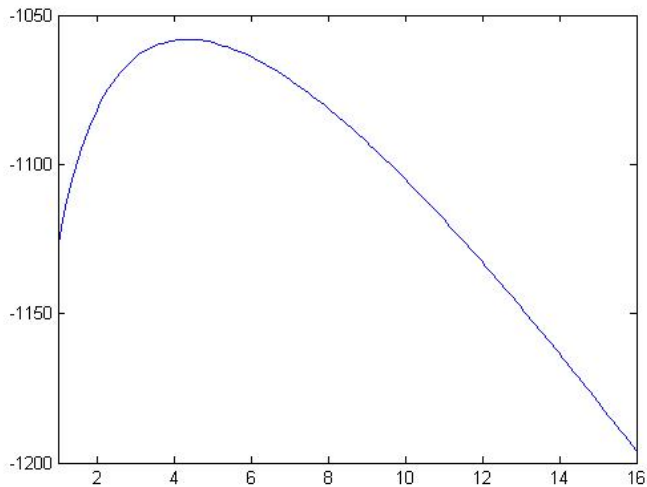
Observed Test Statistics

$$2 \times (\log L(\hat{\alpha}) - \log L(\alpha_0)) = 5.3756 \quad p\text{-value} = 0.0204$$

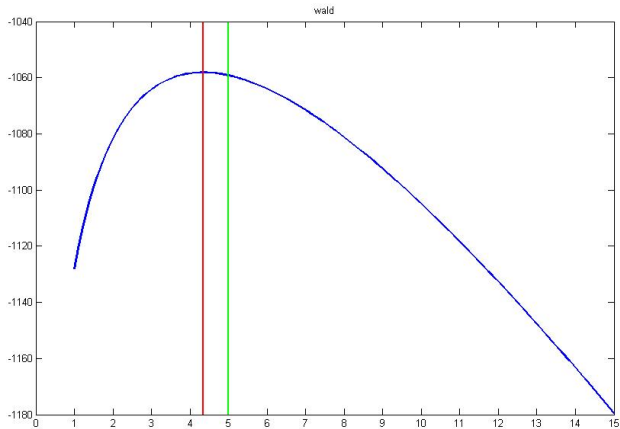
Decision: REJECT

Pareto

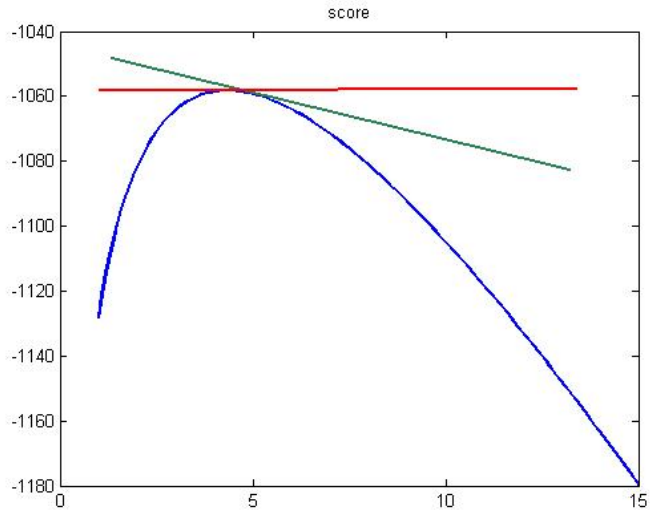




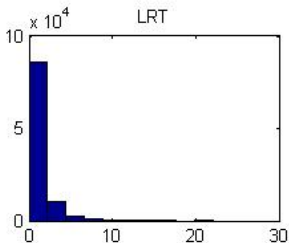
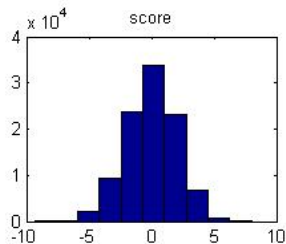
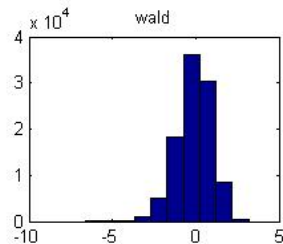
Pareto-wald



Pareto-score



Pareto: 100000 simulations



Pareto: 100 simulations

