TUTORIAL #2
STATISTICS

3/10/2024

Statistics Fall 2024 - TA Session 2
Point Estimation

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Problem 1

Let $X_1, X_2, ..., X_n$ $iid \sim \mathcal{N}(1, \sigma^2)$. Find a method of moments estimator of σ^2, call it $\hat{\sigma^2}$.

Problem 2

Suppose that X is a discrete random variable with the following probability mass function:

X	0	1	2	3
P(X)	$2\theta/3$	$\theta/3$	2(1-θ)/3	(1-θ)/3

Where $0 \leq \theta \leq 1$ is a parameter. The following 10 independent observations : $3, 0, 2, 1, 3, 2, 1, 0, 2, 1$ were taken from such a distribution.

(a) What is the maximum likelihood estimate of θ?

(b) Find the MoM estimate for θ. Is it different from 1/2?

Problem 3

The Pareto distribution has the following probability density function:

$$f(x; \theta) = \theta \, \alpha^\theta \, x^{-\theta-1}, \text{ for } x \geq \alpha \ , \theta > 1$$

Where α and θ are positive parameters of the distribution. Assume that α is known and that $X_1 ... X_n$ is a random sample of size n.

(a) Find the method of moments estimator for θ.

(b) Find the maximum likelihood estimator for θ. Does this estimator differ from that found in part (a)?

(c) Estimate θ based on these data: 3, 5, 2, 3, 4, 1, 4, 3, 3, 3.

Problem 4

Let $X_1, X_2, ..., X_n$ be a random sample from a Bernoulli distribution with parameter p. If p is restricted so that we know that $\frac{1}{2} \leq p \leq 1$, find the MLE of this parameter.

PROBLEM 1

$X_1, X_2, \dots, X_n$ IID $\sim N(1, \sigma^2)$

FIND A MoM ESTIMATOR FOR σ^2

$$E(X) = 1$$

$$E[X^2] = VAR[X] + (E[X])^2$$

$$\stackrel{!}{=} \sigma^2 + 1^2 = \sigma^2 + 1$$

$$\overline{X}^2 = \frac{1}{N} \sum_{i=1}^{N} X_i^2$$

$$E[X^2] = \overline{X}^2$$

$$\sigma^2 + 1 = \overline{X}^2$$

$$\hat{\sigma}^2 = \left(\frac{1}{N} \sum_{i=1}^{N} X_i^2 \right) - 1$$

PROBLEM 2

x	0	1	2	3
$P(x)$	$\frac{2\theta}{3}$	$\frac{\theta}{3}$	$\frac{2(1-\theta)}{3}$	$\frac{1-\theta}{3}$

$$L(\theta) = P(X=3)\,P(X=0)\,P(X=2)\,P(X=1)\,P(X=3)\,P(X=2)\,P(X=1)\,P(X=0)\,P(X=2)\,P(X=1)$$

$$= P(X=0)^2\,P(X=1)^3\,P(X=2)^3\,P(X=3)^2$$

$$= \left(\frac{2\theta}{3}\right)^2 \left(\frac{\theta}{3}\right)^3 \left(\frac{2(1-\theta)}{3}\right)^3 \left(\frac{1-\theta}{3}\right)^2$$

$$\ell(\theta) = \log L(\theta) =$$

$$= 2\left(\log\left(\frac{2}{3}\right) + \log(\theta)\right) + 3\left(\log\left(\frac{1}{3}\right) + \log\theta\right)$$

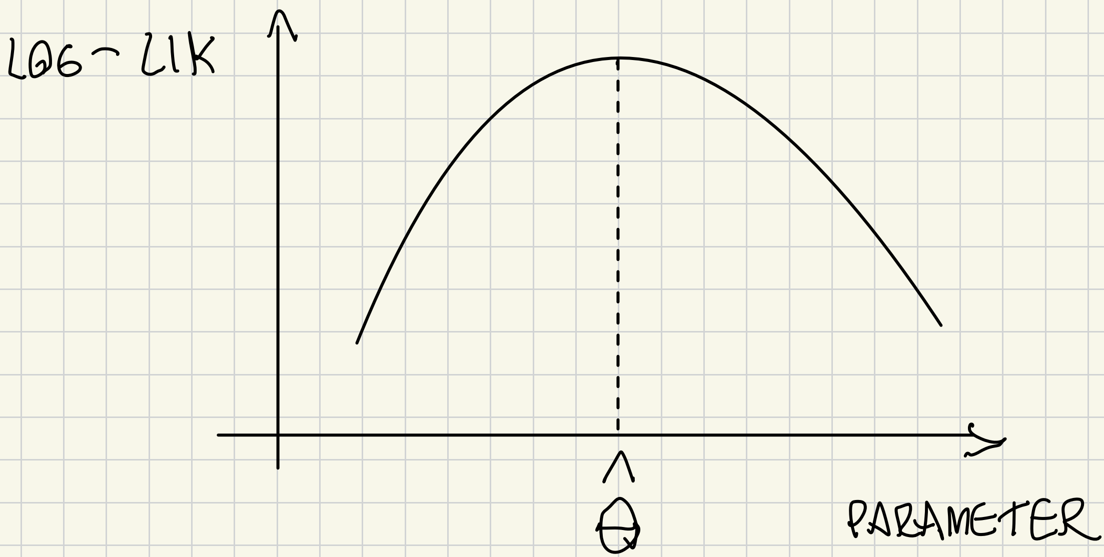
$$+ 3\left(\log\left(\frac{2}{3}\right) + \log(1-\theta)\right) + 2\left(\log\left(\frac{1}{3}\right) + \log(1-\theta)\right)$$

$$= 5\log(\theta) + 5\log(1-\theta) + \text{CONSTANT}$$

$$\ell'(\theta) = \frac{5}{\theta} - \frac{5}{1-\theta}$$

$$\ell'(\theta) = 0 \implies \hat{\theta} = \frac{1}{2}$$

$$\ell''(\theta) = -\frac{5}{\theta^2} - \frac{5}{(1-\theta)^2} < 0$$



FIND MoM ESTIMATE FOR θ

$$\mu_1 = m_1$$

$$\mu_1 = E(X) = 0\left(\frac{2\theta}{3}\right) + 1\left(\frac{\theta}{3}\right) + 2\left(\frac{2(1-\theta)}{3}\right) + 3\left(\frac{1-\theta}{3}\right)$$

$$= \frac{-6\theta + 7}{3}$$

$$m_1 = \frac{1}{N}\sum_{i=1}^{N} X_i$$

$$= \frac{15}{10} = \frac{3}{2}$$

$$\frac{-6\theta + 7}{3} = \frac{3}{2}$$

$$-12\theta + 14 = 9$$

$$\hat{\theta} = \frac{5}{12}$$

PROBLEM 3

$\mu_1 = m_1$

$$\mu_1 = E(X) = \int_{\alpha}^{+\infty} x\,\theta\,\alpha^{\theta}\,x^{-\theta-1}\,dx$$

$$= \int_{\alpha}^{\infty} \theta\,\alpha^{\theta}\,x^{-\theta}\,dx$$

$$= \theta\,\alpha^{\theta}\,\frac{x^{-\theta+1}}{-\theta+1}\Bigg|_{\alpha}^{+\infty}$$

$$= 0 - \theta\,\alpha^{\theta}\,\frac{\alpha^{-\theta+1}}{-\theta+1}$$

$$= \frac{\theta\,\alpha}{\theta-1}$$

$$\bar{X} = \frac{\theta\,\alpha}{\theta-1}$$

$$\hat{\theta} = \frac{\bar{X}}{\bar{X}-\alpha}$$

$$L(\theta) = \prod_{i=1}^{N} \theta \, \alpha^{\theta} \, x_i^{-\theta-1}$$

$$= \theta^{m} \alpha^{m\theta} \prod_{i=1}^{N} x_i^{-\theta-1}$$

$$\ell(\theta) = \log L(\theta) =$$

$$= m \log(\theta) + m\theta \log(\alpha) - (\theta+1) \sum_{i=1}^{N} \log(x_i)$$

$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{m}{\theta} + m \log(\alpha) - \sum_{i=1}^{N} \log(x_i)$$

$$\frac{\partial \ell(\theta)}{\partial \theta} = 0$$

$$\hat{\theta} = \frac{m}{\sum \log(x_i) - m \log(\alpha)}$$

(c) 3, 5, 2, 3, 4, 1, 4, 3, 3, 3

$$n = 10$$

$$\bar{x} = 3.1$$

$$\sum \log(x_i) = 10.57$$

$$\text{MoM}: \hat{\theta} = \frac{\bar{x}}{\bar{x} - \alpha} = \frac{3.1}{3.1 - \alpha}$$

$$\text{MLE}: \hat{\theta} = \frac{n}{\sum \log(x_i) - n \log \alpha}$$

$$= \frac{10}{10.57 - 10 \log(\alpha)}$$

PROBLEM 4

$$L(p) = \prod_{i=1}^{N} p^{x_i} (1-p)^{(1-x_i)}$$

$$\ell(p) = \sum_{i=1}^{N} \left[x_i \log(p) + (1-x_i)\log(1-p) \right]$$

$$\ell'(p) = \sum_{i=1}^{N} \left[\frac{x_i}{p} - \frac{1-x_i}{1-p} \right]$$

$$\ell'(p) = 0$$

$$\sum_{i=1}^{N} \left[\frac{x_i}{p} - \frac{1-x_i}{1-p} \right] =$$

$$\sum_{i=1}^{N} \left[\frac{(1-p)x_i - p(1-x_i)}{p(1-p)} \right] =$$

$$\sum_{i=1}^{N} \left[\frac{x_i - \cancel{p x_i} - p + \cancel{x_i p}}{p(1-p)} \right] =$$

$$\sum_{i=1}^{N}\left[\frac{x_i-p}{p(1-p)}\right] =$$

BECAUSE

$$\bar{x} = \frac{1}{m}\sum_{i=1}^{N} X_i$$

$$= \frac{m(\bar{x}-p)}{p(1-p)} = 0$$

$$m\bar{x} - mp = 0$$

$$\bar{x} = p < 1$$

$$\boxed{\frac{1}{2} \leq p < 1}$$

IF $\bar{x} < \frac{1}{2}$ WHAT IS THE MLE OF p?

$$\hat{p} = \text{MAX}\left\{\frac{1}{2}, \bar{x}\right\}$$