1^{st} Assignment Statistics due October 9^{th} 2024

Maura Mezzetti

Exercise 1

The life time of a light bulb is uniformly distribution between 600 and 1000 hours. You have just bought 5 light bulbs, What is the probability that the lowest observed life time is greater than 750 hours?

Exercise 2

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as Beta distribution.

$$f(x|\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 \leq x \leq 1$$

Show that the following statistics

$$T = \frac{1}{n}\left(\sum_{i=1}^{n} log(1 - X_i)\right)^3$$

is a sufficient statistics for parameter β.

Exercise 3

Let $(X_1, \ldots, X_n)$ be a random sample of i.i.d. random variables distributed as

$$f(x;\theta) = \theta(1+x)^{-(\theta+1)} I_{(0,\infty)}(x) \qquad \theta > 1$$

1. Find the Method of Moments Estimator (MOM) and the Maximum Likelihood Estimator (MLE) of θ
2. Find the MLE of $1/\theta$

Exercise 4

An electrical circuit consists of three batteries X_1, X_2, X_3 connected in series to a lightbulb Y. We model the battery lifetimes as independent and identically distributed Exponential(λ) random variables (such that $E(X_i) = \lambda$). Our experiment to measure the lifetime of the lightbulb is stopped when any one of the batteries fails. Hence, the only random variable we observe is $Y = min(X_1, X_2, X_3)$.

1. Determine the distribution of the random variable Y.
2. Compute $\hat{\lambda}_{MLE}$, the maximum likelihood estimator of λ.
3. Determine the mean square error of $\hat{\lambda}_{MLE}$.
4. Use the Cramer-Rao lower bound to prove that $\hat{\lambda}_{MLE}$ is the minimum variance unbiased estimator of λ.

Exercise 5

The double exponential distribution is

$$f(x|\theta) = \frac{1}{2}e^{-|x-\theta|}, \qquad -\infty < x < \infty$$

For an iid sample of size $n = 2m + 1$, find the mle of θ.

Exercise 6

Let $(Y_1, \ldots, Y_n)$ be i.i.d. Exponential(θ),

$$f(y|\theta) = \frac{1}{\theta}exp\left(-\frac{y}{\theta}\right), \quad y > 0,\, \theta > 0$$

Suppose that the n survival times Y_i are right censored at time $c > 0$, i.e., we observe:

$$X_i = \begin{cases} Y_i & if \quad Y_i < c \\ c & if \quad Y_i \geq c \end{cases}$$

1. Find $\hat{\theta}(Y_1, \ldots, Y_n)$ the maximum-likelihood estimate of θ based on the original sample of survival times $(Y_1, \ldots, Y_n)$.
2. Determine the distribution of $\hat{\theta}(Y_1, \ldots, Y_n)$ and give formulas for the mean and variance of this distribution.
3. Derive the likelihood function of $(X_1, \ldots, X_n)$.
4. Find $\hat{\theta}(X_1, \ldots, X_n)$ the maximum-likelihood estimate of θ based on the sample of censored survival times $(X_1, \ldots, X_n)$.

Exercise 7

Suppose that income Y is distributed as a Pareto distribution:

$$f(y) = \alpha y^{-(\alpha+1)}$$

for $y \geq 1$ and $\alpha > 1$

- It is quite common to not observe all incomes, but only those that are higher than some threshold (so-called truncated variables). Assume that you observe only those individuals with an income greater than or equal to $9,000$\$, and their income is described by a random variable $Y^\star$. How is $Y^\star$ distributed?
- Your have a sample of size n drawn from the population of persons with incomes greater than or equal to $9,000$\$. What is the MLE of α?
- What is asymptotic distribution of the estimator in previous point?