Exercise 4

Let $(X_1, \dots, X_n)$ be independent identically distributed random variables with p.d.f.

$$f(x) = \frac{1}{\theta c^{1/\theta}} x^{\frac{1-\theta}{\theta}} \qquad 0 \leq x \leq c$$

where the boundary c is known and θ is a positive parameter (unknown). Assume $E(log(X)) = ln(c) - \theta$.

1. Find a sufficient statistics for θ
2. Find $\hat{\theta}_{MLE}$ maximum likelihood estimator (MLE) for θ and discuss properties of this estimator.
3. Compute the score function and the Fisher information.
4. Specify asymptotic distribution of $\hat{\theta}_{MLE}$.

Exercise 4

$$
\begin{aligned}
f(x) &= \frac{1}{\theta c^{1/\theta}} x^{\frac{1-\theta}{\theta}} I_{[0,c]}(x) \\
f(x_1, x_2, \ldots, x_n) &= \frac{1}{\theta^n c^{n/\theta}} \prod x_i^{\frac{1-\theta}{\theta}} I_{[0,c]}(x) \\
L(\theta | x_1, x_2, \ldots, x_n) &= \frac{1}{\theta^n c^{n/\theta}} \prod x_i^{\frac{1-\theta}{\theta}} \qquad 0 \leq x_i \leq c \\
logL(\theta | x_1, x_2, \ldots, x_n) &= -nlog(\theta) - \frac{n}{\theta} log(c) + \frac{1-\theta}{\theta} \sum_i log(x_i)
\end{aligned}
$$

Exercise 4: Sufficiency?

$$\begin{aligned}
f(x) &= \frac{1}{\theta c^{1/\theta}} x^{\frac{1-\theta}{\theta}} I_{[0,c]}(x) \\
f(x_1, x_2 \ldots, x_n) &= \frac{1}{\theta^n c^{n/\theta}} \prod x_i^{\frac{1-\theta}{\theta}} I_{[0,c]}(x) \\
L(\theta | x_1, x_2, \ldots, x_n) &= \frac{1}{\theta^n c^{n/\theta}} \prod x_i^{\frac{1-\theta}{\theta}} \qquad 0 \leq x_i \leq c \\
logL(\theta | x_1, x_2, \ldots, x_n) &= -nlog(\theta) - \frac{n}{\theta} log(c) + \frac{1-\theta}{\theta} \sum_i log(x_i)
\end{aligned}$$

Exercise 4

$$
\begin{aligned}
f(x) &= \frac{1}{\theta c^{1/\theta}} x^{\frac{1-\theta}{\theta}} I_{[0,c]}(x) \\
logL(\theta) &= -nlog(\theta) - \frac{n}{\theta} log(c) + \frac{1-\theta}{\theta} \sum_i log(x_i) \\
\frac{\partial logL(\theta)}{\partial \theta} &= -\frac{n}{\theta} + \frac{n}{\theta^2} log(c) - \frac{1}{\theta^2} \sum_i log(x_i) \\
\frac{\partial^2 logL(\theta)}{\partial^2 \theta} &= \frac{n}{\theta^2} - \frac{2n}{\theta^3} log(c) + \frac{2}{\theta^3} \sum_i log(x_i)
\end{aligned}
$$

Exercise 4:MLE

$$\frac{\partial log L(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{n}{\theta^2} log(c) - \frac{1}{\theta^2} \sum_i log(x_i)$$

$$\frac{\partial^2 log L(\theta)}{\partial^2 \theta} = \frac{n}{\theta^2} - \frac{2n}{\theta^3} log(c) + \frac{2}{\theta^3} \sum_i log(x_i)$$

Exercise 4:MLE

$$\begin{aligned}
\frac{\partial logL(\theta)}{\partial\theta} &= -\frac{n}{\theta} + \frac{n}{\theta^2} log(c) - \frac{1}{\theta^2}\sum_i log(x_i) \\
\frac{\partial^2 logL(\theta)}{\partial^2\theta} &= \frac{n}{\theta^2} - \frac{2n}{\theta^3} log(c) + \frac{2}{\theta^3}\sum_i log(x_i) \\
\frac{\partial logL(\theta)}{\partial\theta} &= 0 \\
\frac{n}{\theta} &= \frac{n}{\theta^2} log(c) - \frac{1}{\theta^2}\sum_i log(x_i) \\
\hat{\theta} &= log(c) - \frac{\sum_i log(x_i)}{n} \\
\frac{\partial^2 logL(\theta)}{\partial^2\theta}|_{\hat{\theta}} &= -\frac{n}{\hat{\theta}^2} < 0
\end{aligned}$$

Exercise 4:Score Function and Fisher Information

$$
\begin{aligned}
\frac{\partial logL(\theta)}{\partial\theta} &= -\frac{n}{\theta} + \frac{n}{\theta^2}log(c) - \frac{1}{\theta^2}\sum_i log(x_i) \\
nI(\theta) &= -E\left(\frac{\partial^2 logL(\theta)}{\partial^2\theta}\right) \\
nI(\theta) &= -E\left(\frac{n}{\theta^2} - \frac{2n}{\theta^3}log(c) + \frac{2}{\theta^3}\sum_i log(x_i)\right) \\
nI(\theta) &= -\left(\frac{n}{\theta^2} - \frac{2n}{\theta^3}log(c) + \frac{2}{\theta^3}\sum_i E(log(x_i))\right) \\
nI(\theta) &= -\left(\frac{n}{\theta^2} - \frac{2n}{\theta^3}log(c) + \frac{2}{\theta^3}\sum_i (log(c) - \theta)\right) \\
nI(\theta) &= \frac{n}{\theta^2}
\end{aligned}
$$

Exercise 4:Asympotic distribution of the MLE

$$\hat{\theta} \sim N\left(\theta_0, \frac{\theta_0^2}{n}\right)$$

Exercise 4

$$f(x) = \frac{1}{\theta c^{1/\theta}} x^{\frac{1-\theta}{\theta}} \qquad 0 \leq x \leq c$$

- Suppose that from a random sample of 100 random variables you obtain that the realization of the ML estimator is equal to 2. Furthermore assume $log(c) = 4$.
  Complete ONE of the following questions:
  1. Find the Wald test statistic for testing $H_0 : \theta = 1.5$ versus $H_1 : \theta \neq 1.5$, specify the distribution and verify the null hypothesis.
  2. Find the Score test statistic for testing $H_0 : \theta = 1.5$ versus $H_1 : \theta \neq 1.5$, specify the distribution and verify the null hypothesis.
  3. Find the Likelihood Ratio test statistic for testing $H_0 : \theta = 1.5$ versus $H_1 : \theta \neq 1.5$, specify the distribution and verify the null hypothesis.

Exercise 4: Wald

$$f(x) = \frac{1}{\theta c^{1/\theta}} x^{\frac{1-\theta}{\theta}} \qquad 0 \leq x \leq c$$

- Suppose that from a random sample of 100 random variables you obtain that the realization of the ML estimator is equal to 2. Furthermore assume $log(c) = 4$.

Exercise 4: Wald

$$\begin{aligned}
\frac{\hat{\theta} - \theta_0}{\sqrt{\frac{\hat{\theta}^2}{n}}} &\sim N(0,1) \\
\frac{\hat{\theta} - 1.5}{\sqrt{\frac{2^2}{100}}} &\sim N(0,1) \\
\frac{2 - 1.5}{\sqrt{\frac{2^2}{100}}} &= 2.5 \\
p - value &= 2 \times P(Z > 2.5) = 0.0124 < 0.05
\end{aligned}$$

Exercise 4: Score

$$f(x) = \frac{1}{\theta c^{1/\theta}} x^{\frac{1-\theta}{\theta}} \qquad 0 \leq x \leq c$$

- Suppose that from a random sample of 100 random variables you obtain that the realization of the ML estimator is equal to 2. Furthermore assume $log(c) = 4$.

Exercise 4: Score

$$
\begin{aligned}
\frac{u(\theta_0)}{\sqrt{\frac{n}{\theta_0^2}}} &\sim N(0,1) \\
u(\theta_0) &= -\frac{n}{\theta_0} + \frac{n}{\theta_0^2}\left(log(c) - \frac{\sum_i log(x_i)}{n}\right) \\
u(\theta_0) &= -\frac{n}{\theta_0} + \frac{n}{\theta_0^2} \times 2 \\
u(\theta_0) &= -\frac{100}{1.5} + \frac{100}{1.5^2} \times 2 \\
u(\theta_0) &= -\frac{100}{1.5} + \frac{100}{1.5^2} \times 2 \\
u(\theta_0) &= 22
\end{aligned}
$$

Exercise 4: Score

$$\begin{aligned}
\frac{22.22}{\sqrt{\frac{100}{1.5^2}}} &\sim N(0,1) \\
&= 3.33 \\
p - value &= 2 \times P(Z > 3.33) < 0.01
\end{aligned}$$

Exercise 4: Likelihood Ratio test

$$
\begin{aligned}
logL(\theta) &= -nlog(\theta) - \frac{n}{\theta} log(c) + \frac{1-\theta}{\theta} \sum_i log(x_i) \\
logL(\theta) &= -nlog(\theta) - \frac{n}{\theta} \left(log(c) - \frac{\sum_i log(x_i)}{n} \right) - \sum_i log(x_i)
\end{aligned}
$$

$$
\left(log(c) - \frac{\sum_i log(x_i)}{n} \right) = \hat{\theta} = 2
$$

$$
\sum_i log(x_i) = n \times (log(c) - 2)
$$

$$
\sum_i log(x_i) = 100 \times (4 - 2) = 200
$$

Exercise 4: Likelihood Ratio test

$$
\begin{aligned}
logL(\theta) &= -nlog(\theta) - \frac{n}{\theta} log(c) + \frac{1-\theta}{\theta} \sum_i log(x_i) \\
logL(\theta) &= -nlog(\theta) - \frac{n}{\theta} \left(log(c) - \frac{\sum_i log(x_i)}{n} \right) - \sum_i log(x_i) \\
logL(\hat{\theta}) &= -nlog(\hat{\theta}) - \frac{n}{\hat{\theta}} \left(log(c) - \frac{\sum_i log(x_i)}{n} \right) - \sum_i log(x_i) \\
logL(\hat{\theta}) &= -100 \times log(2) - \frac{100}{2} \times 2 - 200 \\
logL(\hat{\theta}) &= -369.3147
\end{aligned}
$$

Exercise 4: Likelihood Ratio test

$$
\begin{aligned}
logL(\theta) &= -nlog(\theta) - \frac{n}{\theta}log(c) + \frac{1-\theta}{\theta}\sum_i log(x_i) \\
logL(\theta) &= -nlog(\theta) - \frac{n}{\theta}\left(log(c) - \frac{\sum_i log(x_i)}{n}\right) - \sum_i log(x_i) \\
logL(\theta_0) &= -nlog(\theta_0) - \frac{n}{\theta_0}\left(log(c) - \frac{\sum_i log(x_i)}{n}\right) - \sum_i log(x_i) \\
logL(\theta_0) &= -100 \times log(1.5) - \frac{100}{1.5} \times 2 - 200 \\
logL(\theta_0) &= -373.8798
\end{aligned}
$$

Exercise 4: Likelihood Ratio test

$$
\begin{aligned}
logL(\theta) &= -nlog(\theta) - \frac{n}{\theta} log(c) + \frac{1-\theta}{\theta} \sum_i log(x_i) \\
logL(\theta) &= -nlog(\theta) - \frac{n}{\theta} \left(log(c) - \frac{\sum_i log(x_i)}{n} \right) - \sum_i log(x_i) \\
logL(\hat{\theta}) &= -369.3147 \\
logL(\theta_0) &= -373.8798 \\
& \quad 2 \times (logL(\hat{\theta}) - logL(\theta_0)) = 9.13 \\
p - value &= P(\chi 9.13) = 0.0025
\end{aligned}
$$

Large Sample Test

Wald Test

$$\frac{\hat{\theta} - \theta_0}{\sqrt{\frac{\hat{\theta}^2}{n}}} \sim N(0, 1)$$

Score Test

$$\frac{u(\theta_0)}{\sqrt{\frac{n}{\theta_0^2}}} \sim N(0, 1)$$

Large Sample Test

Wald Test

$$\frac{\hat{\theta} - \theta_0}{\sqrt{\frac{\hat{\theta}^2}{n}}} \sim N(0,1)$$

Score Test

$$\frac{u(\theta_0)}{\sqrt{\frac{n}{\theta_0^2}}} \sim N(0,1)$$

Likelihood ratio test

$$-2 \times \left(logL(\theta_0) - logL(\hat{\theta}) \right) \sim \chi^2$$

Large Sample Test

Wald Test

$$\frac{\hat{\theta} - \theta_0}{\sqrt{\frac{\hat{\theta}^2}{n}}} \sim N(0, 1)$$

Score Test

$$\frac{u(\theta_0)}{\sqrt{\frac{n}{\theta_0^2}}} \sim N(0, 1)$$

Likelihood ratio test

$$-2 \times \left(logL(\theta_0) - logL(\hat{\theta}) \right) \sim \chi^2$$

Large Sample Test Statistics

Wald Test Statistcs

$$\frac{log(c) - \frac{\sum_i log(x_i)}{n} - \theta_0}{\sqrt{\frac{\hat{\theta}^2}{n}}}$$

Score Test Statistics

$$\sqrt{\frac{\theta_0^2}{n}} \times \left(-\frac{n}{\theta_0} + \frac{n}{\theta_0^2} log(c) - \frac{1}{\theta_0^2} \sum_i log(x_i) \right)$$

Large Sample Test Statistics

Wald Test Statistcs

$$\frac{log(c) - \frac{\sum_i log(x_i)}{n} - \theta_0}{\sqrt{\frac{\hat{\theta}^2}{n}}}$$

Score Test Statistics

$$\sqrt{\frac{\theta_0^2}{n}} \times \left(-\frac{n}{\theta_0} + \frac{n}{\theta_0^2} log(c) - \frac{1}{\theta_0^2} \sum_i log(x_i) \right)$$

Large Sample Test Statistics

Likelihood Ratio Test Statistics

$$
\begin{aligned}
logL(\theta) &= -nlog(\theta) - \frac{n}{\theta}\left(log(c) - \frac{\sum_i log(x_i)}{n}\right) - \sum_i log(x_i) \\
-2\left(logL(\theta_0) - logL(\hat{\theta})\right) &= -nlog\left(\frac{\hat{\theta}}{\theta_0}\right) - n\left(\frac{1}{\hat{\theta}} - \frac{1}{\theta_0}\right)\left(log(c) - \frac{\sum_i log(x_i)}{n}\right) \\
\hat{\theta} &= log(c) - \frac{\sum_i log(x_i)}{n}
\end{aligned}
$$