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Calculus

Problem Set 1
Solutions

$\hookrightarrow$ **Topics**

Integral calculus, derivatives, sign, domain and level curves of a function.

Exercise 1

In probability you have studied the concepts of p.d.f. and the expected value. In order to verify that a function is a probability density function, you have to check that its integral is equal to 1. Moreover, in order to compute the expected value, you have to compute an integral.

Consider the p.d.f. of an exponential distribution:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}.$$

You can prove by your self that this is a density.

Compute the integral $E(X) = \int x\lambda e^{-\lambda x}$.

Compute the integral $E(X) = \int x^2\lambda e^{-\lambda x}$.

Compute the Variance of the exponential distribution

Solution

$$\int_0^{+\infty} x\lambda e^{-\lambda x}\mathrm{d}x = \qquad \text{multiply and divide by lambda}$$

$$= \frac{1}{\lambda}\int_0^{+\infty} \lambda x e^{-\lambda x}\lambda \mathrm{d}x \qquad \text{define} \lambda x = y,\ \lambda \mathrm{d}x = \mathrm{d}y$$

$$= \frac{1}{\lambda}\int_0^{+\infty} y e^{-y}\mathrm{d}y \qquad \text{by the fundamental theorem of calculus and using the integration by parts}$$

$$= \frac{1}{\lambda}\left[-\frac{(1+y)}{e^y}\right]_0^{+\infty} =$$

$$= \frac{1}{\lambda}\left[-0 - \left(-\frac{1}{1}\right)\right] = \frac{1}{\lambda}$$

Second moment:

$$\int_0^{+\infty} x^2 \lambda e^{-\lambda x} \mathrm{d}x =$$ multiply

$$= \frac{1}{\lambda^2} \int_0^{+\infty} \lambda^2 x^2 e^{-\lambda x} \lambda \mathrm{d}x$$ define

$$= \frac{1}{\lambda^2} \int_0^{+\infty} y^2 e^{-y} \mathrm{d}y$$ by the fundamental t

and using the in

$$= \frac{1}{\lambda^2} \left(\left[-\frac{y^2}{e^y} \right]_0^{+\infty} + 2 \underbrace{\int_0^{+\infty} y e^{-y} dy}_{=[-(1+y)e^{-y}]_0^\infty \text{from the E(X) derivation}} \right) =$$

$$= \frac{1}{\lambda^2} [(-0+0)+2] = \frac{2}{\lambda^2}$$

Variance

$$Var(X) = E(X^2) - E(X)^2 = \frac{1}{\lambda}$$

Exercise 2

Quiz 5

Exercise

Find the domain of the following functions:

a) $f(x, y) = \ln(1 - y^2 - x^2)$

b) $f(x, y) = \ln(x^2 - 1) + \ln(1 - y^2)$

c) $f(x, y) = \ln(x \ln(\frac{1}{x+y}))$

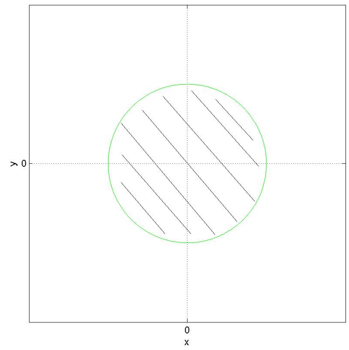
d) $f(x, y) = \ln(y^2 + x - 1)$

e) $f(x, y) = \ln(x^2 - y^2 - 1)$

Solution

a) $f(x, y) = \ln(1 - y - x)$, $D = \{(x, y) \in \mathbb{R}^2 | 1 - y^2 - x^2 > 0\}$

The domain is the interior of the unitary circle without the boundary



b) $f(x,y) = \ln(x^2-1)+\ln(1-y^2)$, $D = \{(x,y) \in \mathbb{R}^2 | x^2-1 > 0 \wedge 1-y^2 > 0\}$

$$\begin{cases} x^2 - 1 > 0 \\ 1 - y^2 > 0 \end{cases}.$$

$D = \{(x,y) \in \mathbb{R}^2 | x < -1 \vee x > 1 \wedge -1 < y < 1\}$

c) $f(x,y) = \ln(x \ln(\frac{1}{x+y}))$, $D = \{(x,y) \in \mathbb{R}^2 | x \ln(\frac{1}{x+y}) \wedge \frac{1}{x+y} > 0$

$$\begin{cases} x \ln(\frac{1}{x+y} > 0 & , \begin{cases} x > 0 \\ \frac{1}{x+y} > 0 \end{cases} \\ \frac{1}{x+y} > 0 & , x + y > 0 \wedge y > -x \end{cases}.$$

(Show it graphically).

d) $f(x,y) = \ln(y^2 + x - 1)$, $D = \{(x,y) \in \mathbb{R}^2 | y^2 + x - 1 > 0\}$
Notice that $x = 1-y^2$ is the equation of a parabola with horizontal axis of symmetry and intercept of the x axis equal to 0 and the intercepts of the y axis are equal to $1, -1$.
The domain of the function studied is on the right of the parabola.

e) $f(x,y) = \ln(x^2 - y^2 - 1)$, $D = \{(x,y) \in \mathbb{R}^2 | x^2 - y^2 - 1 > 0\}$
Notice that $x^2 - y^2 = 1$ is the equation of an hyperbola. Asymptotes: $y = x$, $y = -x$. Intercepts of the x axis 1 and -1.

Exercise 3
Find the domain, the sign and the zero level curve of the following functions

a) $f(x,y) = \ln(y^2 + x - 1)$

b) $f(x,y) = \ln(x^2 - y^2 - 1)$

c) $f(x,y) = \ln \frac{x^{-}4y^2-1}{16x^2+9y^2-1}$

Solution

c) Domain: $D = \{(x,y) \in \mathbb{R}^2 | \frac{x^{-}4y^2-1}{16x^2+9y^2-1} > 0\}$
We have to study separately the sign of the numerator and the denominator. The numerator corresponds to the equation of an hyperbola with asymptotes $x = \pm 2y$ and intercepts $x = \pm 1$.
The denominator corresponds to an ellipsis with intercepts $x = 0, y = \pm\frac{1}{3}$ and $y = 0, x = \pm\frac{1}{4}$.
Study when the numerator and the denominator are both positive or negative. You will find that the area inside the ellipsis and external of the hyperbola is the domain of the function.

Reminder

Differentations rules:

linearity of differentiation $h(x) = af(x)+bg(x)$, $h'(x) = af'(x)+bg'(x)$.

product of two functions $h(x) = f(x)g(x)$, $h(x) = f'(x)g(x)+f(x)g'(x)$.

chain rule (function of a function $h(x) = f(g(x))$, $h(x) = f'(g(x))g'(x)$.

power rule $f(x) = x^n$, $f'(x) = nx^{n-1}$.

quotient rule $h(x) = \frac{f(x)}{g(x)}$, $h'(x) = \frac{f'(x)g(x)-g'(x)f(x)}{g^2(x)}$.

exponential $h(x) = c^f(x)$, $h'(x) = c^f(x)\ \ln c\ a$.

logarithmic $h(x) = log_c x$, $h'(x) = \frac{1}{xlnc}$.

Integral calculus

linearity $\int af(x)dx = a\int f(x)dx$, $\int(f+g)dx = \int fdx + \int gdx$.

antiderivative An antiderivative of a function $f(x)$ is a function $F(x)$ whose derivative is the original F:$F' = f$. F is also called the indefinite integral of f:

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ $(n \neq -1)$.
- $\int \frac{1}{x} dx = lnx + C$.
- $\int e^x dx = e^x + C$
- $\int e^{f(x)} f'(x)dx = e^{f(x)} + C$
- $\int (f(x))^n f'(x)dx = \frac{1}{n+1}(f(x))^{n+1} + C$
- $\int \frac{1}{f(x)} f'(x)dx = \ln f(x) + C$

integration by parts $\int u(x)\cdot v'(x)dx = u(x)\cdot v(x) - \int u'(x)\cdot v(x)dx$

the fundamental theorem of calculus $\int_a^b f(x)dx = F(b) - F(a)$