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Linear Algebra
Practice October, 29th

$\hookrightarrow$ **Topics**

Partial derivatives. Eigenvalues and eigenvectors. Cholesky decomposition.

Exercise 1

Consider the function

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

and show that $f_{x,y}(0,0) \neq f_{y,x}(0,0)$.
What can you deduce for the mixed derivatives of second order?

Solution

If $(x,y) \neq (0,0)$

$$f_x(x,y) = \frac{\partial}{\partial x}\left[(x,y)\frac{x^2-y^2}{x^2+y^2}\right] = y\frac{x^2-y^2}{x^2+y^2} + (xy)\frac{\partial}{\partial x}\left[\frac{x^2-y^2}{x^2+y^2}\right]$$

Therefore, $f_x(0,y) = -y$ if $y \neq 0$. When $(x,y) \neq (0,0)$ we have

$$f_y(x,y) = \frac{\partial}{\partial y}\left[(x,y)\frac{x^2-y^2}{x^2+y^2}\right] = x\frac{x^2-y^2}{x^2+y^2} + (xy)\frac{\partial}{\partial y}\left[\frac{x^2-y^2}{x^2+y^2}\right]$$

Therefore, $f_y(x,0) = x$ if $x \neq 0$.

$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{f(h,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{f(0,h)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

$$f_{xy}(0,0) = \lim_{h\to 0}\frac{f_y(0+h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{f(h,0)}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

$$f_{yx} = \lim_{h\to 0}\frac{f_x(0,0+h)-f_x(0,0)}{h} = \lim_{h\to 0}\frac{f(0,h)}{h} = \lim_{h\to 0}\frac{-h}{h} = -1$$

Therefore, we showed that $f_{xy}(0,0) = 1 \neq -1 = f_{yx}(0,0)$.
This implies that the mixed derivatives are not continuous in the point (0,0).
Otherwise, by Schwartz Theorem we should have $f_{xy}(0,0) = f_{yx}(0,0)$.

Exercise 2

Compute the eigenvalues and the associated eigenvectors of the following matrices:

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}; B = \begin{pmatrix} 3 & 6 \\ 9 & 18 \end{pmatrix}$$

Solution

Eigenvalues of A: $\lambda = 1$ with eigenspace V_1 = vectors of the form: $(\alpha, -\alpha)$; $\lambda = 6$ with eigenspace V_6 = vectors of the form: $(4\beta, \beta)$.

Eigenvalues of B: $\lambda = 0$ with eigenspace V_0 = vectors of the form: $(-2\alpha, \alpha)$; $\lambda = 21$ with eigenspace V_{21} = vectors of the form: $(3\beta, \beta)$.

Exercise 3

Fix the parameter h so that the matrix

$$C = \begin{pmatrix} h & 1 & 0 \\ 1-h & 0 & 2 \\ 1 & 1 & h \end{pmatrix}$$

has the eigenvalue 1.

Solution

We want $\det(C - \lambda I) = 0$. Therefore,

$$C = \begin{vmatrix} h-\lambda & 1 & 0 \\ 1-h & 0-\lambda & 2 \\ 1 & 1 & h-\lambda \end{vmatrix} = 0$$

If $\lambda = 1$

$$(h-1)\begin{vmatrix} -1 & 2 \\ 1 & h-1 \end{vmatrix} + 1(-1)\begin{vmatrix} 1-h & 2 \\ 1 & h-1 \end{vmatrix} = (h-1)[(1-h)-2)] - [(1-h)(h-1)-2] = 0$$

$$-(h-1)^2 - 2(h-1) - (h-1)^2 + 2 = 0$$

$$2(h-1) = 2$$

$$h = 2$$

Exercise 4

Decompose the following matrix according to the Cholesky decomposition.

$$C = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}$$

We want to find two triangular matrix, such that their product is equal to C.
During the last class we showed that C is p.d. and you can see that it is a symmetric matrix.

$$C = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix} = \begin{pmatrix} g_1 & 0 & 0 \\ g_2 & g_3 & 0 \\ g_4 & g_5 & g_6 \end{pmatrix} \begin{pmatrix} g_1 & g_2 & g_4 \\ 0 & g_3 & g_5 \\ 0 & 0 & g_6 \end{pmatrix}$$

Therefore, we have: $g_1^2 = 6$, then $g_1 = \sqrt{6}$
$g_1 g_2 = -2$, then $g_2 = -\frac{\sqrt{6}}{3}$
$g_1 g_4 = 2$, then $g_4 = \frac{\sqrt{6}}{3}$
$g_2^2 g_3^2 = 5$, then $g_3 = \frac{\sqrt{39}}{3}$
$g_2 g_4 + g_3 g_5 = 0$, then $g_5 = \frac{2}{\sqrt{39}}$
$g_4^2 + g_5^2 + g_6^2 = 7$, then $g_6 = \frac{9}{\sqrt{39}}$

Exercise 5

Suppose that P is projection.

1. Prove that $\tilde{P} = I - P$ is a projection.
2. Prove that Ker(P)=Range($\tilde{P}$).
3. Prove that Range(P)=Ker($\tilde{P}$).
4. $P\tilde{P} = \tilde{P}P = 0$.

Solution

If we have a vector space V a projection $P : V \to V$ gives us a decomposition V =Ker(P)$\oplus$Range(P), namely each vector v has a unique decomposition $v = [v - P(v)] + P(v) = \tilde{P}(v) + P(v) = v_1 + v_2$ where $v_1 \in$Ker(P), $v_2 \in$Range(P) and $v_1 \perp v_2$.

1. The transposition is a linear operator therefore
   $\tilde{P}^t = (I - P)^t = I^t - P^t = I - P = \tilde{P}$.
   $\tilde{P}^2 = (I - P)(I - P) = I - P - P + P^2 = I - P = \tilde{P}$

2. Ker(P)=$\{v|P(v) = 0\}$, Range($\tilde{P}$) $= \{v|v = \tilde{P}(w)$ for some $w\}$. Since $v = \tilde{P}(v) + P(v)$ we have:

   i) $P(v) = 0$ implies $v = \tilde{P}(v)$. This means: $v \in$Ker(P) implies $v \in$Range($\tilde{P}$).

   ii) $v = \tilde{P}(w)$ implies $v = w - P(w)$ therefore $P(v) = P(w) - P^2(w) = 0$. This means: $v \in$Range($\tilde{P}$) implies $v \in$Ker(P)

   From i) and ii) we get Ker(P)=Range($\tilde{P}$).

3. Since $\tilde{\tilde{P}} = P$ and the point 2) we have Ker($\tilde{P}$)=Range($\tilde{\tilde{P}}$)=Range(P).

4. Direct consequence of $P^2 = P$.

Exercise 6

Let X be a $n \times k$ matrix. Suppose that X^tX is non-singular. Define $H = H_X = X(X^tX)^{-1}X^t$.

1. Prove that H is an $n \times n$ matrix.

2. Calculate H_X for

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

3. Prove that for the case 2.) H is a projection.

4. Prove that H is a projection in general.

5. Prove that if $n = k$ then $H = I_n$.

Solution

1. X^t is a $k \times n$ matrix. Therefore X^tX is a $k \times k$ matrix. This implies that $(X^tX)^{-1}$ is a $k \times k$ matrix as well. From this we get that $(X^tX)^{-1}X^t$ is $k \times n$ matrix and therefore the conclusion.

2.
$$X^tX = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad (X^tX)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

3. ....

4. $H^t = (X(X^tX)^{-1}X^t)^t = X(X^tX)^{-1}X^t = H$.
$H^2 = HH = X(X^tX)^{-1}\underbrace{X^tX(X^tX)^{-1}}_{I}X^t = X(X^tX)^{-1}X^t = H.$

5. If $n = k$ then X is a squared matrix. Since we suppose X^tX non-singular we have that X is non-singular ($\det(X^tX) = \det(X)^2$) and so we can write $H = X(X^tX)^{-1}X^t = XX^{-1}(X^t)^{-1}X^t = I_n$