Giulia Pavan, based on Chiara Perricone's notes

Calculus
Problem Set 2
Solutions

$\hookrightarrow$ **Topics**
Limits, polar coordinates, continuity, gradient, first and second partial derivatives, Schwartz's theorem, Hessian matrix, directional derivatives, tangent plane.

Exercise 1
Compute the following limits:

- $$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$$

  In polar coordinates:
  $$x = \rho\cos\theta, \qquad y = \rho\sin\theta,$$
  we have
  $$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{\rho\to 0^+} \frac{\rho\cos\theta\rho^2\sin^2\theta}{\rho^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}}$$
  $$= \lim_{\rho\to 0^+} \rho\underbrace{\cos\theta\sin^2\theta}_{bounded} = 0.$$
  Indeed,
  $$\left|\rho\cos\theta\sin^2\theta\right| \le \rho.$$

- $$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$$

  The function is defined for $x+y\neq 0$
  In polar coordinates:
  $$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y} = \lim_{\rho\to 0^+} \frac{\rho\cos\theta\sin\theta}{\rho(\cos\theta+\sin\theta)} = 0.$$
  Indeed,
  $$\left|\frac{cos\theta\sin\theta}{\cos\theta+\sin\theta}\right| \le \frac{1}{2}.$$

- $$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y2}$$

  substitute polar coordinates as in the previous example and notice that the function you get depends on θ. Therefore, the limits does not exist.

- $$\lim_{(x,y)\to(0,0)} \frac{x^3-2xy+y^2}{x^2+y^2}.$$

  In polar coordinates

  $$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{x^3-2xy+y^2}{x^2+y^2} &= \lim_{\rho\to 0^+} \frac{\rho^3\cos^3\theta - 2\rho^2\cos\theta\sin\theta + \rho^2\sin^2\theta}{\rho^2} \\ &= \lim_{\rho\to 0^+} \rho\cos^3\theta - 2\cos\theta\sin\theta + \sin^2\theta,\end{aligned}$$

  which is clearly not indenpendent of θ. This suggests that the limit does not exist. Indeed, let $y = mx$ and compute the limit on the line passing through $(0,0)$:

  $$\begin{aligned}\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^3-2xy+y^2}{x^2+y^2} &= \lim_{x\to 0} \frac{x^3-2x(mx)+(mx)^2}{x^2+(mx)^2} \\ &= \lim_{x\to 0} \frac{x^3-2mx^2+m^2x^2}{x^2(1+m^2)} \\ &= \lim_{x\to 0} \frac{x-2m+m^2}{1+m^2} \\ &= \frac{m^2-2m}{1+m^2}.\end{aligned}$$

  The limit changes depending on m.

Exercise 2
Verify if the following function is continuous in the point $(0,0)$

$$f(x,y) = \begin{cases} \frac{(x^2-y^2)^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}.$$

In polar coordinates:

$$
\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{\rho\to 0^+} \frac{(\rho^2\sin^2\theta - \rho^2\cos^2\theta)^2}{\rho^2\sin^2\theta + \rho^2\cos^2\theta} \\
&= \lim_{\rho\to 0^+} \frac{\rho^4(\sin^2\theta - \cos^2\theta)^2}{\rho^2} \\
&= \lim_{\rho\to 0^+} \rho^2(\sin^2\theta - \cos^2\theta)^2 = 0 \text{ (uniformly, w.r.t. } \theta) \\
&\neq f(0,0).
\end{aligned}
$$

The function is not continuous in $(0,0)$.
Verify if the following function is continuous in the point $(1,0)$

$$
f(x,y) = \begin{cases} \frac{(x-1)^2 y}{(x-1)^2+y^2} & \text{if } (x,y) \neq (1,0) \\ 0 & \text{if } (x,y) = (1,0) \end{cases} .
$$

In polar coordinates

$$
x = 1 + \rho\cos\theta, \qquad y = \rho\sin\theta
$$

we have

$$
\begin{aligned}
\lim_{(x,y)\to(1,0)} f(x,y) &= \lim_{(x,y)\to(1,0)} \frac{(x-1)^2 y}{(x-1)^2+y^2} \\
&= \lim_{\rho\to 0^+} \frac{\rho^2\cos^2\theta\rho\sin\theta}{\rho^2\cos^2\theta + \rho^2\sin^2\theta} \\
&= \lim_{\rho\to 0^+} \rho\cos^2\theta\sin\theta = 0 \text{ (uniformly, w.r.t. } \theta) \\
&= f(1,0).
\end{aligned}
$$

Exercise 3

Compute the gradient of the following functions:

- $f(x,y) = x^2 + 2xy - y^2$
  $f_x \equiv \frac{\partial f}{\partial x} = 2x + 2y$
  $f_y \equiv \frac{\partial f}{\partial y} = 2x - 2y$
  Thus, $\nabla f(x,y) = (2x + 2y, 2x - 2y)$.

- $f(x,y) = y^2 e^{-x}$
  $f_x = \frac{\partial f}{\partial x} = -y^2 e^{-x}$
  $f_y = \frac{\partial f}{\partial y} = 2y e^{-x}$
  Thus, $\nabla f(x,y) = (-y^2 e^{-x}, 2y e^{-x})$.

- $f(x,y) = |x+y|\sin(x^2+y)$ in $(0,0)$

$$f_x(0,0) \equiv \frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{|\Delta x| \sin(\Delta x^2)}{\Delta x} = 0,$$
$$f_y(0,0) \equiv \frac{\partial f}{\partial y}(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{|\Delta y| \sin(\Delta y)}{\Delta y} = 0$$

Thus, $\nabla f(0,0) = (0,0)$.

Exercise 4 Verify that the first order partial derivative exists in (x_0, y_0), compute it and verify it is continuous in (x_0, y_0)

- $f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.
  in $(0,0)$

$$f_x(0,0) \equiv \frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{(\Delta x)^3 0 - (\Delta x) 0}{(\Delta x)^2 + 0} + 0}{\Delta x} = 0,$$
$$f_y(0,0) \equiv \frac{\partial f}{\partial y}(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{0(\Delta y) - 0(\Delta y)}{0 + (\Delta y)^2} + 0}{\Delta y} = 0$$

Therefore, the derivative exists in $(0,0)$ Compute the first order derivatives. The results are:

$$\frac{\partial f}{\partial x} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$
$$\frac{\partial f}{\partial y} = \frac{-xy^4 - 4x^3y^2 - x^5}{(x^2+y^2)^2}$$

To verify the continuity in $(0,0)$, follow the solutions of exercise 1, substituting x and y with the polar coordinates

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta,$$

Exercise 5 Compute the first and second partial derivatives, write the Hessian matrix and check the validity of Schwartz's theorem for the following functions:

- $f(x,y) = \frac{1}{\sqrt{7x+4y-2}}$

$\frac{\partial f}{\partial x}(x,y) = -\frac{7}{2}(7x+4y-2)^{-\frac{3}{2}}$
$\frac{\partial f}{\partial y}(x,y) = -2(7x+4y-2)^{-\frac{3}{2}}$
$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{147}{4}(7x+4y-2)^{-\frac{5}{2}}$
$\frac{\partial^2 f}{\partial y^2}(x,y) = 12(7x+4y-2)^{-\frac{5}{2}}$
$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 21(7x+4y-2)^{-\frac{5}{2}} = \frac{\partial^2 f}{\partial x \partial y}(x,y)$
$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{bmatrix} = \begin{bmatrix} \frac{147}{4}(7x+4y-2)^{-\frac{5}{2}} & 21(7x+4y-2)^{-\frac{5}{2}} \\ 21(7x+4y-2)^{-\frac{5}{2}} & 12(7x+4y-2)^{-\frac{5}{2}} \end{bmatrix}$

- $f(x,y) = \log(1-x^2-y^2)$ $\frac{\partial f}{\partial x}(x,y) = -\frac{2x}{1-x^2-y^2}$
  $\frac{\partial f}{\partial y}(x,y) = -\frac{2y}{1-x^2-y^2}$
  $\frac{\partial^2 f}{\partial x^2}(x,y) = -\frac{2(1+x^2-y^2)}{(1-x^2-y^2)^2}$
  $\frac{\partial^2 f}{\partial y^2}(x,y) = -\frac{2(1-x^2+y^2)}{(1-x^2-y^2)^2}$
  $\frac{\partial^2 f}{\partial x \partial y}(x,y) = -\frac{4xy}{(1-x^2-y^2)^2} = \frac{\partial^2 f}{\partial x \partial y}(x,y)$

Exercise 6

Compute the directional derivatives of the following function:

- $f(x,y) = x^2 + xy - 2$ in $P(1,0)$ along the vector $\vec{w} = (2,1)$
  Note that $f \in C^\infty(\mathbb{R}^2;\mathbb{R})$, since a polynomial is everywhere infinite continuously differentiable, in particular f is differentiable in $(1,0)$. Hence, we can compute the directional derivatives as the scalar product between the gradient in $(1,0)$ and the direction $\vec{v} = \frac{\vec{w}}{|\vec{w}|} = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$. Now,

  $$\nabla f(x,y) = (2x+y, x) \Rightarrow \nabla f(1,0) = (2,1).$$

  Therefore, the directional derivative is given by:

  $$\frac{\partial f}{\partial v}(1,0) = \nabla f(1,0) \cdot \vec{v} = (2,1) \cdot (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}) = \sqrt{5}.$$

  More generally, you can use the definition

  $$f_v(x_0,y_0) = \lim_{t \to 0} \frac{f(x_0+tv) - f(x_0)}{t}$$

- $f(x,y) = e^x \cos y$ in $P(0,0)$ along the vector $\vec{w} = (1,2)$
  Note that $f \in C^\infty(\mathbb{R}^2;\mathbb{R})$, since exponential and trigonometric functions belong to $C^\infty(\mathbb{R}^2;\mathbb{R})$ and $C^\infty(\mathbb{R}^2;\mathbb{R})$ is an algebra. In particular

f is differentiable in $(0,0)$. Thus, we can compute the directional derivatives as the scalar product between the gradient in $(0,0)$ and the direction $\vec{v} = \frac{\vec{w}}{|\vec{w}|} = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$. Now,

$$\nabla f(x,y) = (e^x \cos y, -e^x \sin y) \Rightarrow \nabla f(0,0) = (1,0).$$

Therefore, the directional derivative is given by:

$$\frac{\partial f}{\partial v}(0,0) = \nabla f(0,0) \cdot \vec{v} = (1,0) \cdot (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) = \frac{1}{\sqrt{5}}.$$