Mathematics
Practice Sessions

Maddalena Mula

A.Y. 2024-25

1 Practice 1 - Tuesday, September 17, 2024 (11:00 - 13:00)

1. Show that $A \perp B \Rightarrow A^c \perp B^c$.

   Sol
   If two events are independent, then

   $$P(A \cap B) = P(A)P(B)$$

   According to De Morgan's Laws, $(A \cup B)^c = (A^c \cap B^c)$.
   Hence

   $$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= [1 - P(A)]\,[1 - P(B)] = P(A^c)P(B^c) \end{aligned}$$

2. Throw a die two times. Which is the probability to get at least one six?

   Sol
   Taking into account the throw of the dice and its output to characterize the event $A = 1(x = 6)$. A_1 and A_2 are independent, and the range of the possible outputs is 36.

   $$P(A_1 + A_2 \geq 1) = P(A_1 = 1) + P(A_2 = 1) - P(A_1 = 1 \cap A_2 = 1)] = \frac{6 + 6 - 1}{36} = \frac{11}{36}$$

   Another possible solution is to describe the problem as a binomial distribution of parameters $n = 2$ and $p = \frac{1}{6}$. Define

X_i, $i = 1, 2$, as the random variable taking value 1 if the outcome is 6 and 0 otherwise, and let $S = \sum_i X_i$. Then

$$\begin{aligned} P(S \geq 1) &= \sum_{k=1}^{2} \binom{2}{k} p^k (1-p)^{2-k} \\ &= \binom{2}{1} p(1-p) + \binom{2}{2} p^2 \\ &= \frac{10}{36} + \frac{1}{36} = \frac{11}{36} \end{aligned}$$

3. Define the events $A = \mathit{ill}$, $B = \mathit{smoker}$ and define the probabilities $P(B) = 0.4$, $P(A \mid B) = 0.25$, $P(A \mid B^c) = 0.07$. What is the probability of being ill? What is the probability of being a smoker given that you are ill?

   Sol
   Since the probability of not being a smoker is 0.6, the probability of being ill is

$$\begin{aligned} P(A) &= P(B^c)P(A \mid B^c) + P(B)P(A \mid B) \\ &= 0.6 \cdot 0.07 + 0.4 \cdot 0.25 = 0.142 \end{aligned}$$

   and the probability of being a smoker since you are ill is

$$\begin{aligned} P(B \mid A) &= \frac{P(B)P(A \mid B)}{P(A)} \\ &= \frac{0.4 \cdot 0.25}{0.142} \approx 0.7 \end{aligned}$$

4. Given a package with three balls, let X be the number of broken balls in the package and $p = 0.2$ the probability for a ball to be broken. (We are assuming that the fact that a ball is broken is independent on the state of the other balls.) Which is the probability that the number of broken balls is at most one?

   Sol

$$P(X = 0) + P(X = 1) = \binom{3}{0} \times 0.8^3 + \binom{3}{1} \times 0.2 \times 0.8^2 = 0.896$$

5. Calculate expectation for the geometric distribution.

   Sol

$$\begin{aligned} \mathrm{E}(X) &= \sum_{x=1}^{\infty} pxq^{x-1} = p\sum_{x=1}^{\infty} xq^{x-1} = p\sum_{x=1}^{\infty} \frac{dq^x}{dq} \\ &= \frac{p}{(1-q)^2} = \frac{1}{p} \end{aligned}$$

6. Calculate expectation, second moment and variance for the Poisson distribution.

Sol
$X \sim Poisson(\lambda)$, then $P(X=x) = \frac{\lambda^x}{x!}e^{-\lambda}$.
Expectation:

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{x=0}^{\infty} x\frac{\lambda^x}{x!}e^{-\lambda} \\
&= \sum_{x=1}^{\infty} x\frac{\lambda^x}{x!}e^{-\lambda} \\
&= \lambda e^{-\lambda}\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \\
&= \lambda
\end{aligned}$$

Second moment and variance:

$$\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{\infty} x^2\frac{\lambda^x}{x!}e^{-\lambda} = \sum_{x=1}^{\infty} x^2\frac{\lambda^x}{x!}e^{-\lambda} \\
&= \lambda\sum_{x=1}^{\infty} x\frac{\lambda^{x-1}}{(x-1)!}e^{-\lambda} \\
&= \lambda\sum_{k=0}^{\infty}(k+1)\frac{\lambda^k}{(k)!}e^{-\lambda} = \lambda\sum_{k=0}^{\infty} k\frac{\lambda^k}{(k)!}e^{-\lambda} + \lambda\sum_{k=0}^{\infty}\frac{\lambda^k}{(k)!}e^{-\lambda} \\
&= \lambda^2+\lambda \\
&\implies \mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = \lambda^2+\lambda-\lambda^2 = \lambda
\end{aligned}$$

- Suppose that a monkey is allowed to randomly hit the keys on a typewriter keyboard for an infinite amount of time. Suppose further that the keyboard has 50 keys. What is the probability that the monkey succeeds in typing the word 'banana'?

  Sol
  The chance of the first six letters forming the word 'banana' is $\frac{1}{50^6}$. Define X_n as the event *not typing 'banana' in any of the first n blocks of 6 letters.* It follows that:

$$P(X_n) = \left(1 - \frac{1}{50^6}\right)^n$$

  from which:

$$\begin{aligned}
\lim_{n\to\infty} P(X_n) &= \lim_{n\to\infty}\left(1-\frac{1}{50^6}\right)^n = 0 \\
\implies \lim_{n\to\infty} P(X_n^c) &= \lim_{n\to\infty} 1 - P(X_n) = 1
\end{aligned}$$

  Hence, the monkey will type 'banana' almost surely. As a matter of fact, the monkey will type any finite text almost surely, if given an infinite amount of time. This is known as the **infinite monkey theorem**.

2 Practice 2- Friday, September 20, 2024 (11:00 - 13:00)

1. Calculate second moment and variance for the geometric distribution.

   Sol
   Recall some facts: $X \sim Geometric(p) \Leftrightarrow P(X = x) = p(1-p)^{x-1} = pq^{x-1}$;
   if $q \in (-1, 1)$, then $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$, and this implies that $\sum_{k=0}^{\infty} \frac{dq^k}{dq} = \sum_{k=0}^{\infty} kq^{k-1} = \frac{1}{(1-q)^2}$.
   In order to solve the question, compute $\mathrm{E}(X)$:

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{x=1}^{\infty} pxq^{x-1} = p\sum_{x=0}^{\infty} xq^{x-1} = p\sum_{x=0}^{\infty} \frac{dq^x}{dq} \\
&= \frac{p}{(1-q)^2} = \frac{1}{p}
\end{aligned}
$$

   Same steps to recover $\mathrm{E}(X^2)$:

$$
\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{\infty} px^2q^{x-1} \\
&= p\sum_{x=0}^{\infty}(x^2 - x + x)q^{x-1} \\
&= pq\sum_{x=0}^{\infty}(x^2 - x)q^{x-2} + p\sum_{x=0}^{\infty} xq^{x-1} \\
&= pq\sum_{x=0}^{\infty}(x^2 - x)q^{x-2} + \frac{1}{p} \\
&= pq\sum_{x=0}^{\infty} x(x-1)q^{x-2} + \frac{1}{p} \\
&= pq\sum_{x=0}^{\infty} \frac{d^2q^x}{dq^2} + \frac{1}{p} = pq\frac{2}{(1-q)^3} + \frac{1}{p} \\
&= q\frac{2}{(1-q)^2} + \frac{1}{p} = \frac{2q}{p^2} + \frac{1}{p} \\
&= \frac{2-p}{p^2}
\end{aligned}
$$

   Using the previous results, we have:

$$
\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\
&= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}
\end{aligned}
$$

2. 28 people booked a flight. The probability that each passenger is coming at the check-in is 0.7. Which is the probability that more than 25 passengers come at the check-in? (We are assuming that each passenger is independent from the others).

Sol
Since a passenger can only either come at the check-in or not, then the distribution is $S \sim Binomial(28, 0.7)$. Then

$$
\begin{aligned}
P(S > 25) &= \sum_{x=26}^{28} \binom{28}{x} 0.7^x 0.3^{28-x} \\
&= \binom{28}{26} 0.7^{26} 0.3^2 + \binom{28}{27} 0.7^{27} 0.3^1 + \binom{28}{28} 0.7^{28} 0.3^0 \\
&= \frac{28 \cdot 27}{2} 0.7^{26} 0.3^2 + 28 \cdot 0.7^{27} 0.3 + 0.7^{28} \approx 0.0157
\end{aligned}
$$

3. Consider a random variable U with a density given by:

$$
f_U(x) = 2 \frac{\log x}{x} 1_{[1,c]}(x)
$$

with $c > 1$. Compute c, $\mathrm{E}(U^2)$ and $P(0 < U < 1)$.

Sol
In order for f_U to be a density, its mass must sum to 1, then (using the change of variable $x = e^y \Rightarrow dx = e^y dy$)

$$
\begin{aligned}
\int_{-\infty}^{+\infty} f_U(x) dx &= 2 \int_1^c \frac{\log x}{x} dx \\
&= 2 \int_0^{\log(c)} y dy = 2 \left[\frac{\log(y)^2}{2} \right]_0^c \\
&= \log(c)^2 = 1 \Rightarrow \log(c) = \pm 1 \Rightarrow c = e
\end{aligned}
$$

The second moment is:

$$
\begin{aligned}
\mathrm{E}(U^2) &= 2 \int_1^e \frac{x^2 \log x}{x} dx = 2 \int_1^e x \log(x) dx \\
&= 2 \left[\frac{x^2 \log(x)}{2} \right]_1^e - 2 \int_1^e \frac{x}{2} dx \\
&= e^2 - \frac{e^2 - 1}{2} = \frac{e^2 + 1}{2}
\end{aligned}
$$

where we used integration by parts.
Finally, $P(0 < U < 1) = P(0 \leq U \leq 1) = \int_0^1 f_U(x) dx = 0$, since $f_U(x) = 0$ for $x \in [0, 1]$.

4.

$$
Re(e^{i\pi}) = e^{i\pi}
$$

Sol
True. Rewriting the exponential in the trigonometric form, we obtain:

$$
\begin{aligned}
e^{i\pi} &= \cos(\pi) + i \sin(\pi) = -1 + i \cdot 0 = -1 \\
Re(e^{i\pi}) &= Re(-1) = -1
\end{aligned}
$$

5.

$$\frac{1}{3-4i} = \dots$$

Sol

$$\frac{1}{3-4i} = \frac{1}{3-4i}\frac{3+4i}{3+4i} = \frac{3+4i}{25}$$

6. Prove that $\text{Re}\left(\frac{1}{i}\right) = 0$
   Prove that $\text{Re}(e^{-i\pi} + 1) = 0$.

   Sol

$$\frac{1}{i} = \frac{1}{i}\frac{i}{i} = \frac{i}{-1} = -i$$
$$Re\left(\frac{1}{i}\right) = Re(-i) = 0$$

   and

$$e^{-i\pi} + 1 = \left[e^{i\pi}\right]^{-1} + 1 = -1^{-1} + 1 = -1 + 1 = 0$$

7. If $X \sim Poisson(\lambda)$, then $\text{E}(X) = \log\left(\frac{1}{P(X=0)}\right)$.

   Sol
   Since $\text{E}(X) = \lambda$ and $P(X=0) = \frac{\lambda^0}{0!}e^{-\lambda}$, then

$$\log\left(\frac{1}{P(X=0)}\right) = \log\left(e^{\lambda}\right) = \lambda = \text{E}(X) \tag{1}$$

3 Practice 3 - Tuesday, September 24, 2023 (11:00 - 13:00)

1. $V \sim Poisson(2)$. Order the following three numbers from the smallest to the biggest.

$$\frac{2}{9} \qquad 2F_V(0) \qquad P(|V - E(V)| \geq 3)$$

   Sol
   Since $F_V(0) = \frac{2^0}{0!}e^{-2} = e^{-2}$, thus, because $e < 3 \to \frac{1}{e} > \frac{1}{3} \to \frac{2}{e^2} > \frac{2}{9}$.
   Also $\text{E}(V) = 2$, then $P(|V - E(V)| \geq 3) = P(|V - \text{E}(V)| \geq 3)$, and by *Chebyshev's inequality*

$$P(|V - E(V)| \geq 3) \leq \frac{\text{E}\left[|V - \text{E}(V)|^2\right]}{3^2} = \frac{\text{Var}(V)}{9} = \frac{2}{9}$$

   To conclude, the order is

$$P(|V - \text{E}(V)| \geq 3) \leq \frac{2}{9} < 2F_V(0)$$

2. Prove directly that, given $X \perp Y$ with $X \sim Poisson(\lambda)$ and $Y \sim Poisson(\mu)$, then

$$X + Y \sim Poisson(\lambda + \mu)$$

Sol
Recalling that by the Binomial Theorem $(a+b)^m = \sum_{j=0}^{m} \binom{m}{j} a^j b^{m-j}$ for $m \geq 0$, the probability function of $X+Y$ is

$$\begin{aligned}
P(X+Y=m) &= P\left(\cup_{j=0}^{m} (X=j) \cap (Y=m-j)\right) \\
&= \sum_{j=0}^{m} P\left((X=j) \cap (Y=m-j)\right) \\
&= \sum_{j=0}^{m} P(X=j)P(Y=m-j) \\
&= \sum_{j=0}^{m} \frac{\lambda^j}{j!} e^{-\lambda} \frac{\mu^{m-j}}{(m-j)!} e^{-\mu} \\
&= \frac{e^{-\lambda-\mu}}{m!} \sum_{j=0}^{m} \lambda^j \mu^{m-j} \frac{m!}{j!(m-j)!} \\
&= \frac{e^{-\lambda-\mu}}{m!} \sum_{j=0}^{m} \binom{m}{j} \lambda^j \mu^{m-j} \\
&= \frac{e^{-\lambda-\mu}}{m!} (\lambda+\mu)^m \\
&\implies X+Y \sim Poisson(\lambda+\mu)
\end{aligned}$$

3. Characteristic function of the Gaussian: general case.

Sol

Note that $X \sim \mathcal{N}(\mu, \sigma^2)$ iff a $Z \sim \mathcal{N}(0,1)$ exists such that $X = \sigma Z + \mu$. Remember that $\varphi_Z(t) = \exp(-t^2/2)$. For a costant R.V. equal to μ we have that $\varphi_\mu(t) = \exp(i\mu t)$. Moreover a constant R.V. is independent of any other R.V. Therefore

$$\varphi_X(t) = \varphi_{\sigma Z + \mu}(t) = \varphi_{\sigma Z}(t) \cdot \varphi_\mu(t) = \varphi_Z(\sigma t) \cdot \varphi_\mu(t) = e^{-\frac{1}{2}\sigma^2 t^2} \cdot e^{i\mu t} = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$$

4. Prove that if $X_n \sim \mathcal{N}(\mu, \frac{1}{n})$ then $X_n \to \mu$ in law as $n \to +\infty$.

Sol
Using the continuity theorem we can derive the conclusion using the characteristic functions. Indeed

$$\varphi_{X_n}(t) = e^{i\mu t - \frac{1}{2}\frac{1}{n}t^2} \to e^{i\mu t} = \varphi_\mu(t) \qquad \text{as} \qquad n \to +\infty$$

5. Using the characteristic function prove that the sum of independent Gaussian r.v. is Gaussian (not true without independence).

   Sol

   Let $X \sim \mathcal{N}(\mu, \sigma^2)$, $Y \sim \mathcal{N}(\tilde{\mu}, \tilde{\sigma}^2)$ and let X, Y be independent. We have that

   $$\varphi_X(t) = e^{it\mu - \frac{\sigma^2}{2}t^2} \qquad \varphi_Y(t) = e^{it\tilde{\mu} - \frac{\tilde{\sigma}^2}{2}t^2}$$

   Then

   $$\varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t) = e^{it(\mu+\tilde{\mu}) - \frac{(\sigma^2+\tilde{\sigma}^2)}{2}t^2}$$

   Since all the distributions with "similar" characteristic functions belong to the same distribution family, then $X + Y \sim \mathcal{N}(\mu + \tilde{\mu}, \sigma + \tilde{\sigma})$.
   Recall that if $X \perp Y \Rightarrow \varphi_{X+Y}(t) = \varphi_X(t)\varphi_Y(t)$ and $X \sim Y \Leftrightarrow \varphi_X(t) = \varphi_Y(t)$.

4 Practice 4 - Friday, September 27, 2024 (11:00 - 13:00)

Determine whether the following claims are **TRUE** or **FALSE**.

1. Given $A, B, C \in \mathcal{F}$, assume $P(A \cap B \cap C) > 0$. Then $P(A \cap B \mid C) = P(A \mid B \cap C)P(B \mid C)$.

   Sol

   $$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} \cdot \frac{P(B \cap C)}{P(C)} = P(A \mid B \cap C)P(B \mid C).$$

2. Let $Z \sim B(n, p)$. This implies that $P(Z \geq 0) > P(Z > 0)$.

   Sol
   FALSE. Recall that the support of a binomial distribution is the set of positive integers, 0 included. Then the problem can be restated as

   $$P(Z > 0) + P(Z = 0) > P(Z > 0)$$

   However, $P(Z = 0) = p^0(1 - p)^n > 0 \iff 1 - p > 0$.
   Therefore, the claim is true if and only if $p < 1$. Since $p \in [0, 1]$, there is one case, $p = 1$, where the claim does not hold.

3. Let $X \sim exp(\lambda)$. This implies that $P(X \geq 0) > P(X > 0)$.

   Sol

FALSE. Recall that the support of an exponential distribution is $\mathbb{R}^+$, and that $P(X \leq k) = \int_0^k \lambda e^{-\lambda x} dx$. Then the problem can be restated as

$$
\begin{aligned}
P(X > 0) + P(X = 0) > P(X > 0) \\
P(X = 0) > 0
\end{aligned}
$$

However, since the exponential distribution is continuous, it has no mass points, and the following holds:

$$P(X = 0) = \int_0^0 \lambda e^{-\lambda x} dx = 0$$

Hence, the claim is false.

4. Let $Z \sim Poisson(\lambda)$. Then $-Z \sim Poisson(\lambda)$.

Sol
FALSE. Recall that the support of a Poisson distribution is the set of positive integers ($k \in N$).

$$P(-Z = k) = P(Z = -k) = 0$$

5. Let $X \sim exp(\lambda)$. This implies that $|X| \sim exp(\lambda)$.

Sol
TRUE.

$$
\begin{aligned}
P(|X| \leq k) &= P(-k \leq X \leq k) \\
&= P(X \leq k) - P(X \leq -k) \\
&= P(X \leq k)
\end{aligned}
$$

6. For any random variable X one has that $t < s$ implies $F_X(t) < F_X(s)$.

Sol
FALSE. Although it is always true that $t < s$ implies $F_X(t) \leq F_X(s)$, the strict inequality is not always the case. Take as an example the following distribution, $U(0, 1)$ and $\tilde{t} = 3 < \tilde{s} = 1000$, the cumulative distribution of U is

$$F_U(u) = \begin{cases} 0 & u \in (-\infty, 0) \\ u & u \in [0, 1) \\ 1 & u \in [1, +\infty) \end{cases}$$

Since $\left\{\tilde{t}, \tilde{s}\right\} \in [1, +\infty]$ then $F_U\left(\tilde{t}\right) = F_U\left(\tilde{s}\right) = 1$.

7. Let $f, g : R \to R$ be densities. Then $h = \frac{1}{3}f + \frac{2}{3}g$ is a density.

Sol
TRUE. Since f and g are densities, then $\int_{\mathbb{R}} f dx = \int_{\mathbb{R}} g dx = 1$, also $f(x)$, $g(x) \geq 0 \, \forall x \in \mathbb{R}$. We have

$$\begin{aligned} \int_{\mathbb{R}} h dx &= \frac{1}{3}\int_{\mathbb{R}} f dx + \frac{2}{3}\int_{\mathbb{R}} g dx \\ &= \frac{1}{3} + \frac{2}{3} = 1 \end{aligned}$$

and h is a linear combination of non-negative functions, then it is non-negative as well.

8. Suppose that $P(A)$, $P(B) > 0$ and $P(A \mid B) = P(B \mid A)$. Then $P(A) = P(B)$.

Sol
FALSE.

$$P(A \mid B) = P(B \mid A) \Rightarrow P(A)P(A \cap B) = P(B)P(A \cap B)$$

In the last equality it is not forbidden that $P(A \cap B) = 0$. In such a case $P(A)$ and $P(B)$ could be any number, also different from each other.

9. For any discrete random variable X it holds $P\left(X = E(X)\right) \neq 0$.

Sol
FALSE. Take as an example $X \sim B(3, 0.5)$ that has $E(X) = \frac{3}{2}$. It is also true that, since its support is N, then $P\left(X = \frac{3}{2}\right) = 0$.

10. Let $X \sim B(n, p)$. Suppose that $P(X = 0) = 1$. This implies that $P(X = n) = 0$.

Sol
TRUE.

$$\begin{aligned} &\sum_{k=0}^{n} P(X = k) = \sum_{k=1}^{n} P(X = k) + P(X = 0) = 1 \\ \Rightarrow &\sum_{k=1}^{n} P(X = k) = 0 \Rightarrow P(X = n) = 0 \end{aligned}$$

11. Let $X \sim B(n, p)$. Then $F_X(n+1) = \psi_X(0)$.

Sol
TRUE. By definition of characteristic function, it holds

$$\psi_X(t) = E(e^{itX}) \Rightarrow \psi_X(0) = E(1) = \sum_{k=0}^{n} P(X = k) = 1$$

Also, since the cumulative distribution is non-decreasing and $F_X \in [0, 1]$ with $F_X(n) = \sum_{k=0}^{n} P(X = k) = 1$, then $1 = F_X(n) \leq F_X(n+1) \leq 1 \Rightarrow F_X(n+1) = 1$.

12. The function $f : R \to R$ defined by $f(x) = \frac{2}{5}x\mathbb{1}(x)_{[0,\sqrt{5}]}$ is a density.

Sol
TRUE. In order to check if f is a density, check its sign

$$f(x) \geq 0 \Leftrightarrow x \in [0, +\infty]$$

which is fine since $\left[0, \sqrt{5}\right] \subset [0, +\infty]$, and if it sums to 1

$$\int_0^{\sqrt{5}} f \, dx = \frac{2}{5}\left[\frac{x^2}{2}\right]_0^{\sqrt{5}} = \frac{5}{5} - 0 = 1$$

13. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then $P(X \leq \mu) = \frac{1}{2}$.

Sol
TRUE. Because $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ is symmetric with respect to μ, it follows

$$P(X \leq \mu) + P(X \geq \mu) = P(X \leq \mu) + P(X \leq \mu) = 1$$

$$\Rightarrow P(X \leq \mu) = \frac{1}{2}$$

5 Practice 5 - Tuesday, October 1, 2024 (11:00 - 13:00)

1. The function $f : R \to R$ is defined by $f(x) = c \, \sin(x) 1_{[0,\pi]}$

   i Fix c so that f is a density

   ii Let X be a random variable such that f is its density: calculate the cumulative distribution function $F_X(t)$

   iii Solve the equation $F_X(t) = \frac{1}{2}$

   Sol

   i For f to be a density it must be positive all over its domain and must sum to 1. The first condition is easily matched for any positive c. For the second one, we have

$$c \int_0^{\pi} \sin(x) dx = 1$$

$$-c\left[\cos(x)\right]_0^{\pi} = 2c \implies c = \frac{1}{2}$$

   ii To calculate the CDF, keep in mind that *before* the lower bound F_X is 0 and *above* the upper bound is 1. Then only calculate what happens inside these bounds.

$$F_X(t) = \frac{1}{2}\int_0^t \sin(x) dx = \frac{1 - \cos(t)}{2}$$

Then

$$F_X(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1-\cos(t)}{2} & t \in (0, \pi) \\ 1 & t \geq \pi \end{cases}$$

iii

$$F_X(t) = \frac{1 - \cos(t)}{2} = \frac{1}{2} \to \cos(t) = 0 \to t = \frac{\pi}{2}$$

2. Suppose that you flip a fair coin which has 0 and 1 on its faces and that you roll, independently, a fair die. Let us denote by X the result of the coin and by Y the result of the die. Let $Z = XY$.

   i Which is the distribution of Z?

   ii Calculate $\mathrm{E}(Z)$

   iii Calculate $\mathrm{Var}(Z)$

   Sol

   i Z has a distribution which combines the features of a die and those of a coin. Hence, when the coin is 1, the die results do not change, while they degenerate to 0 when the coin is 0. Since the events are independent (and to get 0, the only requirement is that the coin be 0) then, for $k \in \{1, 2, 3, 4, 5, 6\}$, the distribution is

$$P(Z = 0) = P(X = 0) = \frac{1}{2}$$
$$P(Z = k) = P(X = 1)P(Y = k) = \frac{1}{2}\frac{1}{6} = \frac{1}{12}$$

   ii To calculate $\mathrm{E}(Z)$

$$\mathrm{E}(Z) = 0 \cdot P(Z = 0) + \sum_{k=1}^{6} k \cdot P(Z = k)$$
$$= \frac{1}{12}\sum_{k=1}^{6} k = \frac{7}{4}$$

   Alternatively, since the two events are independent, $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y] = \frac{21}{6} \cdot \frac{1}{2} = \frac{7}{4}$.

   iii To calculate $\mathrm{Var}(Z)$

$$\mathrm{E}(Z^2) = 0^2 P(Z = 0) + \sum_{k=1}^{6} k^2 P(Z = k)$$
$$= \frac{1}{12}\sum_{k=1}^{6} k^2 = \frac{1}{12}[1 + 4 + 9 + 16 + 25 + 36] = \frac{91}{12}$$

   Thus, the variance is

$$\mathrm{Var}(Z) = \frac{91}{12} - \frac{49}{16} = \frac{364 - 147}{48} = \frac{217}{48}$$

3. Calculate

$$\int_A \left(\frac{x}{2} - xy\right) dxdy$$

   where $A = \left\{(x, y) \in R^2 \mid y > x^2 - 4, y < -x^2 + 4\right\}$.

Sol
The extremes of integration of y are defined in the set A and depend on x, whose extremes are to be found. Check for which values of x the conditions in A are respected, by imposing the inequality

$$-x^2 + 4 > x^2 - 4 \rightarrow -x^2 + 4 > 0 \rightarrow x \in (-2, 2)$$

Now, the integral can be solved by firstly integrating with respect to y

$$\int_{-2}^{2} \left[\int_{x^2-4}^{-x^2+4} \frac{x}{2} - xy \, dy \right] dx$$

Notice that the function xy is odd in y, while $\frac{x}{2}$ is even in y. Moreover, the extremes of integration are opposite. Under such condition, the integral of an odd function is 0, while that of an even function is twice the integral from 0 to the top extreme. Hence, we obtain

$$\int_{x^2-4}^{-x^2+4} \frac{x}{2} - xy \, dy = \int_{y=0}^{-x^2+4} x \, dy = x(-x^2 + 4) = -x^3 + 4x$$

Once again, x^3 and x are odd functions of x and the extremes are opposite, then

$$\int_{-2}^{2} -x^3 + 4x \, dx = 0$$

4. The domain of the function

$$g(x, y) = \sqrt{1 - x^2 - y^2} + \sqrt{-(y + x^2 + 2)}$$

contains the point $(0, 1)$. True or false?

Sol
FALSE. Just plug the coordinates in the function

$$\begin{aligned} g(0, 1) &= \sqrt{1 - 0^2 - 1^2} + \sqrt{-(1 + 0^2 + 2)} \\ &= \sqrt{0} + \sqrt{-3} = \sqrt{-3} \end{aligned}$$

6 Practice 6 - Thursday, October 3, 2024 (11:00 - 13:00)

1. $X \sim Poisson(\lambda)$, $Y \sim Poisson(\mu)$ and $X \perp Y$ implies $\mathrm{E}(X \mid X+Y) = \frac{\lambda}{\lambda+\mu}(X+Y)$. Check the law of iterated expectation.

Sol

$$\mathrm{E}\left[\mathrm{E}\left(X \mid X+Y\right)\right] = \mathrm{E}\left[\frac{\lambda}{\lambda+\mu}(X+Y)\right] = \frac{\lambda}{\lambda+\mu}(\lambda+\mu) = \lambda = \mathrm{E}(X)$$

2. X_1,X_2,...,X_n **i.i.d.r.v.** and $S_n = X_1+X_2+...+X_n$. Prove that $\mathrm{E}(X_1 \mid S_n) = \frac{S_n}{n}$, and check the law of iterated expectation.

   Sol
   Since $\{X_i\}_{i=0}^n$ are **i.i.d.**, $\mathrm{E}[X_j|S_n]$ is the same for all j. Hence

$$\begin{aligned}\mathrm{E}[X_j|S_n] &= \frac{1}{n}\sum_i^n \mathrm{E}[X_i|S_n] \\ &= \frac{1}{n}\mathrm{E}[\sum_i^n X_i|S_n] \\ &= \frac{1}{n}\mathrm{E}[S_n|S_n] \\ &= \frac{1}{n}S_n\end{aligned}$$

   Finally, we check the law of iterated expectation

$$\mathrm{E}\left[\mathrm{E}\left(X_j \mid S_n\right)\right] = \mathrm{E}\left[\frac{S_n}{n}\right] = \frac{n\mu}{n} = \mu = \mathrm{E}(X_j) \quad \forall j$$

3. Calculate

$$\int_B \frac{\sqrt{x}}{y} dxdy$$

   where $B = \left\{(x,y) \in R^2 \mid 1 \le y \le e^{2x}, x \in [1,5]\right\}$.

   Sol

$$\begin{aligned}\int_{x=1}^5 \left[\int_{y=1}^{e^{2x}} \frac{\sqrt{x}}{y} dy\right] dx &= \int_{x=1}^5 \sqrt{x}\left[\ln(e^{2x}) - \ln(1)\right] dx \\ &= \int_{x=1}^5 2x^{\frac{3}{2}}\, dx \\ &= \frac{4}{5}[5^{\frac{5}{2}} - 1] \\ &= \frac{4}{5}[25\sqrt{5} - 1]\end{aligned}$$

4. Calculate

$$\int_C xy\; dxdy$$

   where $C = \left\{(x,y) \in R^2 \mid x^2 + y^2 < 2x\right\}$.

   Sol

To find the extremes, rewrite the set C as $y^2 < 2x - x^2$. Since y^2 is always positive, the condition $2x - x^2 > 0$ must hold, which is satisfied for $x \in (0, 2)$. Applying the square root to y^2, we get $-\sqrt{2x - x^2} < y < \sqrt{2x - x^2}$. Then

$$\int_{x=0}^{2} x \left[\int_{y=-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} y \, dy \right] dx = \int_{x=0}^{2} x \left[\frac{y^2}{2} \right]_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} dx = 0$$

since $f(y) = y$ is an odd function integrated over opposite extremes.

1. $X \sim B(1, p)$, $Y \sim B(1, p)$ and $X \perp Y$ implies $\mathrm{E}(X \mid X + Y) = \frac{1}{2}(X + Y)$. Calculate

   i $\mathrm{Var}(X \mid X + Y)$ and $\mathrm{E}[\mathrm{Var}(X \mid X + Y)]$

   ii $\mathrm{Var}(\mathrm{E}(X \mid X + Y))$

   iii Check the Law of the Total Variance

   Sol

   i Notice that if $X \sim B(1, p)$, then $X^2 \sim B(1, p)$. Now, write the conditional variance as

$$\begin{aligned}
\mathrm{Var}(X \mid X + Y) &= \mathrm{E}(X^2 \mid X + Y) - \mathrm{E}(X \mid X + Y)^2 \\
&= \mathrm{E}(X \mid X + Y) - \mathrm{E}(X \mid X + Y)^2 = \mathrm{E}(X \mid X + Y)[1 - \mathrm{E}(X \mid X + Y)] \\
&= \frac{X + Y}{2} \cdot \frac{2 - X - Y}{2} = \frac{2X + 2Y - X^2 - 2XY - Y^2}{4}
\end{aligned}$$

   Moreover, recall that $\mathrm{E}(X^2) = \mathrm{Var}(X) + \mathrm{E}(X)^2 = p(1 - p) + p^2 = p$. Then

$$\begin{aligned}
\mathrm{E}[\mathrm{Var}(X \mid X + Y)] &= E\left[\frac{2X + 2Y - X^2 - 2XY - Y^2}{4} \right] \\
&= \frac{2p + 2p - p - 2p^2 - p}{4} = \frac{p(1 - p)}{2}
\end{aligned}$$

   ii

$$\begin{aligned}
\mathrm{Var}\left(\mathrm{E}(X \mid X + Y\right) &= \mathrm{Var}\left(\frac{X + Y}{2} \right) \\
&= \frac{\mathrm{Var}\left(X + Y\right)}{4} = \frac{p(1 - p)}{2}
\end{aligned}$$

   iii Recall that the Law of Total Variance states that $\mathrm{Var}(X) = \mathrm{E}[\mathrm{Var}(X \mid X + Y)] + \mathrm{Var}\left(\mathrm{E}(X \mid X + Y)\right)$. Plugging the corresponding elements of our exercise, we obtain

$$\mathrm{E}[\mathrm{Var}(X \mid X + Y)] + \mathrm{Var}\left(\mathrm{E}(X \mid X + Y)\right) = 2\frac{p(1 - p)}{2} = p(1 - p) = \mathrm{Var}(X)$$

7 Practice 7 - Tuesday, October 8, 2024 (11:00 - 13:00)

1. If X is an absolutely continuous random variable with density f_X, then $|X|$ has as density

$$f_{|X|}(x) = \begin{cases} f_X(x) + f_X(-x) & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Sol
True. Since X is absolutely continuous, its density is well-defined. Consider $Y = |X|$, then

$$P(Y \leq 0) = 0 \rightarrow f_{|X|}(0) = 0 \quad \forall x \geq 0$$

Moreover, for $x > 0$,

$$\begin{aligned} P(Y < x) &= P(-x < X < x) = P(X < x) - P(X < -x) \\ \rightarrow f_{|X|}(x) &= \frac{dP(X < x)}{dx} - \frac{dP(X < -x)}{dx} \\ &= f_X(x) + f_X(-x) \quad \forall x > 0 \end{aligned}$$

2. Compute the eigenvalues and the associated eigenvectors of the following matrices

$$A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$$

Sol

$$\begin{aligned} \det(A - \lambda I) &= (5 - \lambda)(2 - \lambda) - 4 \\ &= \lambda^2 - 7\lambda + 6 = 0 \\ &\iff \lambda = \{1, 6\} \end{aligned}$$

To find the eigenvector associated to each eigenvalue, solve

$$\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} &\implies x = 4y \\ &\implies v_6 = \begin{pmatrix} 4\alpha \\ \alpha \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} &\implies x = -y \\ &\implies v_1 = \begin{pmatrix} -\beta \\ \beta \end{pmatrix} \end{aligned}$$

In addition, one can find the spectral decomposition of A, by choosing an arbitrary value for α and β. For instance, let $\alpha = \beta = 1$. Then, A can be written as

$$A = \begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix}^{-1}$$

1. Fix the parameter h so that the matrix

$$D = \begin{pmatrix} h & 1 & 0 \\ 1-h & 0 & 2 \\ 1 & 1 & h \end{pmatrix}$$

has an eigenvalue equal to 1.

Sol

Since the eigenvalue must solve $Z = D - \lambda I = 0$, set $\lambda = 1$, so that

$$Z = \begin{pmatrix} h-1 & 1 & 0 \\ 1-h & -1 & 2 \\ 1 & 1 & h-1 \end{pmatrix}$$

and impose

$$\begin{aligned} \det(Z) &= -(h-1)^2 + 2 + (h-1)^2 - 2(h-1) \\ &= 4 - 2h = 0 \iff h = 2 \end{aligned}$$

2. Find the spectral decomposition of

$$A = \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$$

Sol

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 4 \\ 4 & 3-\lambda \end{pmatrix}$$

By imposing the condition $\det(A - \lambda I) = 0$, we obtain

$$\begin{aligned} \det(A - \lambda I) &= (3-\lambda)^2 - 16 = 0 \\ &\implies \lambda = \{-1, 7\} \end{aligned}$$

The eigenvector corresponding to $\lambda = -1$ is

$$\begin{aligned} 3x + 4y = -x &\implies x = -y \\ &\implies v_{-1} = \begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} \end{aligned}$$

The eigenvector corresponding to $\lambda = 7$ is

$$3x + 4y = 7x \implies x = y$$
$$\implies v_7 = \begin{pmatrix} \beta \\ \beta \end{pmatrix}$$

To make the eigenvector matrix orthonormal, α and β must be such that the norm of the associated eigenvector is equal to 1. Hence, set

$$\alpha^2 + (-\alpha)^2 = 1 \to \alpha = \frac{1}{\sqrt{2}}$$
$$\beta^2 + \beta^2 = 1 \to \beta = \frac{1}{\sqrt{2}}$$

Finally

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

8 Practice 8 - Tuesday, October 15, 2023 (11:00 - 13:00)

1. Find the Choleski decomposition $A = LL^t$ of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 8 & 8 \\ 3 & 8 & 19 \end{pmatrix}$$

and solve the system $LX = b$ where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

Sol
Define

$$L = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$

Then

$$LL^t = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} \begin{pmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{pmatrix}$$
$$= \begin{pmatrix} a^2 & ab & ad \\ ab & b^2 + c^2 & bd + ce \\ ad & bd + ce & d^2 + e^2 + f^2 \end{pmatrix}$$

Then

$$
\begin{aligned}
a^2 = 1 &\to a = 1 \\
ab = 2 &\to b = 2 \\
ad = 3 &\to d = 3 \\
b^2 + c^2 = 8 &\to c = 2 \\
bd + ce = 6 + 2e = 8 &\to e = 1 \\
d^2 + e^2 + f^2 = 9 + 1 + f^2 = 19 &\to f = 3
\end{aligned}
$$

Thus

$$
L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 3 \end{pmatrix}
$$

Finally, we can solve the system by backward substitution:

$$
LX = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x + 2y \\ 3x + y + 3z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}
$$

$$
\implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}
$$

1. Consider the function

$$
f(x, y) = \ln\left(\frac{xy}{(1 + x^2)e^y}\right)
$$

Find

i the domain

ii the stationary points

iii the character of the stationary points (local max, min, saddle)

Sol

i Since the denominator is always strictly positive, then for the logarithm to have positive inputs it only matters that

$$
\mathcal{D} = \{(x, y) \in R^2 \mid xy > 0\}
$$

ii Let us rewrite our function in a more convenient form, keeping all the positive terms together

$$
f(x, y) = \ln\left(\frac{xy}{(1 + x^2)e^y}\right) = \ln(xy) - \ln(1 + x^2) - y
$$

Now, compute the partial derivatives and set them to 0:

$$\frac{\partial f}{\partial x} = \frac{1}{x} - \frac{2x}{1+x^2} = 0 \iff x = \pm 1$$
$$\frac{\partial f}{\partial y} = \frac{1}{y} - 1 = 0 \iff y = 1$$

Since the solution $(-1, 1) \notin \mathcal{D}$, then the only one acceptable is $(1, 1) \in \mathcal{D}$.

iii To characterize the nature of the unique stationary point, compute the second derivatives and evaluate them at $(1, 1)$

$$\frac{\partial^2 f}{\partial x^2}\,|_{x=y=1} = -\frac{1}{x^2} - \frac{2+2x^2-4x}{1+2x^2+x^4} = -1$$
$$\frac{\partial^2 f}{\partial y^2}\,|_{x=y=1} = -\frac{1}{y^2} = -1$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$$

Construct the *Hessian* matrix

$$H = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Since H is symmetric and diagonal, its diagonal elements are its eigenvalues. Because the latter are all negative, $(1, 1)$ is a local maximum.

- Study the stationary points of the function

$$f(x, y) = x^2 + y^3 - xy$$

Sol

$$\frac{\partial f}{\partial x} = 2x - y = 0$$
$$\frac{\partial f}{\partial y} = 3y^2 - x = 0$$

From the second equation, $x = 3y^2$. Plugging this into the first equation, we obtain

$$6y^2 = y \iff y = \left\{0, \frac{1}{6}\right\}$$

$$\implies A = (0, 0) \qquad B = \left(\frac{1}{12}, \frac{1}{6}\right)$$

Let us now compute the second derivatives

$$\frac{\partial^2 f}{\partial x^2} = 2$$
$$\frac{\partial^2 f}{\partial y^2} = 6y$$
$$\frac{\partial^2 f}{\partial xy} = \frac{\partial^2 f}{\partial yx} = -1$$

Construct the *Hessian* matrix and evaluate it at A

$$H(A) = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

The eigenvalues are

$$\begin{aligned} \det(H(A) - \lambda I) &= (2-\lambda)(-\lambda) - 1 \\ &= \lambda^2 - 2\lambda - 1 = 0 \iff \lambda = \{1 \pm \sqrt{2}\} \end{aligned}$$

Since the signs of the eigenvalues are opposite, then A is a saddle point. For B we have

$$H(B) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

which has eigenvalues

$$\begin{aligned} \det(H(B) - \lambda I) &= (2-\lambda)(1-\lambda) - 1 \\ &= \lambda^2 - 3\lambda + 2 - 1 = 0 \iff \lambda = \left\{ \frac{3 \pm \sqrt{5}}{2} \right\} \end{aligned}$$

Since the signs of the eigenvalues are both positive, B is a local minimum.

9 Practice 9 - Tuesday, October 15, 2024 (11:00 - 13:00)

1. Consider the function

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

and show that $f_{x,y}(0,0) \neq f_{y,x}(0,0)$. What can you deduce for the mixed derivatives of second order?

Sol
Let us first rewrite $f(x,y)$ as $f(x,y) = \frac{x^3y - xy^3}{x^2+y^2}$. Then the general partial derivatives are

$$\begin{aligned} \frac{\partial f}{\partial x} = f_x &= \frac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \\ \frac{\partial f}{\partial y} = f_y &= \frac{(x^3 - 3xy^2)(x^2+y^2) - 2y(x^3y - xy^3)}{(x^2+y^2)^2} = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} \end{aligned}$$

In order to compute the derivatives in $(0,0)$, apply the definition:

$$\begin{aligned} \lim_{x\to 0} \frac{\frac{x^3y-xy^3}{x^2+y^2} - 0}{x}\Bigg|_{y=0} &= y\Big|_{y=0} = 0 \\ \lim_{y\to 0} \frac{\frac{x^3y-xy^3}{x^2+y^2} - 0}{y}\Bigg|_{x=0} &= -x\Big|_{x=0} = 0 \end{aligned}$$

Using again the definition, the mixed derivatives are

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial f_y}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} \\ &= \lim_{h\to 0} \frac{f_y(h,0)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{h^5}{h^4} = 1\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial f_x}{\partial x}(0,0) &= \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k} \\ &= \lim_{k\to 0} \frac{f_x(0,k)}{k} \\ &= \lim_{k\to 0} \frac{1}{k} \cdot \frac{-k^5}{k^4} = -1\end{aligned}$$

The mixed derivatives are not equal in $(0,0)$, hence they are not continuous in $(0,0)$. This follows from *Schwarz-Young's theorem.* Indeed, in this case, the *Hessian* matrix is not symmetric in $(0,0)$ (although it is symmetric everywhere else in the domain).

1. Find the (local) maxima and minima of the function

$$f(x,y) = xy - y^2 + 3$$

subject to the constraint

$$g(x,y) = x + y^2 - 1 = 0$$

using

i a parametric representation of the constraint

ii Lagrange multipliers

Are they global?

Sol

i In order to parametrize the constraint and make it always binding, choose $y = t \to x = 1 - t^2$. This allows us to rewrite the maximization problem as

$$\begin{aligned}\max_t h(t) = f(1-t^2, t) &= (1-t^2)t - t^2 + 3 \\ &= -t^3 - t^2 + t + 3\end{aligned}$$

Recall that a sufficient condition for a stationary point x^* to be a local max (min) is that $f''(x^*) < 0$ (> 0). Hence, in order to find the stationary points, we first solve the F.O.C.

$$\begin{aligned}&\frac{dh}{dt} = -3t^2 - 2t + 1 = 0 \\ &\implies t^* = \{-1, \frac{1}{3}\}\end{aligned}$$

and evaluate the second derivative, $\frac{d^2h}{dt^2} = -6t - 2$, at t^*

$$\frac{d^2h(-1)}{dt^2} = 6 - 2 = 4 > 0$$
$$\frac{d^2h(1/3)}{dt^2} = -2 - 2 = -4 < 0$$

Therefore, $(x, y) = (0, -1)$ is a local minimum and $(x, y) = \left(\frac{8}{9}, \frac{1}{3}\right)$ is a local maximum.

ii In order to use the Lagrange multipliers method, define the Lagrangean function

$$\mathcal{L} = xy - y^2 + 3 - \lambda[x + y^2 - 1]$$

and compute the FOC with respect to x, y and λ

$$\frac{\partial \mathcal{L}}{\partial x} = y - \lambda = 0 \rightarrow y = \lambda$$
$$\frac{\partial \mathcal{L}}{\partial x} = x - 2y - 2\lambda y = 0 \rightarrow x = 2y + 2y^2$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = x + y^2 - 1 = 0 \rightarrow x = 1 - y^2$$

Equating the second and third lines, we obtain $y = \left\{-1, \frac{1}{3}\right\}$, from which $x = \left\{0, \frac{8}{9}\right\}$ (by plugging the results into the constraint).

Notice that $\frac{dh}{dt} < 0 \,\forall\, t \in (-\infty, -1) \cup (\frac{1}{3}, +\infty)$ and $\frac{dh}{dt} > 0 \,\forall\, t \in (-1, \frac{1}{3})$. Moreover, $\lim_{t \to +\infty} h(t) = -\infty$ and $\lim_{t \to -\infty} h(t) = +\infty$. Hence, the stationary points are only local max and min for the constrained function.
Also the unconstrained function has no global maximum nor minimum. Indeed, consider the following restriction $y = 1$, so that $f(x, 1) = x + 2$, which clearly has no global maximum nor minimum.

1. Calculate (also using polar coordinates)

$$\int_A 2y \, dxdy$$

where $A = \left\{(x, y) \in R^2 \mid y > 0, (x - 1)^2 + y^2 < 1\right\}$.

Sol
The domain implies that $(x-1)^2 < 1-y^2$, but since the left side is positive then so must be the right side, then $1-y^2 > 0 \rightarrow -1 < y < 1$. Combining with the second condition of the domain, $y > 0$, $0 < y < 1$, and $-\sqrt{1-y^2}+1 < x < \sqrt{1-y^2}+1$.

$$\int_{y=0}^{1} 2y \int_{x=-\sqrt{1-y^2}+1}^{\sqrt{1-y^2}+1} dxdy = \int_{y=0}^{1} 4y\sqrt{1-y^2}dy$$

Using $t^2 = 1 - y^2 \rightarrow 2t\,dt = -2y\,dy$, also the extremes of integration are reversed.

$$\int_{t=1}^{0} -4t^2 dt = \frac{4}{3}$$

If one wants to use polar coordinates then

$$\begin{aligned} x - 1 &= \rho\cos(\theta) \\ y &= \rho\sin(\theta) \end{aligned}$$

Then $y > 0 \to 0 < \theta < \pi$, while $(x-1)^2 + y^2 < 1 \to \rho^2[\cos(\theta)^2 + \sin(\theta)^2] = \rho^2 < 1 \to 0 < \rho < 1$. Recall that for the trigonometric transformation the scale factor is ρ, hence

$$\begin{aligned} \int_{\rho=0}^{1}\int_{\theta=0}^{\pi} 2\rho^2\sin(\theta)d\rho d\theta &= \frac{2}{3}[\rho^3]_0^1[-\cos(\theta)]_0^{\pi} \\ &= \frac{2}{3}[-\cos(\pi) + cos(0)]_0^{\pi} \\ &= \frac{4}{3} \end{aligned}$$

10 Practice 10 - Thursday, October 17, 2024 (11:00 - 13:00)

1. Prove that if A is symmetric and positive definite then it is invertible and A^{-1} is symmetric.

Sol
A is positive definite iff $\lambda_i > 0$. Recalling that the determinant of a matrix is equal to the product of its eigenvalues, it follows that $\det(A) = \prod_i \lambda i > 0$. Hence, A is invertible.
Moreover,

$$\begin{aligned} A &= A^t \\ \implies AA^{-1} &= \left(A^{-1}A\right)^t = I \\ \implies AA^{-1} &= A^t\left(A^{-1}\right)^t = A\left(A^{-1}\right)^t \\ \implies A^{-1} &= \left(A^{-1}\right)^t \end{aligned}$$

that is, A^{-1} is symmetric.

1. Find the spectral decomposition of the matrix

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Sol

$$\begin{aligned} \det(A - \lambda I) = (3-\lambda)^2(6-\lambda) - (6-\lambda) = 0 \\ \lambda = \{2, 4, 6\} \end{aligned}$$

The eigenvector of $\lambda = 2$ solves $3x - y = 2x \to x = y$. For instance, choose $x = 1/\sqrt{2} \to y = 1/\sqrt{2}, z = 0$ and $V_2 = (1/\sqrt{2}, 1/\sqrt{2}, 0)^t$.
The eigenvector of $\lambda = 4$ solves $3x - y = 4x \to x = -y$. For instance, choose $x = 1/\sqrt{2} \to y = -1/\sqrt{2}z = 0$ and $V_4 = (1/\sqrt{2}, -1/\sqrt{2}, 0)^t$.

The eigenvector of $\lambda = 6$ solves $6z = 6z \rightarrow z = z$ and $3x - y = 6x$ with $-x + 3y = 6y \rightarrow x = y = 0$. For instance, choose $z = 1$ and $V_6 = (0, 0, 1)^t$.
The spectral decomposition is then

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Suppose that P is a projection. Prove that

  i $\tilde{P} = I - P$ is a projection

  ii $\text{Ker}(P) = \text{Range}(\tilde{P})$

  iii $\text{Range}(P) = \text{Ker}(\tilde{P})$

  iv $\tilde{P}P = P\tilde{P} = 0$

  v The only invertible projection is P=I

  Sol

  i

  $$\tilde{P}^2 = (I-P)(I-P) = I - 2P + P^2 = I - 2P + P = I - P = \tilde{P}$$

  ii

  $$\text{Ker}(P) = \{v \in V \mid P(v) = 0\}$$
  $$\text{Range}(\tilde{P}) = \left\{v \in V \mid \exists w \in V : \tilde{P}(w) = v\right\}$$

  Any vector $v \in V$ can be rewritten in terms of P and $\tilde{P}$ by $v = Pv + \tilde{P}v$.
  If $v \in \text{Ker}(P)$, then $Pv = 0$, which implies $v = \tilde{P}v \implies v \in \text{Range}(\tilde{P})$.
  On the other hand, if $v \in \text{Range}(\tilde{P})$, then $v = \tilde{P}w = w - Pw$, which implies $Pv = Pw - P^2w = 0 \implies v \in \text{Ker}(P)$.
  Therefore, $\text{Ker}(P) = \text{Range}(\tilde{P})$.

-
  $$\text{Ker}(\tilde{P}) = \left\{v \in V \mid \tilde{P}(v) = 0\right\}$$
  $$\text{Range}(P) = \{v \in V \mid \exists w \in V : P(w) = v\}$$

  If $v \in \text{Ker}(\tilde{P}) \implies \tilde{P}v = 0$, which implies $v = Pv$, that is, $v \in \text{Range}(P)$.
  On the other hand, if $v \in \text{Range}(P) \implies v = Pw = w - \tilde{P}w$, which implies $\tilde{P}v = \tilde{P}w - \tilde{P}^2w = 0$, that is, $v \in\ Ker(\tilde{P})$.
  Therefore, $\text{Ker}(P) = \text{Range}(\tilde{P})$.

-
  $$\tilde{P}P = (1-P)P = P - P^2 = P - P = 0$$
  $$P\tilde{P} = P(1-P) = P - P^2 = P - P = 0$$

-
  $$P = P(PP^{-1}) = (PP)P^{-1} = P^2P^{-1} = PP^{-1} = I$$

- Solve the Cauchy problem

$$\begin{cases} y' = \sqrt{x}y \quad x \geq 0 \\ y(0) = 2 \end{cases}$$

Sol
Recall that $y = y(x)$. Rewrite the problem as

$$\frac{y'}{y} = \sqrt{x}$$
$$\implies \frac{d \ln y}{dx} = \sqrt{x}$$

Then

$$\int_0^x \frac{d \ln y}{dt} dt = \int_0^x \sqrt{t} dt$$
$$\implies \ln y(x) - \ln y(0) = \frac{2}{3}[x^{\frac{3}{2}} - 0^{\frac{3}{2}}]$$
$$\implies \ln y(x) - \ln 2 = \frac{2}{3}x^{\frac{3}{2}}$$
$$\implies \ln \frac{y(x)}{2} = \frac{2}{3}x^{\frac{3}{2}}$$
$$\implies y(x) = 2 \exp\left\{\frac{2}{3}x^{\frac{3}{2}}\right\}$$

11 Practice 11 - Tuesday, October 22, 2024 (11:00 - 13:00)

1. Let X be a $n \times k$ matrix. Suppose that $X^t X$ is non-singular. Define $H = H_X = X(X^t X)^{-1} X^t$.

   i Prove that H is an $n \times n$ matrix

   ii Calculate H_X for

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   iii Prove that for the previous case H is a projection

   iv Prove that H is a projection in general

   v Prove that if $n = k$ then $H = I_n$

Sol

i In terms of dimensions H is

$$(n \times k)(k \times k)(k \times n) = n \times n$$

ii

$$X^tX = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \to (X^tX)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$$

Then

$$\begin{aligned} X(X^tX)^{-1}X^t &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = H \end{aligned}$$

iii To prove that H is a projection matrix, evaluate H^2 and check that $H^2 = H$.

iv To prove that H is a projection in general,

$$\begin{aligned} H^2 &= X(X^tX)^{-1}X^tX(X^tX)^{-1}X^t \\ &= X(X^tX)^{-1}X^t = H \end{aligned}$$

v If $n = k$, then X is a square matrix. Recall that if two matrices A and B are square, then $(AB)^{-1} = B^{-1}A^{-1}$. Thus

$$\begin{aligned} H &= X(X^tX)^{-1}X^t \\ &= XX^{-1}\left(X^t\right)^{-1}X^t = I_n \end{aligned}$$

2. Study the function $F(x, y, z) = x + 3y - z$ under the constraints

$$\begin{aligned} x^2 + y^2 - z &= 0 \\ z - 2x - 4y &= 0 \end{aligned}$$

Sol

Before proceeding, notice that the constrained domain is a closed and bounded set. Indeed, putting together the two constraints yields the circle $x^2 + y^2 - 2x - 4y = 0$, which is closed and bounded. Since the objective function is continuous on $\mathbf{R}^3$, then by Weierstrass it attains a maximum and a minim value. Hence, the critical points of the Lagrangean function must be max and min.

Writing down $\mathcal{L}$ and solving the F.O.C. we obtain

$$\mathcal{L} = x + 3y - z - \lambda[x^2 + y^2 - z] - \mu[z - 2x - 4y]$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= 1 - 2\lambda x + 2\mu = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= 3 - 2\lambda y + 4\mu = 0 \\ \frac{\partial \mathcal{L}}{\partial z} &= -1 + \lambda - \mu = 0 \to \mu = \lambda - 1 \end{aligned}$$

Plugging $\mu = \lambda - 1$ into the first two lines, we obtain $x = 1 - \frac{1}{2\lambda}$ and $y = 2 - \frac{1}{2\lambda}$, and from the second constraint, $z = 10 - \frac{3}{\lambda}$. Combining these results with the first constraint yields the values of λ

$$\left[\frac{-1+4\lambda}{2\lambda}\right]^2 + \left[\frac{-1+2\lambda}{2\lambda}\right]^2 = \frac{10\lambda - 3}{\lambda}$$
$$\iff \lambda = \pm\frac{1}{\sqrt{10}}$$

Then, plugging the points into the objective function, $(x, y, z) = \left(1 + \frac{\sqrt{10}}{2}, 2 + \frac{\sqrt{10}}{2}, 10 + 3\sqrt{10}\right)$ is a (global) minimum and $(x, y, z) = \left(1 - \frac{\sqrt{10}}{2}, 2 - \frac{\sqrt{10}}{2}, 10 - 3\sqrt{10}\right)$ is a (global) maximum.
Finally, notice that the problem could be solved by focusing on the equivalent problem:

$$\begin{aligned} \textit{optimize} \quad & x + 3y - 2x - 4y \\ s.t. \quad & x^2 + y^2 - 2x - 4y = 0 \end{aligned}$$

obtained by putting together the constraints and substituting $z = 2x + 4y$ into the objective function. Simply solve for x and y and then substitute back into the equation for z.

3. Find the maximizer of $f(x, y) = x^2 + y^2$, subject to the constraints $2x + y \le 2$, $x \ge 0$, $y \ge 0$ using
   (a) parameterizations of the segments of the boundary
   (b) using Lagrangian formulation

Sol

i The constraints define a triangle. Also, since

$$\frac{\partial f}{\partial x} = 2x = 0$$
$$\frac{\partial f}{\partial y} = 2y = 0$$

is solved for just one point $(x_0, y_0) = (0, 0)$, which also lies within the boundaries. This means that the maximum must lie on the constraints. Thus, parametrize each segment and solve for each of them.
If $x = 0 \to 0 \le t \le 2 \to f(0, t) = t^2$, then the maximum is at $y = 2$, that is $(x_1, y_1) = (0, 2)$.
If $y = 0 \to 0 \le t \le 1 \to f(t, 0) = t^2$, then the maximum is at $x = 1$, that is $(x_2, y_2) = (1, 0)$.
If $x \ne 0$ and $y \ne 0$, then set $x = t$ and $y = 2 - 2t$, so that $f(t, 2 - 2t) = t^2 + (2 - 2t)^2 = 5t^2 - 8t + 4$. From this, we obtain $t = \frac{4}{5} \implies (x_3, y_3) = \left(\frac{4}{5}, \frac{2}{5}\right)$.
Plugging in f all the results, the maximum is attained at $(x_1, y_1) = (0, 2)$.

ii

$$\mathcal{L} = x^2 + y^2 - \lambda[2x + y - 2] + \mu x + \gamma y$$

Recall that $\lambda, \mu, \gamma \geq 0$.

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= 2x - 2\lambda + \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= 2y - \lambda + \gamma = 0 \\
\lambda[2x + y - 2] &= 0 \\
\mu x &= 0 \\
\gamma y &= 0
\end{aligned}$$

Check case-by-case among the possible values of x and y.
If $x = y = 0 \to \lambda = \mu = \gamma = 0$, all is fine and $(x_0, y_0) = (0, 0)$ is a stationary point.
If $x = 0$ and $y \neq 0 \to \gamma = 0 \to \lambda \neq 0 \to y = 2$ and $\mu = 2\lambda \geq 0$, then $(x_1, y_1) = (0, 2)$.
If $y = 0$ and $x \neq 0 \to \mu = 0 \to \lambda \neq 0 \to x = 1$ and $\gamma \geq 0$, then $(x_2, y_2) = (1, 0)$.
If $x \neq 0$ and $y \neq 0$, then $\mu = \gamma = 0$ and $x = \lambda \neq 0 \to 2x + y - 2 = 0$ with $2y = x$, which is solved by $(x_3, y_3) = \left(\frac{4}{5}, \frac{2}{5}\right)$.
The maximum is attained at (x_1, y_1).