Bachelor of Science in Business & Economics
Mathematics
Academic year 2014-2015
Teachers: P. Gibilisco, A. Carnevale

Lecture 1 – Monday, February 16, 2015 (11:00-13:00)

Introduction to the course.

An example on the applications of mathematics in economics: risk aversion and concavity of utility functions.

Mathematics as art. Suggested readings and activities:

- Eugene Wigner. "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," in Communications in Pure and Applied Mathematics, vol. 13, No. I (February 1960).
- A Mathematicians Lament by Paul Lockhart.
- Chagrin d'école, Daniel Pennac.
- Escher exposition at the Chiostro del Bramante

The three main argument of the course: integration, optimization in several dimensions, linear systems.

A short track to the fundamental theorem of calculus.

- Zeros of a continuous function.
- The image of a closed interval by a continuous function is a closed interval.
- The extreme value theorem (Weierstrass).
- The mean value theorem (Lagrange).
- If $f'(x) = 0$ on an interval the function is constant.
- The fundamental theorem of calculus.
- Antiderivatives: indefinite integration.
- Definite integrals.
- Example and exercises.

Lecture 2 – Wednesday, February 18, 2015 (11:00-13:00)

Linearity of integral.
Antiderivatives for elementary functions.
The change of variable formula.

$$\int \frac{f'(x)}{f(x)}\mathrm{d}x = \log|f(x)| + c$$

Integration by parts.
Improper integrals.
Exercises

- $$\int \left(\frac{1}{\sqrt[3]{x}} + 3\right)\mathrm{d}x = \frac{3}{2}\sqrt[3]{x^2} + 3x + c$$
- $$\int \left(\frac{x-3}{2}\right)^2 \mathrm{d}x = \frac{2}{3}\left(\frac{x-3}{2}\right)^3 + c$$
- $$\int \cos^3 x\mathrm{d}x = -\frac{1}{4}\cos^4 x + c$$
- $$\int \frac{e^x}{e^x+1}\mathrm{d}x = \log(e^x+1) + c$$
- $$\int \tan x\mathrm{d}x = \log|\cos x| + c$$
- $$\int \frac{1}{x\log x}\mathrm{d}x = \log|\log x| + c$$
- $$\int \frac{x+1}{x-1}\mathrm{d}x = x + 2\log|x-1| + c$$
- $$\int x\sin(x^2)\mathrm{d}x = -\frac{1}{2}\cos x^2 + c$$
- $$\int x\cos x\mathrm{d}x = x\sin x + \cos x + c$$

- $$\int xe^{-x}\mathrm{d}x = -e^{-x}(x+1)+c$$
- $$\int \log x \mathrm{d}x = x(\log x - 1) + c$$
- $$\int_1^{+\infty} \frac{1}{x}\mathrm{d}x = +\infty$$

Exercise session 1 – Monday, February 23, 2015 (11:00-13:00)

Definite and indefinite integrals. Integration by parts and integration by substitution.

Exercises:

- $$\int (5x^3 + 2x^2 + 3x)\mathrm{d}x$$
- $$\int 16e^{-4x}\mathrm{d}x$$
- $$\int xe^{3x}\mathrm{d}x$$
- $$\int x\ln x\mathrm{d}x$$
- $$\int (x+1)\sin\frac{x}{2}\mathrm{d}x$$
- $$\int \sqrt{x}\ln x\mathrm{d}x$$
- $$\int \left(\frac{x}{e^x}\right)^2 \mathrm{d}x$$
- $$\int \frac{x}{(4x^2+1)^3}\mathrm{d}x$$

- $$\int \frac{\cos x}{\sin^2 x} \mathrm{d}x$$
- $$\int \frac{\sin \sqrt{x}}{\sqrt{x}} \mathrm{d}x$$
- $$\int e^{2x} \sin(e^{2x}) \mathrm{d}x$$

Exercise session 2 – Wednesday, February 25, 2015 (11:00-13:00)

Integrals of rational functions. Improper integrals.
Exercises:

- $$\int x\sqrt{x+1} \mathrm{d}x$$
- $$\int \frac{2+x}{x-1} \mathrm{d}x$$
- $$\int \frac{\mathrm{d}x}{x^2-3x+2}$$
- $$\int \frac{1-2x}{x^2-2x-15} \mathrm{d}x$$
- $$\int_2^3 \frac{x^2+1}{x^2-1} \mathrm{d}x$$
- $$\int_1^2 \frac{x^2+1}{x^2-1} \mathrm{d}x$$
- $$\int_{-\infty}^0 \frac{e^x}{1+e^x} \mathrm{d}x$$
- $$\int_1^{+\infty} \frac{\ln x}{x^3} \mathrm{d}x$$

- $$\int_{-2}^{3}\left(\frac{1}{\sqrt{x+2}}-\frac{1}{\sqrt{3-x}}\right)\mathrm{d}x$$

Lecture 3 – Friday, February 27, 2015 (11:00-13:00)

Domains of functions $f : \mathbb{R}^2 \to \mathbb{R}$. Examples:

- $$\log\left(\frac{x^2+y^2-1}{x-y}\right)$$
- $$\sqrt{\frac{y-x^2}{2x-y+1}}$$

$\mathbb{R}^n$ as a vector space. Linear combination of vectors. Trivial and non-trivial subspaces of $\mathbb{R}^3$ and of $\mathbb{R}^2$.

The intersection of two subspaces is a subspace (not true for the union).

Linear transformations.

The scalar product. Orthogonality. Modulus of vector.

Exercise session 3 – Monday, March 2, 2015 (11:00-13:00)

Weekly test, 1

Exercise 1.

Compute the following definite integral

$$\int_{2}^{3} x\ln(x^2-1)\mathrm{d}x$$

Exercise 2.

a) Compute the following indefinite integral

$$\int \frac{x-1}{x^2-2x+1}\mathrm{d}x$$

b) Determine whether the following improper integral exists and, if so, evaluate it.

$$\int_{2}^{+\infty} \frac{x-1}{x^2-2x+1}\mathrm{d}x$$

Domain of functions of two variables.
Exercises. Domain of:

- $$f(x,y) = \frac{\sqrt{x}+\sqrt{y}}{x+xy}$$
- $$f(x,y) = \frac{\ln(x^2-2x+1)}{1-x^2-y^2}$$
- $$f(x,y) = \sqrt{\frac{1}{1-x-y}}$$
- $$f(x,y) = \frac{\ln(1-x^2-y)}{y^2-1}$$
- $$f(x,y) = \frac{\sqrt{xy-y^2}}{e^{xy}(2+x-y)}$$

Linear combinations of vectors in $\mathbb{R}^n$. Examples.
A subspace of a vector space contains the vector 0.

Lecture 4 – Wednesday, March 4, 2015 (11:00-13:00)

The vector subspace spanned by a family of vectors. Generators.
Linear dependence and independence.
For a subset B of a vector space V the following conditions are equivalent:
i) B is minimal set of generators;
ii) B is a maximal set of linearly independent vectors.
Bases for vector spaces. All the bases have the same cardinality. Dimension of a vector space.

Exercises. The linear span of a family of vectors is denoted by $\mathrm{Span}(v_1,...,v_n)$ or by $\langle v_1,...,v_n\rangle$.

- Describe $\langle(1,0,0),(0,1,0),(1,1,0)\rangle$.
- Describe $\langle(1,0,0),(0,1,0),(0,0,1)\rangle$.

- Establish if the following sets of vectors are linearly independent or not:

  a) (0,0), (1,-1);

  b) (1,1) (-1,2);

  c) (-1,1), (1,,2), (0,-1);

  d) (1,0,1), (1,1,-1), (2,4,0), (1,0,7);

  e) (1,0,1), (1,1,1), (2,1,2).

- Write the vector (-1,2) as a linear combination of the vectors (1,1) (1,2).

- Write the vector (-1,2) as a linear combination of the vectors (1,1) (1,2), (2,1).

- Write the vector (1,2,-1) as a linear combination of the vectors (1,0,0) (1,1,0), (1,1,1).

- For which k are the vectors $(1, k, k)$ and $-1, 1, 3)$ orthogonal?

Lecture 5 – Friday, March 6, 2015 (11:00-13:00)

Sum and composition of linear transformations.
Matrices and linear transformations. The row by column product.
Algebra of matrices. The transpose of a matrix.
Exercises:

- Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix}$$

  Calculate:

  a) $A - B$

  b) $3A + 2B - 4C$

  c) $2A - B^t + 3C^2$

- Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -2 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Calculate (when possible):

a) AC;

b) $(BC)A$;

c) $B + (CA)$;

d) BA;

e) BA^t;

f) $3A^t + BC$.

Area of a parallelogram. Determinant of 2×2 matrices and its geometric meaning.

Properties of determinant:

- $\det(I) = 1$
- $\det(A^t) = \det(A)$.
- If A has a column (row) of zeros then $\det(A) = 0$
- Summing to a row (column) a multiple of another row (column) does not change the determinant.
- $\det(AB) = \det(A) \cdot \det(B)$.

Exercise session 4 – Monday, March 9, 2015 (11:00-13:00)

Weekly test, 2

Exercise 1.

Determine and draw the domain of

$$f(x, y) = \frac{2\ln(x-1) + 2\ln(y+1)}{xy + y^2}$$

Exercise 2.

Determine and draw the domain of

$$g(x, y) = \frac{e^{\frac{x^2+y^2}{2}}\sqrt{x-2y}}{x^2 + y^2 - 9}$$

Vector spaces, subspaces, linear dependence/independence. Exercises.

- Given
$$A = \{v \in R^3 \,|\, 2nd \text{ coordinate is zero } \}$$
$$B = \{v \in R^3 \,|\, 2nd \text{ coordinate is } 1\}$$
$$C = \{v \in R^3 \,|\, 1st \text{ coordinate is the double of the } 3rd\}$$
determine if A, B, C are subspaces of $\mathbb{R}^3$. Describe $A \cap C$.

- a) Is $\{v = (1, 2)\}$ a basis of $\mathbb{R}^2$? If not complete v to a basis of $\mathbb{R}^2$.

  b) Write $(3, 2)$ as linear combination of v and $e_2 = (0, 1)$.

  c) Find all the vectors that are orthogonal to v in $\mathbb{R}^2$.

- Determine if $S = \{(2, 1, 0),\ (2, 2, 0),\ (0, 1, 0),\ (1, 1, 3)\}$ is a set of linearly independent vectors.

  Extract a maximal subset of linearly independent vectors from S. What is the dimension of their span?

Algebra of matrices. Product of matrices. Examples.

Lecture 6 – Wednesday, March 11, 2015 (11:00-13:00)

The inverse of a 2×2 matrix.
Exercise: calculate the inverse of the matrix
$$A = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$$

How to find the solution of a 2×2 system of linear equations: geometry.
How to find the solution of a 2×2 system of linear equations: by hands (substitution-elimination of variables); using Cramer's rule; using the inverse of a matrix.
Exercise: find the solution of the system
$$\begin{cases} 2x - 3y = 0 \\ x + y = 1 \end{cases}$$
using the afore mentioned techniques.
The determinant for 2×2 matrices as an alternating, multilinear function such that $\det(I) = 1$.
Theorem: on $n \times n$ matrices there exists only one alternating, multilinear function "det" such that $\det(I) = 1$.
Cofactor of a matrix entry.

Theorem: the determinant of a $n \times n$ matrix it is given by the Laplace formula.

Geometric meaning of the determinant.

Calculate the determinant of the matrix

$$A = \begin{pmatrix} -2 & 2 & -3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{pmatrix}$$

Lecture 7 – Friday, March 13, 2015 (11:00-13:00)

Cofactor matrix and adjugate matrix.

The inverse of an $n \times n$ matrix:

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

Find the inverse of the matrix

$$A = \begin{pmatrix} -2 & 2 & -3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{pmatrix}$$

Cramer's rule for an arbitrary linear system.

Solve the system

$$\begin{cases} -2x + 2y - 3z = 1 \\ -x + y + 3z = 0 \\ 2x - z = 0 \end{cases}$$

by hands (substitution-elimination of variables); using Cramer's rule; using the inverse of a matrix.

Properties of determinant (reprise):

- $\det(I) = 1$;
- $\det(A^t) = \det(A)$;
- if A has a column (row) of zeros then $\det(A) = 0$;
- summing to a row (column) a multiple of another row (column) does not change the determinant;
- $\det(AB) = \det(A) \cdot \det(B)$;

- $\det(cA) = c^n \det(A)$;
- If A is triangular then $\det(A) = a_{11} \cdot a_{22} \cdots a_{nn}$.

Exercise. Let

$$A = \begin{pmatrix} -2 & 2 & -3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{pmatrix}$$

Calculate $\det(A)$ transforming A in triangular matrix.

Exercise: is $A + A^t$ symmetric?

Exercise. A matrix is said *idempotent* if $A^2 = A$. Prove that if $AB = A$ and $BA = B$ then A, B are idempotent.

Exercise. Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ k & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

For which $k \in \mathbb{R}$ is A invertible?

Exercise. Let

$$B = \begin{pmatrix} k & k-1 & k \\ 0 & 2k-2 & 0 \\ 1 & k-1 & 2-k \end{pmatrix}$$

For which $k \in \mathbb{R}$ is B invertible?

The solutions of an homogeneous linear system

$$AX = 0$$

form a vector space.

Exercise session 5 – Monday, March 16, 2015 (11:00-13:00)

Weekly test, 3

Exercise 1.

Let $V = \langle (1,0,2), (0,0,1), (1,0,1) \rangle$. What is the dimension of V? Determine a basis of V.

Write, if possible, the vectors $(3,1,0)$ and $(1,0,0)$ as linear combinations of elements of the chosen basis.

Exercise 2.

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 4 & 2 \end{pmatrix}$.

Calculate AB, BA, $\det(AB^2)$, $\det(A^t + B)$.

Linear systems: Cramer's rule and inverse matrix. Exercises

$$\begin{cases} 2x + y = 1 \\ -x - y = -2 \end{cases}$$

Determinant of a 3×3 matrix: Sarrus rule:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a13 + a_{12}a_{23}a_{31} - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{23}a_{32}a_{11})$$

Determinant of

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ 0 & 3 & 1 \end{pmatrix}$$

using Sarrus rule and Laplace theorem.

Determinant of a matrix with a row (or a column) which is a multiple of another row (or column) is zero.

The vector space of the solutions of the homogeneous linear system $AX = 0$ when A is invertible is the zero vector.

Lecture 8 – Wednesday, March 18, 2015 (11:00-13:00)

Minors of a matrix A = determinants of square submatrices.

Theorem. For any matrix A the following are equal:

- size of the largest non-vanishing minor;
- dimension of the vector space generated by columns;
- dimension of the vector space generated by rows.

Rank of a matrix A.

If A is an $n \times k$ matrix then $\text{rank}(A) \leq \min(n, k)$.

Examples: find the rank of the following matrices:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ 2 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 2 \\ 2 & 1 & 0 & 3 \end{pmatrix}$$

How to solve a system of linear equations.

Examples:

$$\begin{cases} 2x + 3y - z = 0 \\ 2x + 3y - z = 1 \end{cases} \qquad \text{No solutions}$$

$$\begin{cases} 2x+3y=0 \\ 2x-3y=0 \end{cases} \qquad \text{Unique solution}$$

$$\begin{cases} 2x-3y+z=0 \\ x+y-z=1 \end{cases} \qquad \infty^1\text{solutions}$$

$$\begin{cases} 2x-3y+z=1 \\ 4x-6y+2z=2 \end{cases} \qquad \infty^2\text{solutions}$$

The Rouché-Capelli theorem: a linear system with n variables

$$AX = b$$

has solutions iff the coefficient matrix and the augmented matrix have the same rank, namely iff Rank(A)=Rank($A|b$) $= p$. The dimension of the space of the solution is $n - p$.

How to find the solutions of linear system using Rouché-Capelli and Cramer theorem.

Example 1. Solve the system

$$\begin{cases} x+2y+z=1 \\ x-y-z=2 \\ 2x+y=3 \end{cases}$$

Steps:

- Rank$A$ =Rank($A|b$) $= p$ =2 and $n = 3$. Therefore there are ∞^{3-2} solutions.
- Choose a non-vanishing minor of the largest possible size. Example

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

- Cancel the rows outside the minor.

$$\begin{cases} x+2y+z=1 \\ x-y-z=2 \end{cases}$$

- Treat the variables outside the minor as "parameters".

$$\begin{cases} x + 2y = 1 - z \\ x - y = 2 + z \end{cases}$$

- Solve the system using Cramer (or any other method)

$$\begin{cases} x = \frac{1}{3}(z+5) \\ y = \frac{1}{3}(-1-2z) \end{cases}$$

- The ∞^1 solutions are given by

$$(\frac{1}{3}(z+5), \frac{1}{3}(-1-2z), z)$$

Example 1. Solve the system

$$\begin{cases} x - 2z = 1 \\ -y + 2z = -1 \\ -x + y = 0 \end{cases}$$

Steps:

- Rank$A$ =Rank$(A|b)$ $= p =$2 and $n = 3$. Therefore there are ∞^{3-2} solutions.

- Choose a non-vanishing minor of the largest possible size. Example

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Cancel the rows outside the minor.

$$\begin{cases} x - 2z = 1 \\ -y + 2z = -1 \end{cases}$$

- Treat the variables outside the minor as "parameters".

$$\begin{cases} x = 1 - 2z \\ -y = -1 - 2z \end{cases}$$

- Solve the system using Cramer (or any other method)

$$\begin{cases} x = 1 + 2z \\ y = 1 + 2z \end{cases}$$

- The ∞^1 solutions are given by

$$(1 + 2z, 1 + 2z, z)$$

Lecture 9 – Friday, March 20, 2015 (11:00-13:00)

Exercise. Find the solutions of the following linear systems.

$$\begin{cases} x + dy + z = 0 \\ dx + dz = 1 \\ y + dz = 0 \end{cases}$$

$$\begin{cases} cx + cy + cz = 0 \\ (c + 2)x + 2y + z = 0 \\ x + cy + z = 1 \end{cases}$$

Educational Intermezzo: Werner Heisenberg, matrix mechanics, non-commutativity and the uncertainty principle.

The diagonal little heaven. Diagonal matrices. Products of diagonal matrices, commutativity. Diagonal matrices with positive or non-negative entries: square root, exponential and logarithm for this class of matrices.

Linear transformations that preserve angles and distances. Example: the symmetry with respect to the y-axis.

Exercise session 6 – Monday, March 23, 2015 (11:00-13:00)

Weekly test, 4
Exercise 1.
Let

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 0 & -3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Write A^t, calculate $\det A^t$. Write explicitly the linear system

$$A^t X = B, \quad \text{where} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and determine its solution, if it exists.

Exercise 2.
Let $A = \begin{pmatrix} 1 & k \\ -1 & k^2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
$a)$ Find the values of k for which A is invertible.
$b)$ For $k = 1$ find the solutions of $AX = B$ where $X = \begin{pmatrix} x \\ y \end{pmatrix}$.

Rank of a matrix. Calculate the rank of

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 & k \\ k & 2 & 1 & 0 \\ 0 & 1 & 0 & k \end{pmatrix}.$$

Rouch-Capelli. Determine the solutions, if they exist, of the system $AX = B_i$ where

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 2 & -5 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Gauss elimination method: the rank of a matrix is invariant under certain operations

- exchange rows (or columns)
- linear combinations of rows (or columns)
- multiplication of a row (or column) by a nonzero scalar.

Dimension of the space of columns of a matrix: determine the dimension and a basis of:

$$V = \langle \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \rangle, \quad W = \langle \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \rangle.$$

Lecture 10 – Wednesday, March 25, 2015 (11:00-13:00)

Transpose matrix and the scalar product:

$$\langle Av, w\rangle = \langle v, A^t w\rangle$$

Orthogonal matrices ($A^t = A^{-1}$).
If A is orthogonal then $\det(A) = \pm 1$.
Orthogonal matrices preserve angles and distances.
Examples of orthogonal matrices: rotations and symmetries in the plane.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues an eigenvectors. Eigenspaces. Eigenvalues for diagonal matrices.
Non-trivial solutions for homogeneous systems.
Eigenvalues as roots of characteristic polynomials.

Lecture 11 – Friday, March 27, 2015 (11:00-12:00)

Complex numbers.
Find the eigenvalues for

$$\begin{pmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find the eigenvalues for

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Symmetric matrices have real eigenvalues.
Find the eigenvalues for

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{pmatrix}$$

Exercise session 7 – Monday, March 30, 2015 (11:00-13:00)

Weekly test, 5

Exercise 1. Determine the rank of

$$A = \begin{pmatrix} k & 0 & 1 & 2 \\ k & 1 & k & 1 \\ k & 0 & k & 2 \end{pmatrix}$$

for $k \in \mathbb{R}$.

Exercise 2. Determine, if they exist, solutions of

$$\begin{cases} 2x - y + x = 3 \\ x + y - z = 1 \\ x - 2y + 2z = 2 \end{cases}$$

Complex numbers. Sum and product of complex numbers. Complex conjugation.

If $z \in \mathbb{C}$ then $z\bar{z}$ and $z - \bar{z}$ are real numbers.

Real polynomials of degree 2 have real or complex conjugate roots.

Eigenvalues, eigenvectors, characteristic polynomial.

Exercise. Find the eigenvalues of the rotation of $\pi/3$ in $\mathbb{R}^2$.

Exercise. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and suppose that there exists an invertible matrix C such that $C^{-1}AC = \begin{pmatrix} \lambda & 0 \\ 0 & 3 \end{pmatrix}$. Which value(s) can take λ?

Exercise. Let $A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$. Which among $v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are eigenvectors of A? What are the corresponding eigenvalues?

Lecture 12 – Wednesday, April 1, 2015 (11:00-13:00)

Topology of $\mathbb{R}$ and $\mathbb{R}^2$: balls; open, closed, bounded, compact sets.

Continuous functions in $\mathbb{R}^2$. The Weierstrass theorem.

Planes in $\mathbb{R}^3$.

What is a good definition of differentiability in dimension $n > 1$? Wanted: differentiability should imply: i) continuity, ii) existence of a tangent plane.

Partial derivatives. Directions. Directional derivatives.
The mother of all counterexamples: the function

$$f(x,y) = \begin{cases} \left(\frac{x^2 y}{x^4+y^2}\right)^2 & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

i) has the partial derivatives in (0,0);
ii) has the directional derivatives in all directions in (0,0);
iii) is discontinuous in (0,0).
Linear transformation $L : \mathbb{R}^2 \to \mathbb{R}$ by scalar product.
Cauchy-Schwartz inequality.

Exercise session 8 – Monday, April 13, 2015 (11:00-13:00)

Weekly test, 6
Exercise 1.
Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

- Determine the eigenvalues of A.
- Compute Ae_1, Ae_2, Ae_3, $A(e_1 + e_2)$, $A(e_1 - e_2)$, where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
- Determine $A(2e_1 + 2e_2 + 3e_3)$ without calculating explicitly the row by column product.

Exercise 2.
Let $z = 3 + i$. Determine all $z_1 \in \mathbb{C}$ such that the real part of $z + z_1$ is 4 and such that $z_1 \bar{z}_1 = 5$.

Partial derivatives of functions of two variables. Exercises.
Determine the domain and the first partial derivatives of

- $f(x,y) = x - \ln(10 + 4y^2)$
- $f(x,y) = \sqrt{x + 3y}$
- $f(x,y) = y^2 + xe^{10+2y}$
- $f(x,y) = \frac{xy}{x-y}$
- $f(x,y) = \frac{x}{y}$
- $f(x,y) = \ln(x^2 + y^2)$

Level curves of a function of two variables. For any $c \in \mathbb{R}$, what kind of curve is $f(x,y) = c$?

Examples.

Level curves of $f(x,y) = x^2 + y^2$ are circles for $c > 0$, the origin for $c = 0$, empty for $c < 0$.

Level curves of $f(x,y) = \ln(\frac{y}{x^2})$ are parabolas for all $c \in \mathbb{R}$.

Lecture 13 – Wednesday, April 15, 2015 (11:00-13:00)

The gradient.

Exercise: find the gradient of $f(x,y) = e^y + \sin(x+y)$ in the point $(\pi/2, 0)$ (Answer (0,1)).

Differentiable functions.

If f is differentiable then $\frac{\partial f}{\partial v}(P_0) = \langle \nabla f(P_0), v \rangle$.

Verify the above formula for the function $g(x,y) = -x^2 - y^2$ in (-1,0). w.r.t. direction (1,0).

Continuity of differentiable functions.

The tangent plane.

Find the tangent plane of the function $f(x,y) = e^y + \sin(x+y)$ in the point $(\pi/2, 0)$ (answer: $y - z + 2 = 0$).

Stationary points.

Exercise. Find the stationary points of the following functions:

$$2x^3 + y^3 - 3x^2 - 3y + 5$$

$$x^2 + y^3 - xy$$

$$x^2 + y^4 + y^2 + z^3 - 2xz$$

Lecture 14 – Friday, April 17, 2015 (11:00-13:00)

A sufficient criterion for differentiability (existence of partial derivatives in a neighborhood of P_0 and their continuity in P_0).

The Schwartz (or Young) theorem: conditions for the symmetry of the Hessian matrix (existence of mixed partial derivatives in a neighborhood of P_0 and their continuity in P_0).

Necessary and sufficient conditions for local maxima and minima using the eigenvalues of the Hessian matrix. Saddle points.

Examples and counterexamples. Study the following functions in the origin (0.0).

$$x^2 - y^4$$
$$x^2 + y^4$$
$$-x^2 - y^4$$

Find the character of the stationary points of the functions:

$$2x^3 + y^3 - 3x^2 - 3y + 5$$
$$x^2 + y^3 - xy$$

Find the stationary points of the following function and discuss the behavior of the function in those points (using two different arguments)

$$h(x, y) = e^{x^2} + xy - y^2 - 5.$$

Consider the function

$$f(x, y) = \log\left(\frac{xy}{(x^2 + 1)(y^2 + 1)}\right).$$

Find: i) the domain; ii) the stationary points; iii) the character of the stationary points (local max, min, saddle). (Hint: consider the symmetries ...)

Exercise session 9 – Monday, April 20, 2015 (11:00-13:00)

Weekly test, 7

Exercise 1.

Determine domain and first partial derivatives of $f(x, y) = (x-y)^2 e^{x-y} + 2\ln(xy)$.

Exercise 2.

Draw the level curves of the function $g(x, y) = ye^{-x}$.

Bonus question: determine, if they exist, points in which the gradient of g $\nabla g = (g_x, g_y)$ is parallel to the y-axis.

Partial derivatives of functions of two variables and stationary points. Exercises.

- Determine the stationary points of $f(x, y) = -(x-4)^2 - y^2$ and their characters. Is there a global maximum/minimum?
- Determine the stationary points of $f(x, y) = x^3 + y^3 + xy$ and their characters.
- Determine the stationary points of $f(x, y) = xye^{-\frac{x^2+y^2}{2}}$ and their characters.

Lecture 15 – Wednesday, April 22, 2015 (11:00-13:00)

Simulation 1

Lecture 16 – Friday, April 24, 2015 (11:00-13:00)

Simulation 2

Lecture 17 – Monday, April 27, 2015 (11:00-13:00)

Simulation 3 (Second part)

Lecture 18 – Wednesday, April 29, 2015 (11:00-13:00)

Simulation 3 (First part)

Study the functions

$$\frac{e^x}{|x| - 2x + 1}$$

$$\frac{x-1}{\log(x-1)}$$

Exercise. Prove that for the function

$$f(x) = x^3 - 3x + 1$$

in the interval $[\sqrt{3}, 0]$ it exist only one point satisfying the Rolle theorem.