Macroeconomics / Global Economics

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Technological Progress and Growth

Using the same notation of the BAG book (chpater 15), consider the following equation describing the accumulation of capital

$$K_{t+1} - K_t = I_t - \delta K_t,$$

where

$$I_t = sY_t,$$

with

$$Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha},$$

We assume that employment and technology grow at given exogenous rates:

$$\begin{aligned} \frac{N_{t+1} - N_t}{N_t} &= g_N, \\ \frac{A_{t+1} - A_t}{A_t} &= g_A. \end{aligned}$$

We can express all variables in terms of effective workers as follows.

Let's start with the production function $Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}$:

$$\begin{aligned} \frac{Y_t}{A_t N_t} &= \frac{K_t^{\alpha}(A_t N_t)^{1-\alpha}}{A_t N_t} \\ \frac{Y_t}{A_t N_t} &= \frac{K_t^{\alpha}(A_t N_t)^{1-\alpha}}{(A_t N_t)^{\alpha}(A_t N_t)^{1-\alpha}} \\ \frac{Y_t}{A_t N_t} &= \frac{K_t^{\alpha}}{(A_t N_t)^{\alpha}} \\ \hat{y}_t &= \hat{k}_t^{\alpha} \end{aligned}$$

where $\hat{y}_t$ and $\hat{k}_t$ denote output and capital per effective workers.

Now take the accumulation equation of capital, an play with it as follows:

$$
\begin{aligned}
K_{t+1} - K_t &= \underbrace{sY_t}_{I_t} - \delta K_t \\
\frac{K_{t+1} - K_t}{A_t N_t} &= s\frac{Y_t}{A_t N_t} - \delta\frac{K_t}{A_t N_t} \\
\frac{K_{t+1}}{A_t N_t} - \frac{K_t}{A_t N_t} &= s\frac{Y_t}{A_t N_t} - \delta\frac{K_t}{A_t N_t} \\
\frac{K_{t+1}}{A_t N_t}\frac{N_{t+1}}{N_{t+1}}\frac{A_{t+1}}{A_{t+1}} - \frac{K_t}{A_t N_t} &= s\frac{Y_t}{A_t N_t} - \delta\frac{K_t}{A_t N_t} \\
\frac{K_{t+1}}{A_{t+1} N_{t+1}}\frac{N_{t+1}}{N_t}\frac{A_{t+1}}{A_t} - \frac{K_t}{A_t N_t} &= s\frac{Y_t}{A_t N_t} - \delta\frac{K_t}{A_t N_t}
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\frac{N_{t+1} - N_t}{N_t} &= g_N \rightarrow \frac{N_{t+1}}{N_t} = 1 + g_N \\
\frac{A_{t+1} - A_t}{A_t} &= g_A \rightarrow \frac{A_{t+1}}{A_t} = 1 + g_A
\end{aligned}
$$

and using the same notation adopted above, then we have

$$
\begin{aligned}
\frac{K_{t+1}}{A_{t+1} N_{t+1}}\frac{N_{t+1}}{N_t}\frac{A_{t+1}}{A_t} - \frac{K_t}{A_t N_t} &= s\frac{Y_t}{A_t N_t} - \delta\frac{K_t}{A_t N_t} \\
\hat{k}_{t+1}(1 + g_N)(1 + g_A) - \hat{k}_t &= s\hat{y}_t - \delta\hat{k}_t
\end{aligned}
$$

which can be equivalently written as

$$
\begin{aligned}
\hat{k}_{t+1}(1 + g_N)(1 + g_A) - \hat{k}_t + g_A\hat{k}_t - g_A\hat{k}_t + g_N\hat{k}_t - g_N\hat{k}_t &= s\hat{y}_t - \delta\hat{k}_t \\
\hat{k}_{t+1}(1 + g_N)(1 + g_A) - \hat{k}_t - g_A\hat{k}_t - g_N\hat{k}_t &= s\hat{y}_t - \delta\hat{k}_t - g_A\hat{k}_t - g_N\hat{k}_t \\
\hat{k}_{t+1}(1 + g_N)(1 + g_A) - \hat{k}_t\left(1 + g_A + g_N\right) &= s\hat{y}_t - \delta\hat{k}_t - g_A\hat{k}_t - g_N\hat{k}_t
\end{aligned}
$$

Finally, note that $(1 + g_N)(1 + g_A) = 1 + g_A + g_N + g_A \times g_N$. Clearly the term $g_A \times g_N \simeq 0$ (both factors refer to growth rates... small numbers...).

Using this approximation $(1+g_N)(1+g_A) \simeq 1+g_A+g_N$ the accumulation equation of capital expressed in effective workers terms becomes

$$\left(\hat{k}_{t+1} - \hat{k}_t\right)(1 + g_A + g_N) = s\hat{y}_t - (\delta + g_A + g_N)\,\hat{k}_t.$$

In steady state $\hat{k}_{t+1} - \hat{k}_t = 0$. Therefore the level of investment (saving) per effective worker needed to maintain a constant level of capital per effective worker is

$$s\hat{y}_{ss} = (\delta + g_A + g_N)\,\hat{k}_{ss}$$

We have so shown the results of page 331 of the BAG book.

Using $\hat{y}_t = \hat{k}_t^{\alpha}$, we then have

$$s\hat{k}_{ss}^{\alpha} = (\delta + g_A + g_N)\,\hat{k}_{ss}$$

which can be solved for $\hat{k}_{ss}$ (the steady-state level of capital per effective workers)

$$\hat{k}_{ss} = \left(\frac{s}{\delta + g_A + g_N}\right)^{\frac{1}{1-\alpha}}$$

Production immediately follows:

$$\begin{aligned} \hat{y}_{ss} &= \hat{k}_{ss}^{\alpha} \\ \hat{y}_{ss} &= \left(\frac{s}{\delta + g_A + g_N}\right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Clearly in steady state all variables expressed in effective worker terms are constant, this implies that:

- the growth rate of output per effective worker is zero
- the growth rate of output per worker is g_A

- the growth rate of output is equal to $g_A + g_N$

Notice in fact that in steady state
$\frac{Y_{t+1}}{A_{t+1}N_{t+1}} = \frac{Y_t}{A_t N_t} = \hat{y}_{ss}$

This implies that the growth rate of output per effective worker is zero ($\frac{Y_{t+1}}{A_{t+1}N_{t+1}} = \frac{Y_t}{A_t N_t} \rightarrow \hat{y}_{t+1} = \hat{y}_t = \hat{y}_{ss}$!!!!)

But also $\frac{Y_{t+1}}{A_{t+1}N_{t+1}} = \frac{Y_t}{A_t N_t} \rightarrow \frac{Y_{t+1}}{N_{t+1}} = \frac{A_{t+1}Y_t}{A_t N_t} \rightarrow \frac{Y_{t+1}}{N_{t+1}} / \frac{Y_t}{N_t} = \frac{A_{t+1}}{A_t} = (1+g_A) \rightarrow$ This implies that the growth rate of output per worker is g_A

Finally
$\frac{Y_{t+1}}{A_{t+1}N_{t+1}} = \frac{Y_t}{A_t N_t} \rightarrow \frac{Y_{t+1}}{Y_t} = \frac{A_{t+1}}{A_t}\frac{N_{t+1}}{N_t} = (1 + g_N)(1 + g_A) \simeq 1 + g_A + g_N$

This implies that the growth rate of output is equal to $g_A + g_N$.