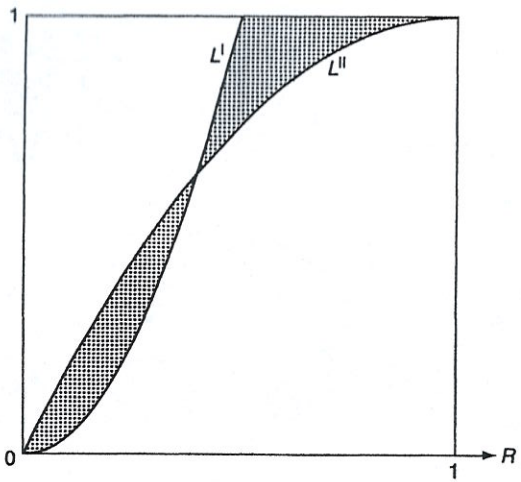

FIGURE 2.4 Distribution of prices in first- and second-price auctions.

Proposition 2.4. *With independently and identically distributed private values, the distribution of equilibrium prices in a second-price auction is a mean-preserving spread of the distribution of equilibrium prices in a first-price auction.*

Proof. The revenue in a second-price auction is just the random variable $R^{II} = Y_2^{(N)}$; the revenue in a first-price auction is the random variable $R^{I} = \beta(Y_1^{(N)})$, where $\beta \equiv \beta^{I}$ is the symmetric equilibrium strategy from Proposition 2.2. So we can write

$$E\left[R^{II} \mid R^{I} = p\right] = E\left[Y_2^{(N)} \mid Y_1^{(N)} = \beta^{-1}(p)\right]$$

But for all y,

$$E\left[Y_2^{(N)} \mid Y_1^{(N)} = y\right] = E\left[Y_1^{(N-1)} \mid Y_1^{(N-1)} < y\right] \tag{2.8}$$

This is because the only information regarding the second highest of N values, $Y_2^{(N)}$, that the event that the highest of N values $Y_1^{(N)} = y$ provides is that the highest of $N-1$ values, $Y_1^{(N-1)}$, is less than y. (See (C.6) in Appendix C for a formal demonstration.)

Using (2.8), we can write

$$\begin{aligned} E\left[R^{II} \mid R^{I} = p\right] &= E\left[Y_1^{(N-1)} \mid Y_1^{(N-1)} < \beta^{-1}(p)\right] \\ &= \beta\left(\beta^{-1}(p)\right) \\ &= p \end{aligned}$$

recalling (2.4)

Since $E\left[R^{II} \mid R^{I} = p\right] = p$, there exists a random variable Z such that the distribution of R^{II} is the same as that of $R^{I} + Z$ and $E\left[Z \mid R^{I} = p\right] = 0$. Thus, L^{II} is a mean-preserving spread of L^{I}. ■

2.5 RESERVE PRICES

In the analysis so far, the seller has played a passive role. Indeed, we have implicitly assumed that the seller parts with the object at whatever price it will fetch. In many instances, sellers reserve the right to not sell the object if the price determined in the auction is lower than some threshold amount—say, $r > 0$. Such a price is called the *reserve price*. We now examine what effect such a reserve price has on the expected revenue accruing to the seller.

RESERVE PRICES IN SECOND-PRICE AUCTIONS

Suppose that the seller sets a "small" reserve price of $r > 0$. Since the price at which the object is sold can never be lower than r, no bidder with a value $x < r$ can make a positive profit in the auction. In a second-price auction, a reserve price makes no difference to the behavior of the bidders; it is still a weakly dominant strategy to bid one's value. The expected payment of a bidder with value r is now just $rG(r)$, and the expected payment of a bidder with value $x \geq r$ is

$$m^{II}(x, r) = rG(r) + \int_r^x y g(y)\, dy \tag{2.9}$$

since the winner pays the reserve price r whenever the second-highest bid is below r.

RESERVE PRICES IN FIRST-PRICE AUCTIONS

Now consider a first-price auction with a reserve price $r > 0$. Once again, since the price is at least r, no bidder with a value $x < r$ can make a positive profit. Furthermore, if β^{I} is a symmetric equilibrium of the first-price auction with reserve price r, it must be that $\beta^{I}(r) = r$. This is because a bidder with value r wins only if all other bidders have values less than r and, in that case, can win with a bid of r itself. In all other respects, the analysis of a first-price auction is unaffected, and in a manner analogous to Proposition 2.2 we obtain that a symmetric equilibrium bidding strategy for any bidder with value $x \geq r$ is

$$\begin{aligned} \beta^{I}(x) &= E[\max\{Y_1, r\} \mid Y_1 < x] \\ &= r\frac{G(r)}{G(x)} + \frac{1}{G(x)} \int_r^x y g(y)\, dy \end{aligned}$$

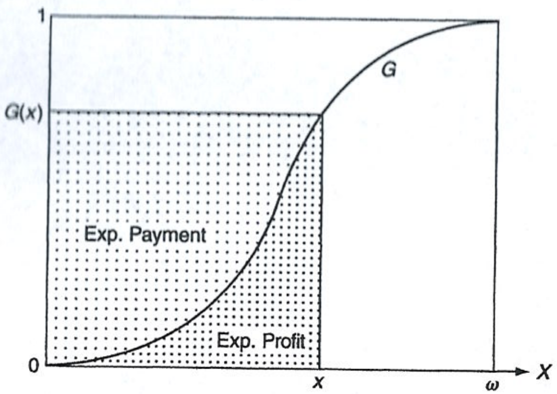

FIGURE 2.3 Payments and profits in first- and second-price auctions.

In a first-price auction, the winner pays what he or she bid, and thus the expected payment by a bidder with value x is

$$m^{\mathrm{I}}(x) = \mathrm{Prob[Win]} \times \text{Amount bid} = G(x) \times E[Y_1 \mid Y_1 < x] \tag{2.5}$$

which is the same as in a second-price auction (see (2.1)). Figure 2.3 depicts both the expected payment and the expected payoff of a bidder with value x in either auction. Because the expected revenue of the seller is just the sum of the *ex ante* (prior to knowing their values) expected payments of the bidders, this also implies that the expected revenues in the two auctions are the same. Let us see why.

The *ex ante* expected payment of a particular bidder in either auction is

$$\begin{aligned} E\left[m^A(X)\right] &= \int_0^\omega m^A(x) f(x)\,dx \\ &= \int_0^\omega \left(\int_0^x y g(y)\,dy\right) f(x)\,dx \end{aligned}$$

where $A = \mathrm{I}$ or II. Interchanging the order of integration, we obtain that

$$\begin{aligned} E\left[m^A(X)\right] &= \int_0^\omega \left(\int_y^\omega f(x)\,dx\right) y g(y)\,dy \\ &= \int_0^\omega y(1-F(y))\,g(y)\,dy \end{aligned} \tag{2.6}$$

The expected revenue accruing to the seller $E\left[R^A\right]$ is just N times the *ex ante* expected payment of an individual bidder, so

$$\begin{aligned} E\left[R^A\right] &= N \times E\left[m^A(X)\right] \\ &= N \int_0^\omega y(1-F(y))\,g(y)\,dy \end{aligned}$$

But now notice that the density of $Y_2^{(N)}$, the second highest of N values, $f_2^{(N)}(y) = N(1-F(y)) f_1^{(N-1)}(y)$ (see Appendix C), and since $f_1^{(N-1)}(y) = g(y)$, we can write

$$\begin{aligned} E\left[R^A\right] &= \int_0^\omega y f_2^{(N)}(y)\,dy \\ &= E\left[Y_2^{(N)}\right] \end{aligned} \tag{2.7}$$

In either case, the expected revenue is just the expectation of the second-highest value. Thus, we conclude that *the expected revenues of the seller in the two auctions are the same*. For future reference, we record this fact in the following proposition.

Proposition 2.3. *With independently and identically distributed private values, the expected revenue in a first-price auction is the same as the expected revenue in a second-price auction.*

The fact that the expected selling prices in the two auctions are equal is all the more striking because in specific realizations of the values the price at which the object is sold may be greater in one auction or the other. With positive probability, the revenue R^{I} in a first-price auction exceeds R^{II}, the revenue in a second-price auction, and vice versa. For instance, when values are uniformly distributed and there are only two bidders, the equilibrium strategy in a first-price auction is $\beta^{\mathrm{I}}(x) = \frac{1}{2}x$. If the realized values are such that $\frac{1}{2}x_1 > x_2$, then the revenue in a first-price auction is greater than that in a second-price auction. On the other hand, if $\frac{1}{2}x_1 < x_2 < x_1$, the opposite is true. Thus, while the revenue may be greater in one auction or another depending on the realized values, we have argued that *on average* the revenue to the seller will be the same.

Actually, we can say more about the distribution of prices in the two auctions. It is clear that the revenues in a second-price auction are more variable than in its first-price counterpart. In the former, the prices can range between 0 and ω; in the latter, they can only range between 0 and $E[Y_1]$. A more precise result can be formulated along the following lines. Let L^{I} denote the distribution of the equilibrium price in a first-price auction and likewise, let L^{II} be the distribution of prices in a second-price auction. Then L^{II} is a *mean-preserving spread* of L^{I}—from the perspective of the seller, a second-price auction is *riskier* than a first-price auction (see Appendix B). Every risk-averse seller prefers the latter to the former (assuming, of course, that bidders are risk-neutral).[1] Figure 2.4 depicts the two distributions in the case of uniformly distributed values with two bidders. Since the two distributions have the same mean, the two shaded regions are, as they must be, equal in area.

[1]This is also equivalent to the statement that L^{I} dominates L^{II} in the sense of *second*-order stochastic dominance. Again, see Appendix B.

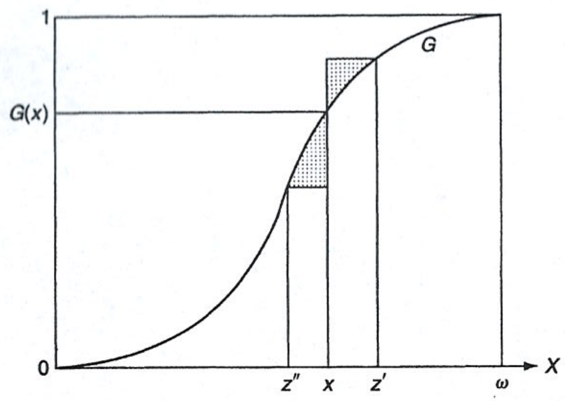

FIGURE 2.1 Losses from over- and underbidding in a first-price auction.

(The preceding argument shows that bidding an amount $\beta(z') > \beta(x)$ rather than $\beta(x)$ results in a loss equal to the shaded area to the right in Figure 2.1; similarly, bidding an amount $\beta(z'') < \beta(x)$ results in a loss equal to the area to the left.)

We have thus argued that if all other bidders are following the strategy β, a bidder with a value of x cannot benefit by bidding anything other than $\beta(x)$, and this implies that β is a symmetric equilibrium strategy. ■

The equilibrium bid can be rewritten as

$$\beta^{\mathrm{I}}(x) = x - \int_0^x \frac{G(y)}{G(x)}\,dy$$

by using (A.2) in Appendix A again. This shows that the bid is, naturally, less than the value x. Since

$$\frac{G(y)}{G(x)} = \left[\frac{F(y)}{F(x)}\right]^{N-1}$$

the degree of "shading" (the amount by which the bid is less than the value) depends on the number of competing bidders and as N increases, approaches 0. Thus, for fixed F, as the number of bidders increases, the equilibrium bid $\beta^{\mathrm{I}}(x)$ approaches x.

It is instructive to derive the equilibrium strategies explicitly in a few examples.

Example 2.1. *Values are uniformly distributed on* $[0,1]$.

If $F(x) = x$, then $G(x) = x^{N-1}$ and

$$\beta^{\mathrm{I}}(x) = \frac{N-1}{N}x$$

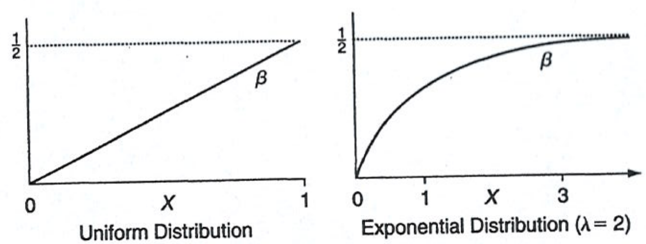

FIGURE 2.2 Equilibria of two-bidder symmetric first-price auctions.

In this case, the equilibrium strategy calls upon a bidder to bid a constant fraction of his value. For the case of two bidders, the equilibrium bidding strategy is depicted in the left-hand panel of Figure 2.2. ▲

Example 2.2. *Values are exponentially distributed on* $[0,\infty)$, *and there are only two bidders.*

If $F(x) = 1 - \exp(-\lambda x)$, for some $\lambda > 0$, and $N = 2$, then

$$\begin{aligned}\beta^{\mathrm{I}}(x) &= x - \int_0^x \frac{F(y)}{F(x)}\,dy \\ &= \frac{1}{\lambda} - \frac{x\exp(-\lambda x)}{1-\exp(-\lambda x)}\end{aligned}$$

As a particular instance, consider the case where $\lambda = 2$ so that $E[X] = \frac{1}{2}$. The equilibrium bidding strategy in this case is depicted in the right-hand panel of Figure 2.2. The figure highlights the fact that with the exponentially distributed values, even a bidder with a very high value—say, \$1 million—will not bid more than 50 cents! This seems counterintuitive at first—the bidder is facing the risk of a big loss by not bidding higher—but is explained by the fact that the probability that the bidder with a high value will lose in equilibrium is infinitesimal. Indeed, for a bidder with a value of \$1 million, it is smaller than $10^{-400000}$. This fact, together with the assumption that bidders are risk neutral, implies that bidders with high values are willing to bid very small amounts. Formally, the fact that no bidder bids more than $\frac{1}{2}$ is a consequence of the property that for all x,

$$\beta^{\mathrm{I}}(x) = E[Y_1 \mid Y_1 < x] \leq E[Y_1]$$

and when there are only two bidders, the latter is the same as $E[X]$. ▲

2.4 REVENUE COMPARISON

Having derived symmetric equilibrium strategies in both the second- and first-price auctions, we can now compare the selling prices—the revenues accruing to the seller—in the two formats.

As before, if there is more than one bidder with the highest bid, the object goes to each such bidder with equal probability.

In a first-price auction, equilibrium behavior is more complicated than in a second-price auction. Clearly, no bidder would bid an amount equal to his or her value, since this would only guarantee a payoff of 0. Fixing the bidding behavior of others, at any bid that will neither win for sure nor lose for sure, the bidder faces a simple trade-off. An increase in the bid will increase the probability of winning while, at the same time reducing the gains from winning. To get some idea about how these effects balance off, we begin with a heuristic derivation of symmetric equilibrium strategies.

Suppose that bidders $j \neq 1$ follow the symmetric, increasing, and differentiable equilibrium strategy $\beta^{\mathrm{I}} \equiv \beta$. Suppose bidder 1 receives a signal, $X_1 = x$, and bids b. We wish to determine the optimal b.

First, notice that it can never be optimal to choose a bid $b > \beta(\omega)$, since in that case, bidder 1 would win for sure and could do better by reducing his bid slightly, so he still wins for sure but pays less. So we need only consider bids $b \leq \beta(\omega)$. Second, a bidder with value 0 would never submit a positive bid, since he would make a loss if he were to win the auction. Thus, we must have $\beta(0) = 0$.

Bidder 1 wins the auction whenever he submits the highest bid—that is, whenever $\max_{i \neq 1} \beta(X_i) < b$. Since β is increasing, $\max_{i \neq 1} \beta(X_i) = \beta(\max_{i \neq 1} X_i) = \beta(Y_1)$, where, as before, $Y_1 \equiv Y_1^{(N-1)}$, the highest of $N-1$ values. Bidder 1 wins whenever $\beta(Y_1) < b$ or equivalently, whenever $Y_1 < \beta^{-1}(b)$. His expected payoff is therefore

$$G\left(\beta^{-1}(b)\right) \times (x - b)$$

where, again, G is the distribution of Y_1. Maximizing this with respect to b yields the first-order condition:

$$\frac{g\left(\beta^{-1}(b)\right)}{\beta'\left(\beta^{-1}(b)\right)}(x-b) - G(\beta^{-1}(b)) = 0 \tag{2.2}$$

where $g = G'$ is the density of Y_1.

At a symmetric equilibrium, $b = \beta(x)$, and thus (2.2) yields the differential equation

$$G(x)\beta'(x) + g(x)\beta(x) = xg(x) \tag{2.3}$$

or equivalently,

$$\frac{d}{dx}(G(x)\beta(x)) = xg(x)$$

and since $\beta(0) = 0$, we have

$$\begin{aligned} \beta(x) &= \frac{1}{G(x)} \int_0^x y g(y)\, dy \\ &= E[Y_1 \mid Y_1 < x] \end{aligned}$$

The derivation of β is only heuristic because (2.3) is merely a necessary condition: We have not formally established that if the other $N-1$ bidders follow β, then it is indeed optimal for a bidder with value x to bid $\beta(x)$. The next proposition verifies that this is indeed correct.

Proposition 2.2. *Symmetric equilibrium strategies in a first-price auction are given by*

$$\beta^{\mathrm{I}}(x) = E[Y_1 \mid Y_1 < x] \tag{2.4}$$

where Y_1 is the highest of $N-1$ independently drawn values.

Proof. Suppose that all but bidder 1 follow the strategy $\beta^{\mathrm{I}} \equiv \beta$ given in (2.4). We will argue that in that case it is optimal for bidder 1 to follow β also. First, notice that β is an increasing and continuous function. Thus, in equilibrium the bidder with the highest value submits the highest bid and wins the auction. It is not optimal for bidder 1 to bid a $b > \beta(\omega)$. The expected payoff of bidder 1 with value x if he bids an amount $b \leq \beta(\omega)$ is calculated as follows. Denote by $z = \beta^{-1}(b)$ the value for which b is the equilibrium bid—that is, $\beta(z) = b$. Then we can write bidder 1's expected payoff from bidding $\beta(z)$ when his value is x as follows:

$$\begin{aligned} \Pi(b,x) &= G(z)[x - \beta(z)] \\ &= G(z)x - G(z)E[Y_1 \mid Y_1 < z] \\ &= G(z)x - \int_0^z y g(y)\, dy \\ &= G(z)x - G(z)z + \int_0^z G(y)\, dy \\ &= G(z)(x - z) + \int_0^z G(y)\, dy \end{aligned}$$

where the fourth equality is obtained as a result of integration by parts. (Alternatively, see formula (A.2) in Appendix A.)

We thus obtain that

$$\Pi(\beta(x),x) - \Pi(\beta(z),x) = G(z)(z - x) - \int_x^z G(y)\, dy \geq 0$$

regardless of whether $z \geq x$ or $z \leq x$.

willing to pay for the object. Each X_i is independently and identically distributed on some interval $[0,\omega]$ according to the increasing distribution function F. It is assumed that F admits a continuous density $f \equiv F'$ and has full support. We allow for the possibility that the support of F is the nonnegative real line $[0,\infty)$ and if that is so, with a slight abuse of notation, write $\omega=\infty$. In any case, it is assumed that $E[X_i]<\infty$.

Bidder i knows the realization x_i of X_i and only that other bidders' values are independently distributed according to F. Bidders are risk neutral; they seek to maximize their expected profits. All components of the model other than the realized values are assumed to be commonly known to all bidders. In particular, the distribution F is common knowledge, as is the number of bidders.

Finally, it is also assumed that bidders are not subject to any liquidity or budget constraints. Each bidder i has sufficient resources so if necessary, he or she can pay the seller up to his or her value x_i. Thus, each bidder is both willing and able to pay up to his or her value.

We emphasize that the distribution of values is the same for all bidders, and we will refer to this situation as one involving *symmetric* bidders.

In this framework, we examine two major auction formats:

I. A first-price sealed-bid auction, where the highest bidder gets the object and pays the amount he bid

II. A second-price sealed-bid auction, where the highest bidder gets the object and pays the second highest bid

Each of these auction formats determines a game among the bidders. A strategy for a bidder is a function $\beta_i:[0,\omega]\to\mathbb{R}_+$, which determines his or her bid for any value. We will typically be interested in comparing the outcomes of a symmetric equilibrium—an equilibrium in which all bidders follow the same strategy—of one auction with a symmetric equilibrium of the other. Given that bidders are symmetric, it is natural to focus attention on symmetric equilibria. We ask the following questions:

What are symmetric equilibrium strategies in a first-price auction (I) and a second-price auction (II)?

From the point of view of the seller, which of the two auction formats yields a higher expected selling price in equilibrium?

2.2 SECOND-PRICE AUCTIONS

Although the first-price auction format is more familiar and even natural, we begin our analysis by considering second-price auctions. The strategic problem confronting bidders in second-price auctions is much simpler than that in first-price auctions, so they constitute a natural starting point. Also recall that in the private values framework, second-price auctions are equivalent to open ascending price (or English) auctions.

In a second-price auction, each bidder submits a sealed bid of b_i, and given these bids, the payoffs are:

$$\Pi_i=\begin{cases} x_i-\max_{j\neq i} b_j & \text{if } b_i>\max_{j\neq i} b_j \\ 0 & \text{if } b_i<\max_{j\neq i} b_j \end{cases}$$

We also assume that if there is a tie, so $b_i=\max_{j\neq i} b_j$, the object goes to each winning bidder with equal probability. Bidding behavior in a second-price auction is straightforward.

Proposition 2.1. *In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to $\beta^{\mathrm{II}}(x)=x$.*

Proof. Consider bidder 1, say, and suppose that $p_1=\max_{j\neq 1} b_j$ is the highest competing bid. By bidding x_1, bidder 1 will win if $x_1>p_1$ and not if $x_1<p_1$ (if $x_1=p_1$, bidder 1 is indifferent between winning and losing). Suppose, however, that he bids an amount $z_1<x_1$. If $x_1>z_1\geq p_1$, then he still wins, and his profit is still x_1-p_1. If $p_1>x_1>z_1$, he still loses. However, if $x_1>p_1>z_1$, then he loses, whereas if he had bid x_1, he would have made a positive profit. Thus, bidding less than x_1 can never increase his profit but in some circumstances may actually decrease it. A similar argument shows that it is not profitable to bid more than x_1. ■

It should be noted that the argument in Proposition 2.1 relied neither on the assumption that bidders' values were independently distributed nor the assumption that they were identically so. Only the assumption of private values is important, and Proposition 2.1 holds as long as this is the case.

With Proposition 2.1 in hand, let us ask how much each bidder expects to pay in equilibrium. Fix a bidder—say, 1—and let the random variable $Y_1\equiv Y_1^{(N-1)}$ denote the highest value among the $N-1$ remaining bidders. In other words, Y_1 is the highest-order statistic of $X_2,X_3,\ldots,X_N$ (see Appendix C). Let G denote the distribution function of Y_1. Clearly, for all y, $G(y)=F(y)^{N-1}$. In a second-price auction, the expected payment by a bidder with value x can be written as

$$\begin{aligned} m^{\mathrm{II}}(x) &= \text{Prob[Win]}\times E[\text{2nd highest bid}\mid x \text{ is the highest bid}] \\ &= \text{Prob[Win]}\times E[\text{2nd highest value}\mid x \text{ is the highest value}] \\ &= G(x)\times E[Y_1\mid Y_1<x] \end{aligned} \tag{2.1}$$

2.3 FIRST-PRICE AUCTIONS

In a first-price auction, each bidder submits a sealed bid of b_i, and given these bids, the payoffs are

$$\Pi_i=\begin{cases} x_i-b_i & \text{if } b_i>\max_{j\neq i} b_j \\ 0 & \text{if } b_i<\max_{j\neq i} b_j \end{cases}$$

Chapter | Two

Private Value Auctions: A First Look

We begin the formal analysis by considering equilibrium bidding behavior in the four common auction forms in an environment with independently and identically distributed private values. In the previous chapter we argued that the open descending price (or Dutch) auction is strategically equivalent to the first-price sealed-bid auction. When values are private, the open ascending price (or English) auction is also equivalent to the second-price sealed-bid auction, albeit in a weaker sense. Thus, for our purposes, it is sufficient to consider the two sealed-bid auctions.

This chapter introduces the basic methodology of auction theory. We postulate an informational environment consisting of (1) a valuation structure for the bidders—in this case, that of private values—and (2) a distribution of information available to the bidders—in this case, it is independently and identically distributed. We consider different auction formats—in this case, first- and second-price sealed-bid auctions. Each auction format now determines a game of incomplete information among the bidders and, keeping the informational environment fixed, we determine a Bayesian-Nash equilibrium for each resulting game. When there are many equilibria, we usually select one on some basis—dominance, perfection, or symmetry—but make sure that the criterion is applied uniformly to all formats. The relative performance of the auction formats on grounds of revenue or efficiency is then evaluated by comparing the equilibrium outcomes in one format versus another.

2.1 THE SYMMETRIC MODEL

There is a single object for sale, and N potential buyers are bidding for the object. Bidder i assigns a value of X_i to the object—the maximum amount a bidder is