Integration

A. Fabretti

Mathematics 2
A.Y. 2015/2016

Table of contents

Indefinite Integral

Suppose to a have a function f and we want to look for a function F such that

$$F'(x) = f(x)$$

we say that we are looking for an *anti-derivative* of f and we call F an indefinite integral of f.

Definition

If $F(x)$ is an antiderivative of f, the most general anti-derivative of f is called the *indefinite integral* and denoted

$$\int f(x)dx = F(x) + c$$

where c is a constant, called the *constant of integration*. Function f is called the *integrand*.

Indefinite Integral: observations

Roughly speaking Integration is the “inverse” process wrt Differentiation.
The solution to the problem of integration is not a one definite function F but a class of function $F(x) + c$ all having the same derivative f

Some Important Integrals (1)

Let $a \neq -1$, since we know that $D(x^n) = nx^{n-1}$, we have this integration formula which follow immediately from derivatives rules

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C$$

If $a = -1$, $x^a = \frac{1}{x}$ and we know that $D(\ln x) = \frac{1}{x}$, hence

$$\int \frac{1}{x} dx = \ln |x| + C$$

we need to insert the absolute value of x, since the function $\frac{1}{x}$ is definite for all $x \neq 0$, while ln is definite only for $x > 0$.

Some Important Integrals(1):Examples

1. $\int x dx = \frac{1}{2}x^2 + C$

2. $\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{1}{-3+1} x^{-3+1} + C = -\frac{1}{2x^2} + C$

3. $\int \sqrt{x} dx = \int x^{1/2} dx = \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + C = \frac{2}{3} x^{\frac{3}{2}} + C$

Some Important Integrals (2)

Consider the exponental function e^x, we know that $D(e^x) = e^x$, which follows immediately

$$\int e^x dx = e^x + C$$

or more generally

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

since we can write $a^x = e^{(\ln a)x}$ with $a > 0$ and $a \neq 1$

$$\int a^x dx = \frac{1}{\ln a} a^x + C$$

Some Immediate Integrals

Consider the chain rule $D[f(g(x))] = f'(g(x))g'(x)$, it follows immediately

$$\int f'(x)f^n(x)dx = \frac{1}{n+1}f^{n+1}(x) + C$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Some Immediate Integrals:Examples

1. $\int \frac{\ln(x)}{x} dx = \frac{1}{2} \ln^2 x + C$

2. $\int 3x^2 e^{x^3} dx = e^{x^3} + C$

3. $\int \frac{5x^4 - 2x}{x^5 - x^2 + 3} dx = \ln |x^5 - x^2 + 3| + C$

Integral Properties

As derivatives, Integral are linear hence

$$\int af(x)dx = a\int f(x)dx \qquad a \quad \text{is a constant}$$

$$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$$

jointing the two properties, we have the more general one

$$\int [a_1 f_1(x) + \cdots + a_n f_n(x)]dx = a_1 \int f_1(x)dx + \cdots + a_n \int f_n(x)dx$$

Examples

1. $\int (3x^4 + 5x^2 + 2)dx =$

2. $\int \left(\frac{3}{x} - 8e^{-4x}\right) dx =$

3. $\int (a + bq + cq^2)dq =$

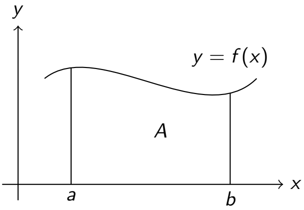
4. In the manufacture of a product, the marginal cost of producing x units is $C'(x)$ and fixed costs are $C(0)$. Find the total cost function C(x) when

$$C'(x) = 3x + 4 \qquad C(0) = 40$$

$$C'(x) = ax + b \qquad C(0) = C_0$$

Area and Definite Integral

How to compute the area A under the graph of a continuous and nonnegative function f over the interval $[a, b]$?



Riemann Integral

Let f be a bounded function in the interval $[a, b]$ and let n be a natural number. Subdivide $[a, b]$ into n parts by choosing points $a = x_0 < x_1 < \cdots < x_n = b$.
We have defined a partition of the interval $[a, b]$, denoted by P.
Put $\Delta x_i = x_{i+1} - x_i$, $i = 0, 1, ..., n-1$.
Denote by m_i the infimum value and by M_i the superior value of f under the interval $[x_i, x_{i+1}]$.
We want to "approximate" the area under the plot of f between a and b using "small" rectangles.

Lower Riemann sum

Denote by m_i the infimum value of f under the interval $[x_i, x_{i+1}]$. We define the lower Riemann sum of f with respect to the partition P by

$$s(f, P) = \sum_{i=0}^{n-1} m_i \Delta x_i$$

$s(f, P)$ is the sum of the signed areas of rectangles that lie below the graph of f.

Upper Riemann sum

Denote by M_i the supremum value of f under the interval $[x_i, x_{i+1}]$. We define the upper Riemann sum of f with respect to the partition P by

$$S(f, P) = \sum_{i=0}^{n-1} M_i \Delta x_i$$

$S(f, P)$ is the sum of the signed areas of rectangles that lie above the graph of f.

Riemann Integral

Denoting with A the area under the graph of f we have

$$m(b-a) \leq s(f,P) \leq A \leq S(f,P) \leq M(b-a)$$

Suppose to take any possible partition P of $[a,b]$ and consider the supremum value of $s(f,P)$ and the infimum value of $S(f,P)$ over all possible partitions and denote them the lower integral $L(f)$ and the upper integral $U(f)$ respectively.

Riemann Integral

Definition

A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ if it is bounded and its upper integral $U(f)$ and lower integral $L(f)$ are equal. In that case, the Riemann integral of f on $[a,b]$, denoted by

$$\int_a^b f(x)dx$$

is the value of $U(f)$ or $L(f)$.

Definite Integral Properties

Given f a continuous function in an interval that contains a, b and c, then

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$
4. $\int_a^b f(x) = \int_a^c f(x)dx + \int_c^b f(x)dx$

The mean value Theorem

Theorem

Given f a continuous function on an interval $[a, b]$, there exists c in (a, b) such that

$$f(c)(b-a) = \int_a^b f(x)dx.$$

The theorem says that if f is a continuous function the area under f between a and b is equal to the area of a rectangle high $f(c)$, for some c in between a and b and base $b-a$. Or in other words that $f(c)$ is the mean "height" of function f between a and b.

The fundamental Theorem of calculus

Theorem

Given f a continuous function on an interval I, we define the integral function

$$G(x) = \int_a^x f(t)dt;$$

the integral function G is differentiable and

$$G'(x) = f(x).$$

Consider a function F such that $F' = f$ hence

$$\int_a^b f(x)dx = F(b) - F(a)$$

Sketch of the Proof of The fundamental Theorem of calculus

(Part I) Given the definition of $G(x)$ show that $G'(x) = f(x)$ using the derivative definition.

(Part II) Since F and G are antiderivatives of f they differentiable and differ by a constant, $F(x) = G(x) + k$, then

$$\begin{aligned} F(b) - F(a) &= G(b) + k - (G(a) + k) = G(b) - G(a) \\ &= \int_a^b f(x)dx - \int_a^a f(x)dx = \int_a^b f(x)dx. \end{aligned}$$

Examples

1. $\int_1^2 2x+1 \; dx = x^2+x\,\big]_1^2 = 4+2-1-1=4$

2. $\int_0^1 \frac{e^x}{e^x+1} \; dx = \ln(e^x+1)\,\big]_0^1 = \ln(e^1+1)-\ln(e^0+1) = \ln\left(\frac{e+1}{2}\right)$

3. $\int_2^{1+e} \frac{t-1}{t} \; dt = \int_2^{1+e} 1-\frac{1}{t} \; dt = t-\ln|t|\,\big]_2^{1+e} = e-1-\ln\left(\frac{e+1}{2}\right)$

4. $\int_1^2 \frac{1+x^2e^{x^2}}{x} dx =$

5. $\int_e^{2e} \frac{\ln(\sqrt{x})}{x} dx =$

Integration by parts

A convenient rule for computing antiderivatives considers the converse of the production rule

$$D[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

which gives

$$\int (f'(x)g(x) + f(x)g'(x))dx = f(x)g(x)$$

hence we get the rule, know as *integration by parts*,

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

which holds both for definite and indefinite integral.

Integration by parts:Examples

Solve $\int xe^{2x}dx$.

We can see e^{2x} as the derivative of $\frac{1}{2}e^{2x}$, hence

$$f'(x) = e^{2x}, \qquad g(x) = x, \qquad f(x) = \frac{1}{2}e^{2x}, \qquad g'(x) = 1$$

applying the rule of integration by parts we get

$$\begin{aligned} \int xe^{2x}dx &= x\ \tfrac{1}{2}e^{2x} - \int 1\ \tfrac{1}{2}e^{2x}dx \\ &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}dx \\ &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C \end{aligned}$$

Integration by substitution

Consider the integral $\int f(x)\ dx$ and the substitution $x = u(t)$, we get

$$\int f(u(t))u'(t)\ dt$$

suppose we get the solution

$$\int f(u(t))u'(t)\ dt = G(t) + C.$$

If $u(t)$ is invertible, we have $t = u^{-1}(x)$ and the original integral is solved

$$\int f(x)\ dx = G(u^{-1}(x)) + C.$$

Integration by substitution: Observations

Integration by substitution is a useful methods to solve integral which can appear as complicated, however when substitution is "smart" this method lead straightforward solutions, because substitution gives a new "shape" to the problem which can be easier to be solved

In the case of a definite integral we can also transform the integration interval or work with the indefinite integral and if $g(t)$ is invertible at the end when we find the solution to the original (indefinite) integral solve the definite one.

Integration by substitution: Example

Example:

$$\int_0^1 \frac{e^{2x}}{e^x+1}\, dx$$

set $1+e^x = t$, $x=\ln(t-1)$ hence (we consider the indefinite integral)

$$\int \frac{e^{2x}}{e^x+1}\, dx = \int \frac{(t-1)^2}{t}\frac{1}{t-1}dt = \int \frac{t-1}{t}\, dt = t - \ln|t|$$

replacing t with $1+e^x$ we get

$$\int_0^1 \frac{e^{2x}}{e^x+1}\, dx = 1+e^x - \ln(1+e^x)\,\Big|_0^1 = -2-\ln(2)$$

Exercises

Solve the following integrals

1. $\int_1^3 \frac{1}{\sqrt{x}+1} \; dx$

2. $\int \ln\left(\frac{x-x^2}{x^3}\right) \; dx$

3. $\int_1^e \sqrt{x}\ln(x) \; dx$

4. $\int_2^3 \sqrt{2x+1} \; dx$

Improper Integral

In statistics and in economics it is common to encounter integrals over an infinite interval.

An improper integral is the limit (if it exists) of a definite integral as an endpoint of the integration interval goes to a value for which the function is not specified or goes to ∞ or $-\infty$. We write

$$\int_a^{\infty} f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx$$

Improper Integral: example

The exponential distribution in statistics is defined by the density function $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and λ a positive constant
We show that $\int_0^\infty f(x)\ dx = 1$. Set $b > 0$ and compute

$$\int_0^b \lambda e^{-\lambda x}\ dx = -e^{-\lambda x}\Big|_0^b = -e^{-\lambda b} + 1$$

the limit of $-e^{-\lambda b} + 1$ as $b \to \infty$ is 1.

Exercises

Prove the following integrals converge and find their values

1. For $c > 0$,
$$\int_{-\infty}^{\infty} xe^{-cx^2}\ dx$$
2. For $a > 1$
$$\int_{1}^{\infty} \frac{1}{x^a}\ dx$$

Consumer and Producer Surplus(1)

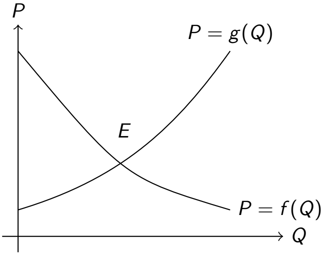
Let $P = f(Q)$ be the demand curve and $P = g(Q)$ the supply curve. The equilibrium price P^* is the one which induces consumers to purchase precisely the same aggregate amount that producers are willing to offer at that price. It holds

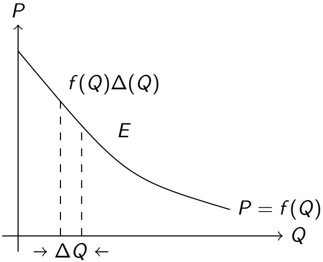
$$g(Q) = P(Q)$$

this equation identifies a point $E = (Q^*, P^*)$

Consumer and Producer Surplus(2)

There are consumers who are willing to pay more than P^* per unit. The total amount saved by all such consumers is called the consumer surplus.





Consumer and Producer Surplus(3)

For those who are willing to buy the commodity at price P^* or higher, the total amount they are willing to pay is the total area below the demand curve over the interval $[0, Q^*]$, that is $\int_0^{Q^*} f(Q)dQ$. If all consumers together buy Q^* units of commodity the total cost is P^*Q^* (which represents the area of the rectangle with base Q^* and height P^*).
The consumer surplus is

$$CS = \int_0^{Q^*} [f(Q) - P^*]dQ$$

which equals the total amount consumers are willing to pay for Q^* minus what the actually pay.

Producer Surplus

Exercise: Are you able to derive the Producer Surplus?

The total revenue the producers actually receive minus what makes them willing to supply Q^*.

$$PS = \int_0^{Q^*} [P^* - g(Q)]dQ.$$

Exercise on CS and PS

Exercise

Suppose the demand curve is $P = f(Q) = 50 - 0.1Q$ and the supply curve is $P = g(Q) = 0.2Q + 20$. Find the equilibrium and compute the consumer and producer surplus.

Hint and solution

Find Q^* and P^* solving $f(Q) = g(Q)$, hence compute CS and PS.

$$Q^* = 100, \quad P^* = 40, \quad CS = 500 \quad PS = 1000.$$

Present value of a flow

Suppose $P(t)$ is the annual rate at which income is flowing at time t. Given r the interest rate. Consider a period between a and b, if we partition the time interval with subintervals $[t_{i-1}, t_i]$, for $i = 1, .., n$ with $t_0 = a$ and $t_n = b$ we know that we can approximate the income in any subinterval as $P(t_i)(t_i - t_{i-1})$. The discount factor will be e^{-rt_i}.
Using integral to get the present value of the flow on the whole period

$$\int_a^b e^{-rt} P(t) dt$$

An example of improper integral in economics

Denoting by $c(t)$ consumption at time t, by U the instantaneous utility function and r a discount rate. An integral

$$\int_{t_0}^{\infty} U(c(t))e^{-rt}dt$$

represents the present value of future cumulative utility coming from consumption.

Exercises

Ex1 Find the area under the graph of the function

$$f(x) = \frac{1}{\sqrt{x}} + \frac{1}{x} + \frac{1}{x^2}$$

with $x \in [1, 4]$.

Ex2 Find the area between the graph of the function and the x axis

$$f(x) = \begin{cases} \frac{x^2+x}{5} & 0 \leq x < \pi \\ sinx & \pi \leq x \leq 2\pi \end{cases}$$