Static Games of Complete Information

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General Information (1)

- **Schedule**

  Mo (14:00 - 16:00), Tu (14:00 - 16:00), We (14:00 - 16:00).

- **Office Hours**: by appointment.

- **References**:

  1. Gibbons, R. (1992), Game theory for applied economists, Princeton University Press.
  2. Lecture slides.

- **Topics covered**:

  - Prof. Ferrari: Static and Dynamic Games of Complete Information;
  - Prof. Pommey: Repeated Games, Static and Dynamic Games of Incomplete Information.

General Information (2)

- **Practice Sessions**: Thursday 9:00-12:00 with Professor Bozzoli.
- **Exam**:
  - Written test (one open question/exercise for each Professor);
  - Grade is averaged (50%) with the one in **Industrial Organisation**. You need grade ≥ 18 in both modules to pass the exam.
  - Midterm after the end of Game Theory module. If passed (grade ≥ 18) allows to take I.O. part only in the pre-exam or first call of Winter Session.
  - Midterm grade expires if exam is either not passed or the grade is not accepted.
  - It is not possible to re-seat the exam in the same session.

Outline

- Introduction
- Normal-Form
- IESDS
- NE
- Cournot
- Bertrand
- TotC

What is Game Theory? (1)

What do these **seemingly opposite** situations have in common?

- Students **successfully cooperate** on a school project.
- Two firms (e.g. Coke and Pepsi), **compete to increase sales**.

What is Game Theory? (2)

1. The group project's **grade** depends on **all members' effort**.
2. Firm's **sales** depend, for example, on the competitor's **marketing strategy**.

In both examples **interdependence** is essential!

- **One person's behaviour** affects **another person's well-being**, either **positively** or **negatively**.
- Situations of interdependence are called **strategic settings** because, in order to decide how best to behave, a person must consider how others around her choose their actions.

What is Game Theory? (3)

- Game theory is the study of **multiperson decision problem** and provides a **methodology** of formally describing **strategic interaction environments**:

  1. Involves two or more **rational players**.
  2. Each player's utility **depends on the decision of the others**.

Remember that **rational player** acts as to **maximise her own utility/satisfaction/payoff**.

Types of Games

- **Four main categories** of games:

    1. Static Games of Complete Info → Nash Equilibrium
    2. Dynamic Games of Complete Info → Subgame-perfect Nash Equilibrium
    3. Static Games of Incomplete Info → Bayesian Nash Equilibrium
    4. Dynamic Games of Incomplete Info → Perfect Bayesian Nash Equilibrium

- Game Theory helps us find the **solution** of a game, i.e. **what we expect** rational players will choose in a game. Different game forms require (partially) **different solution concepts** (equilibrium).

Applications

- Many applications in fields of economics:

1. **Industrial Organisation** (the study of firms' interactions in different market structures).
2. **Labour** (e.g. wages, compensations) and **Financial Economics** (e.g. financial crises).
3. **International Economics** (e.g. negotiations on tariffs and trade barriers).

Assumptions

We begin with games that have the following characteristics:

- **Static**: players choose *simultaneously* what to play (alternatively, each of them does not know the others' choice).
- **Complete Info**: each player's *payoff function* (depending on all players' decisions) is *common knowledge*.

Example: Matching Pennies:

- Two players choose simultaneously **Head** or **Tail** on a penny.
- If pennies **match** (both Head or both Tail), Player 2 gives her penny to Player 1. If pennies **do not match**, Player 1 gives her penny to Player 2.

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- It is convenient to describe simultaneous games in *normal form*. This requires to specify the:
  1. **Players** of the game
  $$N = \{1, 2, ..., i-1, i, i+1, ..., n\}.$$
  2. **Strategies** available to each player (in her set of strategies)
  $$s_i \in S_i.$$
  3. **Payoffs** received by each player for each combination of strategies available to players
  $$u_i = u_i(s_1, ..., s_{i-1}, s_i, s_{i+1}, ..., s_n) = u_i(s_{-i}, s_i).$$

Definition: *The normal-form representation of an n-player game specifies the players, the players' strategy spaces* $S_1, ..., S_n$ *and their payoff functions* $u_1, ..., u_n$. *We denote this game* $G = \{1, 2, ..., N, S_1, ..., S_n, u_1, ..., u_n\}$.

Matching Pennies

- Two players simultaneously choose **Head** or **Tail** on a penny.
- If pennies match (do not match), player 2 (1) gives the penny to 1 (2).

		Player 2	
		H	T
Player 1	H	$(1, -1)$	$(-1, 1)$
	T	$(-1, 1)$	$(1, -1)$

- $N = \{Player\ 1, Player\ 2\}$.
- $S_1 = S_2 = \{H, T\}$.
- $u_1(H, H) = 1, u_1(H, T) = -1, u_1(T, H) = -1, u_1(T, T) = 1$.
- $u_2(H, H) = -1, u_2(H, T) = 1, u_2(T, H) = 1, u_2(T, T) = -1$.

Meeting in Rome

- Marco and Giulia are supposed to meet in Rome, but they do not remember whether in *Termini station* or *Tiburtina station*. They cannot communicate (e.g. broken phone).
- If they meet, they get \$100 each. Otherwise, they get nothing.

Giulia

Marco		Termini	Tiburtina
	Termini	$(100, 100)$	$(0, 0)$
	Tiburtina	$(0, 0)$	$(100, 100)$

- $N = \{Marco, Giulia\}$.
- $S_M = S_G = \{Te, Ti\}$.
- $u_M(Te, Te) = 100, u_M(Te, Ti) = 0, u_M(Ti, Te) = 0, u_M(Ti, Ti) = 100.$
- $u_G(Te, Te) = 100, u_G(Te, Ti) = 0, u_G(Ti, Te) = 0, u_G(Ti, Ti) = 100.$

The Prisoners' Dilemma

- Two friends commit an armed robbery. Police finds the guns in their home and arrests them but does not have full proof of the crime. They are put in separate cells and cannot communicate.
- They choose simultaneously whether to *Confess* (C) or *Not Confess* (NC). If one confesses and the other does not, the first is free and the other is convicted to 9 months (-9). If neither confesses, they are both convicted to 1 month (-1). If both confess, they are convicted to 6 months (-6).

		Prisoner 2	
		Not Confess	Confess
Prisoner 1	Not Confess	$(-1,-1)$	$(-9,0)$
	Confess	$(0,-9)$	$(-6,-6)$

- $N = \{Prisoner\ 1, Prisoner\ 2\}$.
- $S_1 = S_2 = \{C, NC\}$.
- $u_1(C,C) = -6, u_1(C,NC) = 0, u_1(NC,C) = -9, u_1(NC,NC) = -1$.
- $u_2(C,C) = -6, u_2(C,NC) = -9, u_2(NC,C) = 0, u_2(NC,NC) = -1$.

The Battle of Sexes

- *Carl* and *Pat* must choose simultaneously whether to spend the evening at the opera or at a boxe fight. They cannot communicate, they simply decide independently where to go.
- They love being together, so they get a positive payoff only if they end up in the same place. However, *Carl* likes opera more than the fight, and *Pat* likes the fight more than opera.

Carl \ Pat	Opera	Fight
Opera	$(2,1)$	$(0,0)$
Fight	$(0,0)$	$(1,2)$

- $N = \{Carl, Pat\}$.
- $S_C = S_P = \{Opera, Fight\}$.
- $u_C(O,O) = 2, u_C(O,F) = 0, u_C(F,O) = 0, u_C(F,F) = 1$.
- $u_P(O,O) = 1, u_P(O,F) = 0, u_P(F,O) = 0, u_P(F,F) = 2$.

Chicken Game

- Two drivers drive towards each other on a collision course: one must swerve, or both may die, but if one driver swerves and the other does not, the one who swerved will be called a "chicken", i.e. a coward.
- If both drivers swerve, each gets 0. If one driver swerves and the other does not, the one who swerves (the "chicken") gets -1 and the other gets +1. If both go straight, each gets -100 (i.e. they die in the crash).

		Driver 2	
		Swerve	Straight
Driver 1	Swerve	$(0, 0)$	$(-1, 1)$
	Straight	$(1, -1)$	$(-100, -100)$

- $N = \{Driver\ 1, Driver\ 2\}$.
- $S_1 = S_2 = \{Sw, St\}$.
- $u_1(Sw, Sw) = 0, u_1(Sw, St) = -1, u_1(St, Sw) = 1, u_1(St, St) = -100.$
- $u_2(Sw, Sw) = 0, u_2(Sw, St) = 1, u_2(St, Sw) = -1, u_2(St, St) = -100.$

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Strictly Dominant and Dominated Strategies (1)

Strictly Dominant Strategy: A strategy that is better than any other a player might choose, **no matter what** strategy the other player follows.

- In other words, this strategy achieves a higher payoff **irrespective** of the strategy chosen by the other players.

Strictly Dominated Strategy: A strategy such that the player has another strategy that gives a higher payoff **no matter what** the other player does.

- In other words, this strategy achieves a lower payoff **irrespective** of the strategy chosen by the other players.

Strictly Dominant and Dominated Strategies (2)

Definition:

In the normal-form game $G = \{S_1, ..., S_n, u_1, ..., u_n\}$ *let* $s_i^{'}$ *and* $s_i^{''}$ *feasible strategies for player i* ($s_i^{'}, s_i^{''} \in S_i$)*. Strategy* $s_i^{'}$ *is* **strictly dominated** *by strategy* $s_i^{''}$ *if* **for each feasible combination** *of the other players' strategies, i's payoff from playing* $s_i^{'}$ *is* **strictly less** *than i's payoff from playing* $s_i^{''}$*:*

$$u_i(s_1, ..., s_{i-1}, s_i^{'}, s_{i+1}, ..., s_n) < u_i(s_1, ..., s_{i-1}, s_i^{''}, s_{i+1}, ..., s_n)$$

for each $(s_1, ..., s_{i-1}, s_{i+1}, ..., s_n)$ that can be constructed from the other players' strategy spaces $S_1, ..., S_{i-1}, S_{i+1}, ..., S_n$ $(\forall s_j \in S_j, j \neq i)$.

Iterated Elimination of Strictly Dominated Strategies (1)

- If it is **common knowledge** that all players are **rational**, it is possible to **cancel out strictly dominated strategies** in an **iterated fashion** as they will never be played by rational agents.

- Iterated elimination of strictly dominated strategies (IESDS) is the first (and least precise) solution concept we implement.
- For each player, we compare strategies 2 by 2 and check whether one is always better than the other.

Iterated Elimination of Strictly Dominated Strategies (2)

Example 1

Player 2

Player 1		*Left*	*Middle*	*Right*
	Up	$(1, 0)$	$(1, 2)$	$(0, 1)$
	Down	$(0, 3)$	$(0, 1)$	$(2, 0)$

Player 2

Player 1		*Left*	*Middle*	*Right*
	Up	$(1, 0)$	$(1, 2)$	$(0, 1)$
	Down	$(0, 3)$	$(0, 1)$	$(2, 0)$

- Middle gives player 2 a *higher payoff* **irrespective** of what Player 1 chooses ($2 > 1$ and $1 > 0$).
- Since Player 2 is rational, he will **never play** *Right*. The latter is **strictly dominated** by *Middle*.

Iterated Elimination of Strictly Dominated Strategies (3)

Player 2

Player 1	Left	Middle
Up	(1,0)	(1,2)
Down	(0,3)	(0,1)

Player 2

Player 1	Left	Middle
Up	(1, 0)	(1,2)

- Player 1 knows that Player 2 will never play *right*.
- *Up* gives player 1 a *higher payoff* **irrespective** of what Player 2 chooses ($1 > 0$ and $1 > 0$).
- Since Player 1 is rational, he will **never play** *Down*. The latter is **strictly dominated** by *Up*.
- The set of strategies surviving *IESDS* is (*Up*, *Middle*).

Iterated Elimination of Strictly Dominated Strategies (4)

		Player 2		
		Left	*Center*	*Right*
Player 1	*Top*	$(8, 3)$	$(0, 4)$	$(4, 4)$
	Middle	$(4, 2)$	$(1, 5)$	$(5, 3)$
	Bottom	$(3, 7)$	$(0, 1)$	$(2, 0)$

- Middle gives player 1 a *higher payoff* than Bottom **irrespective** of what Player 1 chooses ($4 > 3$, $1 > 0$, and $5 > 2$).
- Since Player 1 is rational, she will **never play** *Bottom*. The latter is **strictly dominated** by *Middle*.

Iterated Elimination of Strictly Dominated Strategies (5)

		Player 2		
		Left	*Centre*	*Right*
Player 1	*Top*	$(8, 3)$	$(0, 4)$	$(4, 4)$
	Middle	$(4, 2)$	$(1, 5)$	$(5, 3)$

- Right gives player 2 a *higher payoff* **irrespective** of what Player 1 chooses ($4 > 3$ and $3 > 2$).
- Since Player 2 is rational, she will **never play** *Left*. The latter is **strictly dominated** by *Right*.

Iterated Elimination of Strictly Dominated Strategies (6)

Player 2

Player 1		*Centre*	*Right*
	Top	$(0, 4)$	$(4, 4)$
	Middle	$(1, 5)$	$(5, 3)$

Player 2

Player 1		*Centre*	*Right*
	Middle	$(1, 5)$	$(5, 3)$

- The set of strategies surviving *IESDS* is $(Middle, Left)$.

Drawbacks (1)

- *IESDS* requires to assume that **it is common knowledge** that players are rational (not only all players are rational, but all players know that all players are rational, and all players know that all players know that all players are rational, *ad infinitum*).
- It is not always a **good solution concept**, as it can lead to very **imprecise predictions**.
- This is not desirable as our task is to predict the outcome of the game **as precisely as possible**.

Drawbacks (2)

		Player 2		
		Left	*Center*	*Right*
Player 1	*Top*	$(0, 4)$	$(4, 0)$	$(5, 3)$
	Middle	$(4, 0)$	$(0, 4)$	$(5, 3)$
	Bottom	$(3, 5)$	$(3, 5)$	$(6, 6)$

- There is **no strictly dominated strategy**!
- We resort to a stronger concept, the one of **Nash Equilibrium**.

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Motivation

- Suppose that game theory makes a **unique prediction** about the strategy each player will choose in a game.

- This prediction is correct only if each player is **willing to choose** that strategy.

Nash Equilibrium

Definition (*pure-strategy NE*):

In the n-player normal-form game $G = \{S_1, ..., S_n, u_1, ..., u_n\}$ *the strategies* $(s_1^*, ..., s_n^*)$ *are a Nash Equilibrium if, for each player* i, s_i^* *is (at least tied for) player* i*'s best response to the strategies specified for the* $n-1$ *other players,* $(s_1^*, ..., s_{i-1}^*, s_{i+1}^*, ..., s_n^*)$*:*

$$u_i(s_1^*, ..., s_{i-1}^*, s_i^*, s_{i+1}^*, ..., s_n^*) \geq u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*)$$

for every $s_i \in S_i$. In other words, s_i^* solves

$$\max_{s_i \in S_i} u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*)$$

Notice: Given the other players' strategies!

Conditions for Nash Equilibrium

Two conditions characterise the concept of *NE*:

1. Each player must be playing the **best response** to the other players' strategies.
2. There is **no profitable deviation** from the strategies chosen in equilibrium. If $s_1^{'}, ..., s_n^{'}$ is not a *NE*, there exists a strategy $s_i^{''}$ such that

$$u_i(s_1^{'}, ..., s_{i-1}^{'}, s_i^{'}, s_{i+1}^{'}, ..., s_n^{'}) \leq u_i(s_1^{'}, ..., s_{i-1}^{'}, s_i^{''}, s_{i+1}^{'}, ..., s_n^{'}).$$

The prediction achieved using this concept is **strategically stable** (also said **self-enforcing**).

NE in Example 1 (1)

		Player 2		
		Left	*Center*	*Right*
Player 1	*Top*	$(0, \underline{4})$	$(\underline{4}, 0)$	$(5, 3)$
	Middle	$(\underline{4}, 0)$	$(0, \underline{4})$	$(5, 3)$
	Bottom	$(3, 5)$	$(3, 5)$	$(\underline{6}, \underline{6})$

Player 1's best response to:

- Left is Middle $(4 > 3 > 0)$.
- Center is Top $(4 > 3 > 0)$.
- Right is Bottom $(6 > 5 = 5)$.

Player 2's best response to:

- Top is Left $(4 > 3 > 0)$.
- Middle is Center $(4 > 3 > 0)$.
- Bottom is Right $(6 > 5 = 5)$.

NE in Example 1 (2)

		Player 2		
		Left	*Center*	*Right*
Player 1	*Top*	$(0, 4)$	$(4, 0)$	$(5, 3)$
	Middle	$(4, 0)$	$(0, 4)$	$(5, 3)$
	Bottom	$(3, 5)$	$(3, 5)$	$(6, 6)$

The strategies $(Bottom, Right)$ are the only *pure-strategy NE* of this game:

- Both players choose their *best response* to the other player's strategy.
- There is no strict incentive to unilaterally deviate from the equilibrium:
  - If Player 1 deviates when Player 2 plays Right, she gets $5 < 6$.
  - If Player 2 deviates when Player 1 play Bottom, she gets $5 < 6$.

NE in Prisoners' Dilemma (1)

		Prisoner 2	
		Not Confess	Confess
Prisoner 1	Not Confess	$(-1, -1)$	$(-9, \underline{0})$
	Confess	$(\underline{0}, -9)$	$(\underline{-6}, \underline{-6})$

Prisoner 1's best response to:

- C is C $(-6 > -9)$.
- NC is C $(0 > -1)$.

Player 2's best response to:

- C is C $(-6 > -9)$.
- NC is C $(0 > -1)$.

NE in Prisoners' Dilemma (2)

		Prisoner 2	
		Not Confess	Confess
Prisoner 1	Not Confess	$(-1, -1)$	$(-9, \underline{0})$
	Confess	$(\underline{0}, -9)$	$(\underline{-6}, \underline{-6})$

The strategies (C, C) are the only *pure-strategy NE* of this game:

- Both players choose their *best response* to the other player's strategy.
- There is no strict incentive to unilaterally deviate from the equilibrium:
  - If Prisoner 1 deviates when Prisoner 2 plays C, she gets $-9 < -6$.
  - If Prisoner 2 deviates when Prisoner 1 plays C, she gets $-9 < -6$.

Pareto-Efficient Strategy Profiles (1)

- There is a major tension in strategic interactions, i.e. a clash between **individual** and **group interests**.

  Definition: Strategy profile s is more (Pareto) efficient than strategy profile s' if all of the players prefer the outcome of s to the outcome of s' and if the preference is strict for at least one player. Formally:

$$u_1(s_1, s_2) \geq u_1(s'_1, s'_2) \text{ and } u_2(s_1, s_2) \geq u_2(s'_1, s'_2), \text{ and}$$

either $u_1(s_1, s_2) > u_1(s'_1, s'_2)$ or $u_2(s_1, s_2) > u_2(s'_1, s'_2)$ (or both).

Pareto-Efficient Strategy Profiles (2)

Definition: Strategy profile s is (Pareto) efficient if there is no other strategy s' that is more efficient, i.e. there is no other strategy profile s' such that:

$$u_1(s_1', s_2') \geq u_1(s_1, s_2) \text{ and } u_2(s_1', s_2') \geq u_2(s_1, s_2), \text{ and}$$

$$\text{either } u_1(s_1', s_2') > u_1(s_1, s_2) \text{ or } u_2(s_1', s_2') > u_2(s_1, s_2) \text{ (or both).}$$

In the Prisoners' Dilemma, (NC, NC) is more efficient than (C, C) and it is also efficient.

Notice that the concept of efficiency is different from the one of dominance.

NE in the Battle of Sexes

		Pat	
		Opera	Fight
Carl	Opera	$(\underline{2}, \underline{1})$	$(0, 0)$
	Fight	$(0, 0)$	$(\underline{1}, \underline{2})$

Carl's best response to:

- O is 0 $(2 > 0)$.
- F is F $(1 > 0)$.

Pat's best response to:

- O is 0 $(1 > 0)$.
- F is F $(2 > 0)$.

NE in the Battle of Sexes

		Pat	
		Opera	Fight
Carl	Opera	$(\underline{2}, \underline{1})$	$(0, 0)$
	Fight	$(0, 0)$	$(\underline{1}, \underline{2})$

The strategies (O, O) and (F, F) are both *pure-strategy NE* of this game:

- Both players choose their *best response* to the other player's strategy.
- There is no strict incentive to unilaterally deviate from the equilibrium:
  - If Carl deviates when Pat plays O (F), she gets $0 < 2$ ($0 < 1$).
  - If Pat deviates when Carl plays O (F), she gets $0 < 1$ ($0 < 2$).

NE in Chicken Game (1)

		Driver 2	
		Swerve	Straight
Driver 1	Swerve	$(0, 0)$	$(\underline{-1}, \underline{1})$
	Straight	$(\underline{1}, \underline{-1})$	$(-100, -100)$

Driver 1's best response to:

- Sw is St $(0 > -1)$.
- St is Sw $(-1 > -100)$.

Driver 2's best response to:

- Sw is St $(0 > -1)$.
- St is Sw $(-1 > -100)$.

NE in Chicken Game (2)

		Driver 2	
		Swerve	Straight
Driver 1	Swerve	$(0, 0)$	$(\underline{-1}, \underline{1})$
	Straight	$(\underline{1}, \underline{-1})$	$(-100, -100)$

The strategies (Sw, St) and (St, Sw) are both *pure-strategy NE* of this game:

- Both players choose their *best response* to the other player's strategy.
- There is no strict incentive to unilaterally deviate from the equilibrium:
    - If Driver 1 deviates when Driver 2 plays Sw (St), she gets $0 < 1$ $(-100 < -1)$.
    - If Driver 2 deviates when Driver 1 plays Sw (St), she gets $0 < 1$ $(-100 < -1)$.

NE in Matching Pennies?

		Player 2	
		H	T
Player 1	H	$(\underline{1}, -1)$	$(-1, \underline{1})$
	T	$(-1, \underline{1})$	$(\underline{1}, -1)$

Player 1's best response to:

- H is H ($1 > -1$).
- T is T ($1 > -1$).

Player 2's best response to:

- H is T ($1 > -1$).
- T is H ($1 > -1$).

There is **no Nash Equilibrium** in pure strategies!

Relation between NE and IESDS

a. If *IESDS* eliminates **all but the strategies** $(s_1^*, \dots, s_n^*)$, then these strategies are the only *NE* of the game;

b. If the strategies $(s_1^*, \dots, s_n^*)$ are a *NE*, then they **survive IESDS**. On the other hand, there may be strategies that survive *IESDS* that are not part of any *NE*.

- *NE* is a stronger solution concept than *IESDS*.
- John Nash was able to show that there exists **at least one NE** in any game, possibly involving **mixed strategies**, that we present in the next slides.

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Assumptions

- Players: two firms

$$N = \{Firm\ i, Firm\ j\}.$$

- Strategies: quantity of a *homogeneous* good to be produced (infinite is not included in the production interval)

$$S_i = S_j = [0, \infty)\ or$$
$$q_i = q_j = [0, \infty).$$

- Inverse market demand is $p(Q) = a - Q$.
- Payoffs: firm's profits

$$\pi_i = [a - Q - c]q_i,$$

where $Q = q_i + q_j$, c is constant *marginal cost* of production, and a is a parameter.

Maximization Problem

- Each firm chooses the quantity that maximises its profits **given the quantity chosen by the other firm** (the other firm's **best-response**):

$$\max_{s_i} u_i(s_i, s_j^*) = \max_{0 \leq q_i < \infty} \pi_i(q_i, q_j^*),$$

$$\max_{s_j} u_j(s_i^*, s_j) = \max_{0 \leq q_j < \infty} \pi_j(q_i^*, q_j).$$

- Replace the equation of profits

$$\max_{q_i} [a - (q_i + q_j^*) - c] q_i,$$

$$\max_{q_j} [a - (q_i^* + q_j) - c] q_j.$$

Best Responses

- Maximisation is given by **first-order conditions**, obtained by taking the partial derivative of each firm's profit with respect to quantity produced and equating it to zero

$$\frac{\delta \pi_i(q_i, q_j^*)}{\delta q_i} = a - 2q_i - q_j^* - c = 0,$$
$$\frac{\delta \pi_j(q_i^*, q_j)}{\delta q_j} = a - 2q_j - q_i^* - c = 0.$$

- Rearranging yields the two firms' best responses to each other's optimum quantity

$$\begin{cases} q_i^* = \dfrac{a - q_j^* - c}{2} \\ q_j^* = \dfrac{a - q_i^* - c}{2} \end{cases}$$

Nash Equilibrium

- This is a system of 2 equations in 2 unknowns. *Pure-strategy NE* is obtained by replacing the second equation into the first

$$
\begin{aligned}
q_i^* &= \frac{1}{2}\left[a - \frac{1}{2}\left(a - q_i^* - c\right) - c\right], \\
4q_i^* &= 2a - a + q_i^* + c - 2c, \\
3q_i^* &= a - c, \\
q_i^* &= \frac{a-c}{3} = q_j^*.
\end{aligned}
$$

- Notice that we need $a > c$ in order for the equilibrium quantity to be positive.

Equilibrium Price

- To find the equilibrium price, replace the NE quantity in demand:

$$p(q_i^*, q_j^*) = a - 2\frac{a-c}{3} = \frac{3a-2a+2c}{3} = \frac{a+2c}{3}.$$

Equilibrium Profits

- To find equilibrium payoffs (profits), we plug equilibrium quantities in the firms' profit function

$$\pi_i^c(q_i^*, q_j^*) = \pi_j^c(q_i^*, q_j^*) = \left[a - \left(\frac{a-c}{3} + \frac{a-c}{3}\right) - c\right]\frac{a-c}{3} =$$

$$= \left[\frac{3a - 3c - 2a + 2c}{3}\right]\frac{a-c}{3} = \left(\frac{a-c}{3}\right)\left(\frac{a-c}{3}\right) = p^c q_i^* =$$

$$= \frac{(a-c)^2}{9}.$$

Monopoly vs Cournot (1)

- Recall that monopoly quantity and profits are given by

$$\max_q \pi(q) = [a - q - c]q.$$

$$\frac{\delta \pi^m(q)}{\delta q} = a - 2q - c = 0,$$

$$q^m = \frac{a-c}{2} \ \textit{and} \quad p_m = a - \frac{a-c}{2} = \frac{a+c}{2}.$$

$$\pi^m(q^m) = \frac{(a-c)^2}{4}.$$

- If firms could split the market *equally* the could produce half the monopoly quantity $\frac{a-c}{4}$ and get half the monopoly profit. This would be **larger than the Cournot profit**.

Monopoly vs Cournot (2)

$$\frac{(a-c)^2}{8} > \frac{(a-c)^2}{9}.$$

- However, notice that half the monopoly quantity **is not the best response** of one firm when the other produces half the monopoly quantity : $\frac{q^m}{2} = \frac{a-c}{4}$ (q_i^D stands for deviation quantity)

$$q_i^D = \frac{1}{2}\left[a - \frac{a-c}{4} - c\right]$$

$$8q_i^D = [4a - a + c - 4c]$$

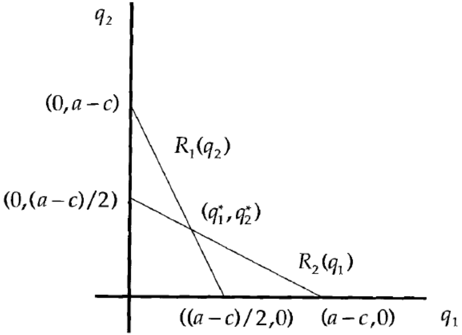
$$q_i^D = \frac{3(a-c)}{8} > \frac{(a-c)}{4}.$$

Monopoly vs Cournot (3)

- Notice that this is larger than half the monopoly quantity. In other words, firms have an **incentive to produce more** as this **increases their profits**.
- Define π^D firm $i's$ deviation profit

$$
\begin{aligned}
\pi_i^D &= \left[a - \frac{3(a-c)}{8} - \frac{(a-c)}{4} - c\right]\frac{3(a-c)}{8} = \\
&= \left[\frac{8a - 3a + 3c - 2a + 2c - 8c}{8}\right]\frac{3(a-c)}{8} = \\
&= \left[\frac{3a-3c}{8}\right]\frac{3(a-c)}{8} = \\
&= \frac{3(a-c)}{8}\frac{3(a-c)}{8} = \\
&= \frac{9}{64}(a-c)^2 > \frac{(a-c)^2}{8} = \frac{\pi^M}{2}.
\end{aligned}
$$

Graphical Interpretation Using Reaction Functions



$$\begin{cases} R_i(q_j) = \dfrac{a - q_j - c}{2} \\ R_j(q_i) = \dfrac{a - q_i - c}{2} \end{cases}$$

Quantities are **strategic substitutes**, i.e. $\frac{\Delta R_i}{\Delta q_j} < 0$ and $\frac{\Delta R_j}{\Delta q_i} < 0$.

Remarks and Extensions

Remark:

- The game can be solved also using *IESDS* (not shown).

Extensions:

- Effect of including **fixed costs** F (i.e. costs independent from quantity) on *NE* equilibrium prices, quantities and profits?
- Effect of the two firms facing **different marginal costs** on *NE* equilibrium prices, quantities and profits?
- Effect of changing the **slope** b of the demand function?

Cournot with Fixed Costs

- In this case profits are reduced by a **fixed cost** F:

$$\pi_i = [a - Q - c]q_i - F,$$
$$\pi_j = [a - Q - c]q_j - F.$$

- Each firm maximises profits **given the quantity chosen by the other firm** (the other firm's **best-response**)

$$\max_{q_i}[a - (q_i + q_j^*) - c]q_i - F,$$
$$\max_{q_j}[a - (q_i^* + q_j) - c]q_j - F.$$

- When taking the first derivative, however, notice that F **cancels out** (it is a constant). F **does not affect** NE quantities. However, it **reduces** NE profits.

Cournot with Asymmetric Costs (1)

- In this case we assume that firm i is **more efficient** than j:

$$c_j > c_i.$$

- To solve the game, sufficient to follow all the steps provided for the general case. Not possible, however, to use the **general formula** to compute NE quantities (actually a formula exists, as shown in the next slides).

- The **more efficient firm** will produce a **higher quantity** and make **larger profits**.

Cournot with Asymmetric Costs (2)

- In this case the first-order conditions become:

$$\frac{\delta\pi_i(q_i, q_j^*)}{\delta q_i} = a - 2q_i - q_j^* - c_i = 0,$$

$$\frac{\delta\pi_j(q_i^*, q_j)}{\delta q_j} = a - 2q_j - q_i^* - c_j = 0.$$

- Rearranging yields the two firms' best responses to each other's optimum quantity

$$\begin{cases} q_i^* = \dfrac{a - q_j^* - c_i}{2} \\ q_j^* = \dfrac{a - q_i^* - c_j}{2} \end{cases}$$

- Equilibrium quantities are **left as an exercise**.

Cournot with n Firms (1)

Suppose now there are n firms in the market. We define:

- q_i: quantity produced by Firm i;
- $q_{-i} = \sum_{j \neq i} q_j$: quantity produced by all other Firms.
- The maximisation problem for Firm i is (all firms face marginal cost c):

$$\max_{q_i} [a - (q_i + q_{-i}^*) - c] q_i,$$

- In this case the first-order conditions are:

$$\frac{\delta \pi_i (q_i, q_{-i}^*)}{\delta q_i} = a - 2q_i - q_{-i}^* - c_i = 0,$$

- Rearranging yields Firm i's best response to the other Firms' quantity:

$$q_i^* = \frac{a - q_{-i}^* - c}{2}.$$

Cournot with n Firms (2)

Notice that this is a **symmetric problem**, i.e.:

$$q_1^* = q_2^* = ... = q_n^* = q^*.$$

This implies:

$$q_{-i}^* = (n-1)q^*.$$

Replace in the reaction function to find the equilibrium quantity:

$$q^* = \frac{a-(n-1)q^*-c}{2} \implies 2q^* = a-(n-1)q^*-c \implies$$

$$2q^* = a - nq^* + q^* - c \implies (n+1)q^* = a-c \implies$$

$$q^* = \frac{a-c}{n+1}.$$

Cournot with n Firms (3)

Replace in demand to find **equilibrium price**:

$$p^* = a - n\left(\frac{a-c}{n+1}\right) = \frac{an + a - an + cn}{n+1} = \frac{a+cn}{n+1}.$$

Finally, **equilibrium profits** are given by:

$$\pi_i = \left(\frac{a+cn}{n+1} - c\right)\frac{a-c}{n+1} = \left(\frac{a+cn-cn-c}{n+1}\right)\frac{a-c}{n+1} = \left(\frac{a-c}{n+1}\right)^2.$$

- Exercise: try to solve the model for $n = 1$ and $n = 2$. Does it remind you of anything we have already seen?

Outline

Assumptions

- Players: two firms

$$N = \{Firm\ i, Firm\ j\}.$$

- Strategies: price of a *homogeneous* good to be produced (infinite is not included in the production interval):

$$S_i = S_j = [0, \infty)\ or$$
$$p_i = p_j = [0, \infty).$$

- Each firm's demand is (Q is market demand):

$$q_i = \begin{cases} 0 & if \quad p_i > p_j \\ \frac{1}{2}Q & if \quad p_i = p_j \\ Q & if \quad p_i < p_j \end{cases}$$

- Payoffs: firm's profits (same marginal cost).

The Bertrand Paradox

The duopolists will set the **same price**, and this will be equal to their **marginal cost**. As a consequence, both firms will make **zero profits**. Let us see why (take ϵ very small number):

- $p_i > p_j > c$: Firm i sells nothing, it could serve all the market by setting $p_i = p_j - \epsilon$. **NO NE**.
- $p_i = p_j > c$: Firm i sells half the market quantity and makes a positive profit. However, it could set $p_i = p_j - \epsilon$ and serve the whole market **NO NE**.
- $c < p_i < p_j$: Firm i serves all the market, but it could increase its profits by increasing p_i to $p_i = p_j - \epsilon$ **NO NE**.
- $p_i < c$: The firm makes negative profits. **NO NE**.

The two firms will **undercut** each other up to the point in which

$$p_i = p_j = c.$$

Differentiated Bertrand

- The way to solve the *paradox* is to assume that the two firms produce **differentiated goods**.
- Each firm's demand is

$$q_i(p_i, p_j) = a - p_i + bp_j.$$

- $b > 0$ is a parameter that captures the **substitutability** between the two goods. When $\Delta p_j > 0$, $q_i > 0$. The effect increases as b becomes larger. If $b = 0$ no effect.
- Firms' profits are

$$\pi_i(p_i, p_j) = [a - p_i + bp_j][p_i - c],$$
$$\pi_j(p_i, p_j) = [a - p_j + bp_i][p_j - c].$$

Maximisation Problem

- Each firm chooses the price that maximises its profits **given the price chosen by the other firm** (the other firm's **best-response**):

$$\max_{s_i} u_i(s_i, s_j^*) = \max_{0 \leq p_i < \infty} \pi_i(p_i, p_j^*),$$

$$\max_{s_j} u_j(s_i^*, s_j) = \max_{0 \leq p_j < \infty} \pi_j(p_i^*, p_j).$$

- Replace the equation of profits:

$$\max_{p_i} [a - p_i + bp_j^*][p_i - c],$$

$$\max_{p_j} [a - p_j + bp_i^*][p_j - c].$$

Best Responses

- Maximisation is given by **first-order conditions**, obtained by taking the partial derivative of each firm's profit with respect to price and equating it to zero:

$$\frac{\delta \pi_i(p_i, p_j^*)}{\delta p_i} = a - 2p_i + bp_j^* + c = 0,$$
$$\frac{\delta \pi_j(p_i^*, p_j)}{\delta p_j} = a - 2p_j + bp_j^* + c = 0.$$

- Rearranging yields the two firms' best responses to each other's optimum quantity:

$$\begin{cases} p_i^* = \dfrac{a + bp_j^* + c}{2} \\ p_j^* = \dfrac{a + bp_i^* + c}{2} \end{cases}$$

Nash Equilibrium

- This is a system of 2 equations in 2 unknowns. *Pure-strategy NE* is obtained by replacing the second equation into the first:

$$
\begin{aligned}
p_i^* &= \frac{1}{2}\left[a + \frac{b}{2}\left(a + bp_i^* + c\right) + c\right], \\
4p_i^* &= 2a + ba + b^2 p_i^* + bc + 2c, \\
(4 - b^2)p_i^* &= 2(a + c) + b(a + c), \\
(4 - b^2)p_i^* &= (a + c)(2 + b), \\
p_i^* &= \frac{(a + c)(2 + b)}{4 - b^2}, \\
p_i^* &= \frac{(a + c)(2 + b)}{(2 - b)(2 + b)}, \\
p_i^* &= \frac{a + c}{2 - b} = p_j^*.
\end{aligned}
$$

- Notice that we need $b < 2$ in order for the equilibrium price to be positive. If $b = 0$ the goods are not substitutes, and the duopolist can act as a monopolist.

Equilibrium Profits

- To find equilibrium payoffs (profits), we plug equilibrium prices in the firms' profit function:

$$\pi_i(p_i^*, p_j^*) = \pi_j(p_i^*, p_j^*) = \left[a - \frac{a+c}{2-b} + b\frac{a+c}{2-b}\right]\left[\frac{a+c}{2-b} - c\right] =$$

$$= \left[a + (b-1)\frac{a+c}{2-b}\right]\left[\frac{a+c}{2-b} - c\right] =$$

$$= \left[\frac{2a - ab + ab + bc - a - c}{2-b}\right]\left[\frac{a-c+bc}{2-b}\right] =$$

$$= \left[\frac{a-c+bc}{2-b}\right]\left[\frac{a-c+bc}{2-b}\right] = \left[\frac{a-c+bc}{2-b}\right]^2$$

- When $b = 0$, each duopolist makes monopoly profits.

Interpretation Using Reaction Functions

Prices are strategic complements:

$$\begin{cases} R_i(p_j) = \dfrac{a + bp_j^* + c}{2} \\ R_j(p_i) = \dfrac{a + bp_i^* + c}{2} \end{cases}$$

Remarks and Extensions

Remark:

- The game can be solved also using *IESDS*. (not shown)

Extensions:

- Effect of including **fixed costs** F on *NE* equilibrium prices, quantities and profits? (same as in Cournot)
- Effect of the two firms facing **different marginal costs** on *NE* equilibrium prices, quantities and profits? (same as in Cournot)

Bertrand with Fixed Costs

- In this case profits are reduced by a **fixed cost** F

$$\max_{p_i}[a - p_i + bp_j^*][p_i - c] - F,$$

$$\max_{p_j}[a - p_j + bp_i^*][p_j - c] - F.$$

- Each firm maximises profits **given the quantity chosen by the other firm** (the other firm's **best-response**):

$$\max_{p_i}[a - p_i + bp_j^*][p_i - c] - F,$$

$$\max_{p_j}[a - p_j + bp_i^*][p_j - c] - F.$$

- When taking the first derivative, however, notice that F **cancels out** (it is a constant). F **does not affect** NE quantities. However, it **reduces** NE profits.

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- In this case we assume that firm i is **more efficient** than j:

$$c_j > c_i.$$

- To solve the game, sufficient to follow all the steps provided for the general case. Not possible, however, to use the **general formula** to compute NE quantities.

- The more efficient firm will make **larger profits**.

Bertrand with Asymmetric Costs (2)

- In this case the first-order conditions become:

$$\frac{\delta\pi_i(p_i, p_j^*)}{\delta p_i} = a - 2p_i + bp_j^* + c_i = 0,$$
$$\frac{\delta\pi_j(p_i^*, p_j)}{\delta p_j} = a - 2p_j + bp_j^* + c_j = 0.$$

- Rearranging yields the two firms' best responses to each other's optimum quantity:

$$\begin{cases} p_i^* = \dfrac{a + bp_j^* + c_i}{2} \\ p_j^* = \dfrac{a + bp_i^* + c_j}{2} \end{cases}$$

- Equilibrium prices are **left as an exercise**.

Outline

- Introduction
- Normal-Form
- IESDS
- NE
- Cournot
- Bertrand
- TotC

The Tragedy of the Commons

"If citizens respond only to private incentives, public goods will be **underprovided** *and public resources* **overutilised***."*

David Hume (1739)

Rules of the Game (1)

- **Players**: n farmers in a village graze goats on the common green.
- **Strategies**: number of goats to own $[0, G_{max}]$ (G_{max} maximum number of goats).
  - The *ith* farmer owns g_i goats;
  - The total number of goats in the village is

$$G = g_1 + ... + g_i + ... + g_n.$$

- Farmers choose the number of goats to own *simultaneously*.

Rules of the Game (2)

- **Payoffs**: value of grazing a goat minus its constant per-unit cost c:
  - Value is positive only if $G < G_{max}$, the *maximum amount* of goats that can survive on the green:

$$\begin{cases} V(G) > 0 \;\; if \;\; G < G_{max} \\ V(G) = 0 \;\; if \;\; G > G_{max} \end{cases}$$

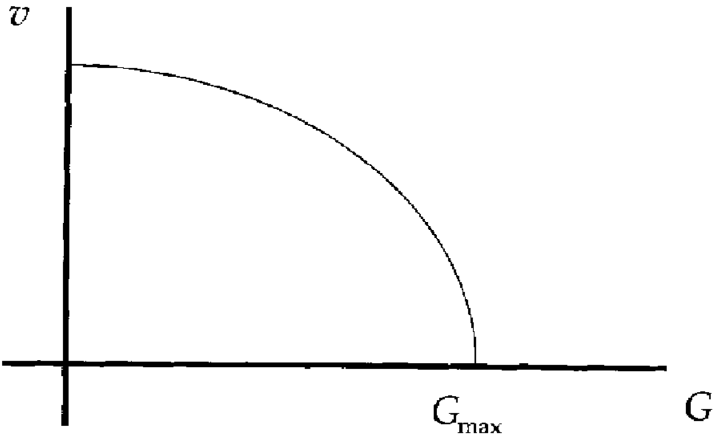
  - *Marginal effect* of adding a goat is *negative* on $V(G)$. The negative effect is *increasing* in the number of goats:

$$\frac{\delta V(G)}{\delta G} < 0.$$
$$\frac{\delta^2 V(G)}{\delta G^2} < 0.$$

  - Eah player's payoffs are:

$$g_i V(G) - cg_i.$$

Decreasing and Diminishing Marginal Returns



Individual Maximisation Problem

- Farmer i maximises its payoffs by the number of goats **given** that the other farmers are playing their equilibrium strategies:

$$\max_{g_i} \; g_i V(g^*_{-i}, g_i) - c g_i.$$

- We take the *first derivative* wrt g_i and equate it to zero

$$V(g^*_{-i}, g_i) + g_i V^{'}(g^*_{-i}, g_i) - c = 0.$$

- As the solution is *symmetric* (farmers have the same strategies and payoff functions) we can replace $g_i = \frac{G^*}{n}$:

$$V(G^*) + \frac{G^*}{n} V^{'}(G^*) - c = 0.$$

Maximisation by Social Planner

- In contrast, the *social planner* (assumed to be benevolent), would maximise **social** benefits from grazing

$$\max_{0 \leq G \leq \infty} \quad GV(G) - cG.$$

- We take the *first derivative* wrt G and equate it to zero

$$V(G^{**}) + G^{**}V^{'}(G^{**}) - c = 0.$$

Comparison of Equilibria

- As both *first-order conditions* equal zero, the following equality must hold

$$V(G^*) + \frac{G^*}{n} V^{'}(G^*) = V(G^{**}) + G^{**} V^{'}(G^{**}).$$

- We claim that $G^* > G^{**}$. To prove this we use **contradiction**, i.e. we assume $G^* < G^{**}$. This implies
  1. $V(G^*) > V(G^{**})$ since $V' < 0$.
  2. $V'(G^*) > V'(G^{**})$ since $V'' < 0$ (less negative).
  3. $\frac{G^*}{n} < G^{**}$ since $G^* < G^{**}$.
- As a consequence, the left-hand side of the equation is **strictly larger**!
- This contradicts the equality, so it must be $G^* > G^{**}$. Farmers graze **too many goats** because each farmer considers only his own incentives.