MATHEMATICS 2

Exam

13/09/2018, A.Y. 2017/2018

Lastname .... Name ....

1) *(8 p.ts)* Solve the following integral

$$\int_{\frac{\pi}{2}}^{2\pi} e^{-x} \cos x dx$$

Solution

Let us solve through intgration by parts the following indefinite integral

$$\int e^{-x} \cos x dx = \sin x e^{-x} - \int \sin x(-e^{-x}) dx = \sin x e^{-x} + \int \sin x e^{-x} dx =$$

$$\sin x e^{-x} - \cos x e^{-x} - \int - \cos x(-e^{-x}) dx = \sin x e^{-x} - \cos x e^{-x} - \int \cos x e^{-x} dx$$

Putting together forst and last member we obtain the following equation in our unknown $\int e^{-x} \cos x dx$

$$\int e^{-x} \cos x dx = \sin x e^{-x} - \cos x e^{-x} - \int e^{-x} \cos x dx$$

hence if we solve it for $\int e^{-x} \cos x dx$, we obtain

$$\int e^{-x} \cos x dx = \frac{e^{-x}}{2}(\sin x - \cos x) + c$$

By the Fundamental Theorem of Calculus we can then obtain the requested definite integral

$$\int_{\frac{\pi}{2}}^{2\pi} e^{-x} \cos x dx = \left[\frac{e^{-x}}{2}(\sin x - \cos x)\right]_{\frac{\pi}{2}}^{2\pi} = -\frac{1}{2}(e^{-2\pi} + e^{-\frac{\pi}{2}})$$

2) *(10 p.ts)* Given the following linear system

$$\begin{cases} x + z = 1 \\ 3x + 2y = 0 \\ -x + 3z = 0 \\ 4x + 2y - z = k \end{cases}$$

discuss and find solutions as k changes.

Solution

Consider incomplete and complete matrix associated to the given system

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ -1 & 0 & 3 \\ 4 & 2 & -1 \end{pmatrix} \quad A|b = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 2 & -1 & k \end{pmatrix}$$

By Rouché Capelli Theorem the system admits solutions if and only if $rkA = rkA|b$. By observing that

$$\begin{vmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix}$$

we can conclude that $rkA = 3$. Therefore for the system to be compatible we need to find k values such that $rkA|b = 3$ and this happens if $detA|b = 0$, hence

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 2 & -1 & k \end{vmatrix} = k \begin{vmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} - \begin{vmatrix} 3 & 2 & 0 \\ -1 & 0 & 3 \\ 4 & 2 & -1 \end{vmatrix} = 8k - 4$$

hence $detA|b = 0$ if and only if $k = \frac{1}{2}$. The system is incompatible if $k \neq \frac{1}{2}$.
If $k = \frac{1}{2}$, by Rouché Capelli the originary system is equivalent to the following system

$$\begin{cases} x + z = 1 \\ 3x + 2y = 0 \\ -x + 3z = 0 \end{cases}$$

whose solution is

$$\begin{cases} x = \frac{3}{4} \\ y = -\frac{9}{8} \\ z = \frac{1}{4} \end{cases}$$

3) *(10 p.ts)* Given the function in $\mathbb{R}^2$

$$f(x,y) = (x^2 + xy + 2y^2)e^x$$

find maximum and/or minimum and say if they are local or global.

Solution

Let us find stationary points by equating to zero first partial derivatives

$$\begin{cases} f_x = (x^2 + xy + 2y^2)e^x + (2x + y)e^x = (x^2 + xy + 2y^2 + 2x + y)e^x = 0 \\ f_y = (x + 4y)e^x = 0 \end{cases}$$

from the second equation we obtain that $x = -4y$, we can therefore substitute this information in the first equation

$$[(-4y)^2 + (-4y)y + 2y^2 + 2(-4y) + y)]e^{-4y} = 0$$

by expanding all products we obtain

$$16y^2 - 4y^2 + 2y^2 - 8y + y = 14y^2 - 7y = 7y(2y - 1) = 0$$

this last equation implies that $y = 0$ and hence $x = 0$, and that $y = \frac{1}{2}$ and hence $x = -2$.
Let us analyze the nature of these two couples of points by evaluating second order conditions, and by calculating the Hessian matrix

$$\begin{cases} f_{xx} = (x^2 + xy + 2y^2 + 2x + y)e^x + (2x + y + 2)e^x = (x^2 + xy + 2y^2 + 4x + 2y + 2)e^x \\ f_{yx} = f_{xy} = (x + 4y)e^x + e^x = (x + 4y + 1)e^x \\ f_{yy} = 4e^x \end{cases}$$

The Hessian therefore is

$$H(x,y) = \begin{pmatrix} (x^2 + xy + 2y^2 + 4x + 2y + 2)e^x & (x + 4y + 1)e^x \\ (x + 4y + 1)e^x & 4e^x \end{pmatrix}.$$

As

$$|H(0,0)| = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7 > 0$$

and $f_{xx}(0,0) = 2 > 0$, hence $(0,0)$ is a Local Minimum.
As

$$|H(-2, \frac{1}{2})| = \begin{vmatrix} -\frac{3}{2}e^{-2} & e^{-2} \\ e^{-2} & 4e^{-2} \end{vmatrix} = -7e^{-4} < 0$$

hence $(-2, \frac{1}{2})$ is a Saddle point.