

Static Games of Incomplete Information

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Outline

- Introduction
- Assumptions
- Normal-Form
- Equilibrium
- Bayesian Cournot
- An Auction

- **Static:** players choose *simultaneously* what to play (alternatively, each of them does not know the others' choice)
- **Incomplete Info:** At least one player is **uncertain** about another player's payoff (the other player's **type**)
- In other words, payoffs are **not common knowledge**, at least for one player. These games are called **Bayesian** from the Bayes rule used in probability theory
- **Example:** *Sealed-bid* auctions. Players know their own valuations of the auctioned good, but not the others'

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Strategies in Static Games

- Recall that we defined strategies in dynamic games as a
 - Complete plan of action
 - Specifying the **actions** of the players at **every decision node** where they may be called to act

- Actions and strategies for player i **coincide** in static games:

$$S_i = A_i$$

- As a consequence, a static game of **complete** info can be written in the following **normal-form**

$$G = \{1, 2, \dots, N, A_1, \dots, A_n, u_1(a_1, a_2, \dots, a_n), \dots, u_n(a_1, a_2, \dots, a_n)\}.$$

- In Bayesian games, however, we need to introduce **types**

Types and Payoffs (1)

- We use **types** to incorporate the fact that at least one player is **uncertain** about the others' payoff function
- Player i 's payoff when her type is t_i is

$$u_i(a_1, \dots, a_n; t_i).$$

- **Example:** suppose Player i can be of **two types**. We define player i 's **type space** as

$$T_i = \{t_{i1}, t_{i2}\}.$$

Player i thus has two payoff functions

$$u_i(a_1, \dots, a_n; t_{i1}) \text{ and } u_i(a_1, \dots, a_n; t_{i2}).$$

Types and Payoffs (2)

- Potentially, **different actions** may be available for **different types** of player i
- The type space for **all the other players** but i is

$$T_{-i} = \{T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_N\}.$$

- The actual types for **all the other players** but i are

$$t_{-i} = \{t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_N\}.$$

Beliefs

- Player i has some **beliefs** about the other players' type.
- These are a **probability distribution** over the all other players types given that player i is type t_i

$$p_i(t_{-i}/t_i).$$

- Assume t_{-i} is **independent** from t_i , so

$$p_i(t_{-i}).$$

is **the same for all players**

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Normal-Form in Bayesian Games

Definition: *The normal-form representation of a N -player static Bayesian game specifies the **players**, their **action spaces** A_1, \dots, A_n , their **type spaces** T_1, \dots, T_N , their beliefs p_1, \dots, p_N , and their **payoff functions** u_1, \dots, u_n .*

$$G = \left\{ \{N\}; \{A_{i \in N}\}; \{T_{i \in N}\}; p_{i \in N}(t_{-i \in N}); u_{i \in N}(A_{i \in N}; t_i) \right\}.$$

It is possible to represent Bayesian games in **extensive form**

- To do so, we include another player, **Nature**
- In the first stage Nature **draws types** $t_i = (t_1, \dots, t_n)$ from T_i
- Each player is **not aware** of the other players' type

Two Technical Points

1. Incomplete info can also mean that players have some **private info** about the game
 - In this case, the payoffs of all players depend on the information that the others have...
 - ...and types are identified by information
 - Payoffs depend on all types

$$u_i(a_1, \dots, a_n; t_1, \dots, t_n).$$

2. Beliefs $p_i(t_{-i}/t_i)$ are computed using **Bayes Rule**

Bayes Rule

- Before nature draws types, each player has a **prior probability** distribution over the type of each player
- Let A and B be two events. The probability of A **given that** B has occurred is

$$P(A/B) = \frac{P(A, B)}{P(B)},$$

where $P(A, B)$ is the probability that A and B **occur jointly** and $P(B)$ is the probability of the **conditioning event**

- We can apply this rule to beliefs

$$P_i(t_{-i}/t_i) = \frac{P(t_{-i}, t_i)}{P(t_i)} = \frac{P(t_{-i}, t_i)}{\sum_{t_{-i} \in t_i} P(t_{-i}, t_i)}.$$

where $\sum_{t_{-i} \in t_i} P(t_{-i}, t_i)$ is the sum for each other player of the probability that the other are t_{-i} when player i is t_i

- The situation is easier for us since we assume

$$P(t_{-i}/t_i) = P(t_{-i}).$$

Strategies in Bayesian Games (1)

Definition: *In the static Bayesian Game*

$$G = \{A_1, \dots, A_N; T_1, \dots, T_N; P_1, \dots, P_N; u_1, \dots, u_N\},$$

a **strategy** for player i is a function $s_i(t_i)$, where for each type t_i in T_i , $s_i(t_i)$ specifies the **action** from the feasible set A_i that type t_i would choose if drawn by Nature.

- A strategy is a function **from types to actions**:

$$s_i : T_i \rightarrow A_i.$$

- It is an action for **each type** of the player

Strategies in Bayesian Games (2)

- A strategy must specify **what player i would do when she is of type t_i**
- This is needed as the other players base their own decisions on their beliefs concerning player i 's type
- Strategies can be:
 1. **Pooling**: all types choose the **same action**
 2. **Separating**: each type chooses a **different action**

Example

		Player 2	
		L	R
Player 1	T	(4, 3)	(3, 1)
	B	(3, 6)	(2, 3)
		$p_1 = 2/3$	

		Player 2	
		L	R
Player 1	T	(3, 3)	(1, 6)
	B	(1, 1)	(5, 3)
		$p_2 = 1/3$	

- $N = \{\text{Player 1}, \text{Player 2}\}$
- $T_1 = \{1\}$
- $T_2 = \{1, 2\}$
- $P_1(T_2 = 1) = 2/3$: Player 1's belief that Player 2 is type 1
- $P_1(T_2 = 2) = 1/3$: Player 1's belief that Player 2 is type 2
- $A_1 = \{T, B\}$ and $A_2 = \{L, R\}$
- $S_1 = \{T, B\}$ and $S_2 = \{LL, LR, RL, RR\}$

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Bayesian NE

Definition: *In a static Bayesian game*

$$G = \{A_1, \dots, A_n; T_1, \dots, T_n; P_1, \dots, P_n; u_1, \dots, u_n\},$$

the strategies $s^ = (s_1^*, \dots, s_n^*)$ are a **pure-strategy Bayesian Nash Equilibrium** if for each player i and for each of i 's types t_i in T_i , $s_i^*(t_i)$ solves*

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i(s_1^*(t_1), \dots, s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), \dots, s_n^*(t_n))$$

That is, no player wants to change his or her strategy, even if the change involves only one action by one type

Example

		Player 2	
		L	R
Player 1	T	(4, 3)	(3, 1)
	B	(3, 6)	(2, 3)
		$p_1 = 2/3$	

		Player 2	
		L	R
Player 1	T	(3, 3)	(1, 6)
	B	(1, 1)	(5, 3)
		$p_2 = 1/3$	

To find the *BNE* of this game we proceed as follows:

1. Compute **expected payoffs** for Player 1
2. Create a new "**merged**" matrix including expected payoffs for Player 1 and payoffs for both types of Player 2
3. Find **best responses** for Player 1 and both types of Player 2

Example - Player 1's Expected Payoffs

- Player 1 does not know whether $t_2 = 1$ or $t_2 = 2$
- She just has some **beliefs** $P_1(t_2 = 1)$ and $P_1(t_2 = 2)$
- Best response will be based on **expected payoffs** from playing the actions T and B given the actions of each type of Player 2

$$\nu(T, LL) = \frac{2}{3} \times 4 + \frac{1}{3} \times 3 = \frac{11}{3}$$

$$\nu(T, LR) = \frac{2}{3} \times 4 + \frac{1}{3} \times 1 = 3$$

$$\nu(T, RL) = \frac{2}{3} \times 3 + \frac{1}{3} \times 3 = 3$$

$$\nu(T, RR) = \frac{2}{3} \times 3 + \frac{1}{3} \times 1 = \frac{7}{3}$$

$$\nu(B, LL) = \frac{2}{3} \times 3 + \frac{1}{3} \times 1 = \frac{7}{3}$$

$$\nu(B, LR) = \frac{2}{3} \times 3 + \frac{1}{3} \times 5 = \frac{11}{3}$$

$$\nu(B, RL) = \frac{2}{3} \times 2 + \frac{1}{3} \times 1 = \frac{5}{3}$$

$$\nu(B, RR) = \frac{2}{3} \times 2 + \frac{1}{3} \times 5 = 3$$

Example - New Matrix

		Player 2			
		<i>LL</i>	<i>LR</i>	<i>RL</i>	<i>RR</i>
Player 1	<i>T</i>	$\frac{11}{3}; (3, 3)$	$3; (3, 6)$	$3; (1, 3)$	$\frac{7}{3}; (1, 6)$
	<i>B</i>	$\frac{7}{3}; (6, 1)$	$\frac{11}{3}; (6, 3)$	$\frac{5}{3}; (3, 1)$	$3; (3, 3)$

- Player 2 know perfectly her type, no need to compute expected payoffs
- In this new matrix we have **three payoffs**, one (expected) for Player 1 and two for each type of Player 2
- To find the *BNE*, look for the best response of Player 1 and both types of Player 2

Example - Best Responses

		Player 2			
		LL	LR	RL	RR
Player 1	T	$\frac{11}{3}; (3, 3)$	$3; (3, 6)$	$3; (1, 3)$	$\frac{7}{3}; (1, 6)$
	B	$\frac{7}{3}; (6, 1)$	$\frac{11}{3}; (6, 3)$	$\frac{5}{3}; (3, 1)$	$3; (3, 3)$

- Player 1. Best response to:
 - LL is T
 - LR is B
 - RL is T
 - RR is B
- Player 2, type 1. Best response to:
 - T is L
 - B is L
- Player 2, type 2. Best response to:
 - T is R
 - B is R
- The only actions such that Player 1 and both types of Player 2 do not have any incentive to deviate are (B, LR) , the *BNE*

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Assumptions (1)

- Players: two firms

$$N = \{\textit{Firm 1}, \textit{Firm 2}\}.$$

- Strategies: quantity of a *homogeneous* good to be produced (infinite is not included in the production interval)

$$q_1 = q_2 = [0, \infty).$$

- Inverse market demand is $p(q) = a - Q$
- Payoffs: firm's profits

$$\pi_1 = [a - Q - c]q_1,$$

where $Q = q_1 + q_2$

Baseline: Complete Info

- Firm 1's marginal cost c is observed by both firms
- Firm 2's marginal cost is also observed by both firms and can be **high** (c_H) or **low** (c_L), with $c_H > c_L$

Baseline: Maximisation Problem

- Firm 2 maximises:

$$\max_{q_2} [a - (q_2 + q_1^*) - c_H] q_2 \text{ when } MC = c_H, \text{ and}$$

$$\max_{q_2} [a - (q_2 + q_1^*) - c_L] q_2 \text{ when } MC = c_L.$$

- Firm 1 maximises profits given Firm 2's type

$$\max_{q_1} [a - (q_1 + q_2^*(c_H)) - c] q_1 \text{ when } MC = c_H, \text{ and}$$

$$\max_{q_1} [a - (q_1 + q_2^*(c_L)) - c] q_1 \text{ when } MC = c_L.$$

Baseline: Best Responses

- If $MC = c_H$

$$\frac{\delta \pi_2(q_1^*, q_2)}{\delta q_2(c_H)} = a - 2q_2^*(c_H) - q_1^* - c_H = 0 \rightarrow q_2^* = \frac{a - q_1^* - c_H}{2}$$

$$\frac{\delta \pi_1(q_1, q_2^*(c_H))}{\delta q_1} = a - 2q_1^* - q_2^*(c_H) - c = 0 \rightarrow q_1^* = \frac{a - q_2^*(c_H) - c}{2}$$

- If $MC = c_L$

$$\frac{\delta \pi_2(q_1^*, q_2)}{\delta q_2(c_L)} = a - 2q_2^*(c_L) - q_1^* - c_L = 0 \rightarrow q_2^* = \frac{a - q_1^* - c_L}{2}$$

$$\frac{\delta \pi_1(q_1, q_2^*(c_L))}{\delta q_1} = a - 2q_1^* - q_2^*(c_L) - c = 0 \rightarrow q_1^* = \frac{a - q_2^*(c_L) - c}{2}$$

Baseline: Nash Equilibrium

- If $MC = c_H$

$$q_1^* = \frac{a - \left(\frac{a - q_1^* - c_H}{2} \right) - c}{2} = \frac{a - 2c + c_H}{3}$$

$$q_2^* = \frac{a - \left(\frac{a - 2c + c_H}{3} \right) - c_H}{2} = \frac{a + c - 2c_H}{3}$$

- If $MC = c_L$

$$q_1^* = \frac{a - \left(\frac{a - q_1^* - c_L}{2} \right) - c}{2} = \frac{a - 2c + c_L}{3}$$

$$q_2^* = \frac{a - \left(\frac{a - 2c + c_L}{3} \right) - c_L}{2} = \frac{a + c - 2c_L}{3}$$

Introducing Incomplete Info

- Results in baseline are as **Cournot** with **complete info** and asymmetric costs
- Firm 1's marginal cost c is observed by both firms
- Firm 2's marginal cost is **private information**. Firm 1 only has **beliefs** θ and $(1 - \theta)$ that marginal cost is **high** or **low**

$$\begin{cases} c_H & \text{with probability } \theta \\ c_L & \text{with probability } 1 - \theta \end{cases}$$

with $c_L < c_H$.

- Notice that now Firm 2's **best response** depends on its marginal cost. We will thus have $q_2(c_H)$ and $q_2(c_L)$

Maximisation Problem

- Firm 2 maximises:

$$\max_{q_2} [a - (q_2 + q_1^*) - c_H] q_2 \text{ when } MC = c_H, \text{ and}$$

$$\max_{q_2} [a - (q_2 + q_1^*) - c_L] q_2 \text{ when } MC = c_L.$$

- Firm 1 does not know Firm 2's cost function, so it maximises **expected profits** given its beliefs

$$\max_{q_1} \theta [(a - q_1 - q_2^*(c_H)) - c] q_1 + (1 - \theta) [(a - q_1 - q_2^*(c_L)) - c] q_1.$$

Firm 2's Best Responses

- To compute *BRs* for Firm 2, take the derivative of each profit function with respect to q_2 and equate it to zero

$$\frac{\delta \pi_2(q_1^*, q_2)}{\delta q_2(c_H)} = a - 2q_2^*(c_H) - q_1^* - c_H = 0 \text{ and}$$

$$\frac{\delta \pi_2(q_1^*, q_2)}{\delta q_2(c_L)} = a - 2q_2^*(c_L) - q_1^* - c_L = 0.$$

- Rearranging yields Firm 2's best response to Firm 1's quantity for c_H and c_L

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2}$$

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2}$$

Firm 1's Best Responses

- To compute *BR* for Firm 1, take the derivative of **expected profits** with respect to q_1 and equate it to zero

$$\frac{\delta \pi_1(q_1, q_2^*(c_H), q_2^*(c_L))}{\delta q_1} =$$

$$= \theta[a - 2q_1^* - q_2^*(c_H) - c] + (1 - \theta)[a - 2q_1^* - q_2^*(c_L) - c] = 0.$$

- Collect the terms with q_1^* and bring to the right

$$2\theta q_1^* + 2(1 - \theta)q_1^* = \theta[a - q_2^*(c_H) - c] + (1 - \theta)[a - q_2^*(c_L) - c]$$

$$2\theta q_1^* + 2q_1^* - 2\theta q_1^* = \theta[a - q_2^*(c_H) - c] + (1 - \theta)[a - q_2^*(c_L) - c]$$

$$q_1^* = \frac{\theta[a - q_2^*(c_H) - c] + (1 - \theta)[a - q_2^*(c_L) - c]}{2}.$$

Best Responses

- Firm 1's *BR* is a weighted average (with weights $\theta, (1 - \theta)$) of the best responses for $q_2^*(c_H)$ and $q_2^*(c_L)$
- We have a system of **three equations** in **three unknowns**

$$\begin{cases} q_1^* = \frac{\theta[a - q_2^*(c_H) - c] + (1 - \theta)[a - q_2^*(c_L) - c]}{2} \\ q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} \\ q_2^*(c_L) = \frac{a - q_1^* - c_L}{2} \end{cases}$$

Eq. Quantity for Firm 1

- Replace $q_2^*(c_H)$ and $q_2^*(c_L)$ in q_1^* :

$$q_1^* = \frac{\theta[a - (\frac{a - q_1^* - c_H}{2}) - c] + (1 - \theta)[a - (\frac{a - q_1^* - c_L}{2}) - c]}{2},$$

$$4q_1^* = \theta[2a - a + q_1^* + c_H - 2c] + (1 - \theta)[2a - a + q_1^* + c_L - 2c]$$

$$4q_1^* = 2\theta a - \theta a + \theta q_1^* + \theta c_H - 2\theta c + 2a - a + q_1^* + c_L - 2c - 2\theta a + \theta a - \theta q_1^* - \theta c_L + 2\theta c$$

- Simplify and bring the terms with q_i^* to the left

$$4q_1^* - \theta q_1^* + \theta q_1^* - q_1^* = 2\theta a - \theta a + \theta c_H - 2\theta c + 2a - a + c_L - 2c - 2\theta a + \theta a - \theta c_L + 2\theta c$$

$$3q_1^* = 2a - a + \theta c_H + c_L - 2c - \theta c_L$$

$$q_1^* = \frac{a - 2c + \theta c_H + c_L - \theta c_L}{3} = \frac{a - 2c + \theta c_H + (1 - \theta)c_L}{3}$$

- q_1^* includes a weighted average of Firm 2's costs, where weight are beliefs

Eq. Quantity for Firm 2 (c_H)

- Replace q_1^* in $q_2^*(c_H)$:

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} = \frac{a - \left(\frac{a - 2c + \theta c_H + (1 - \theta)c_L}{3} \right) - c_H}{2},$$

$$6q_2^*(c_H) = 3a - a - 3c_H + 2c - \theta c_H - (1 - \theta)c_L$$

$$6q_2^*(c_H) = 2a + 2c - 3c_H - \theta c_H + (\theta - 1)c_L$$

$$q_2^*(c_H) = \frac{2a + 2c - 3c_H}{6} + \frac{(\theta - 1)c_L - \theta c_H}{6}$$

- Mathematical trick: subtract c_H from the first ratio and add it to the second and simplify

$$q_2^*(c_H) = \frac{2a + 2c - 4c_H}{6} + \frac{(\theta - 1)c_L - (\theta - 1)c_H}{6}$$

$$q_2^*(c_H) = \frac{a + c - 2c_H}{3} + \frac{(1 - \theta)}{6}(c_H - c_L)$$

Eq. Quantity for Firm 2 (c_L)

- Replace q_1^* in $q_2^*(c_L)$:

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2} = \frac{a - \left(\frac{a - 2c + \theta c_H + (1 - \theta)c_L}{3} \right) - c_L}{2},$$

$$6q_2^*(c_L) = 3a - a - 3c_L + 2c - \theta c_H - (1 - \theta)c_L$$

$$6q_2^*(c_L) = 2a + 2c - 3c_L - \theta c_H - (1 - \theta)c_L$$

$$q_2^*(c_L) = \frac{2a + 2c - 3c_L}{6} - \frac{\theta c_H + (1 - \theta)c_L}{6}$$

- Mathematical trick: subtract c_L from the first ratio and add it to the second and simplify

$$q_2^*(c_L) = \frac{2a + 2c - 4c_L}{6} - \frac{\theta c_H + \theta c_L}{6}$$

$$q_2^*(c_L) = \frac{a + c - 2c_L}{3} - \frac{\theta}{6}(c_H - c_L)$$

Eq. Quantities and Comparison with Baseline

- The *BNE* of this game is

$$q_2^*(c_H) = \frac{a + c - 2c_H}{3} + \frac{(1 - \theta)}{6}(c_H - c_L)$$

$$q_2^*(c_L) = \frac{a + c - 2c_L}{3} - \frac{\theta}{6}(c_H - c_L)$$

The **red part was not present** in the complete info case

- As $c_H > c_L$, it follows that

$$\frac{(1 - \theta)}{6}(c_H - c_L) > 0 \text{ and } -\frac{\theta}{6}(c_H - c_L) < 0$$

- When info is incomplete, the **inefficient** type c_H produces **more** and the **efficient** type c_L produces **less**
- This occurs as Firm 1 is uncertain about Firm 2's type and it maximises **expected profits**
- As uncertainty disappears ($\theta = 0$ or $\theta = 1$) we are back (of course) to the complete info case

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A Sealed-Bid Auction: Rules

- Two **bidders** $i = 1, 2$
- v_i : **Valuation** of the good by bidder i **distributed uniformly** on $[0, 1]$. It is **private info** of bidder i
- $b_i \geq 0$: Player i 's bid must be **non-negative**

Three possible outcomes:

- If $b_i > b_j$ player i gets $v_i - b_i$
- If $b_i = b_j$ flip of a coin (probability is $1/2$)
- If $b_i < b_j$ player i gets 0

Normal-Form

- $A_i = [0, \infty]$
- $T_i = [0, 1]$: Uniform distribution
- $p_i(v_j/v_i) = p(v_j)$: Types are **valuations** and are **independent**
- Payoffs are:

$$u_i(b_1, b_2; v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ (v_i - b_i)/2 & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

- A **strategy** is a **function** from the valuations to the bid

$$b_i(v_i)$$

Expected Payoff and Maximisation

- Expected payoffs for bidder i are ($Prob$ stands for probability):

$$\begin{aligned}
 & (v_i - b_i) Prob\{b_i > b_j(v_j)\} \\
 & + \\
 & \frac{(v_i - b_i)}{2} Prob\{b_i = b_j(v_j)\} \\
 & + \\
 & 0 Prob\{b_i < b_j(v_j)\}
 \end{aligned}$$

- The maximisation problem is:

$$\max_{b_i} (v_i - b_i) Prob\{b_i > b_j(v_j)\} + \frac{(v_i - b_i)}{2} Prob\{b_i = b_j(v_j)\}$$

Linear Bids (1)

- Assume that bids are a **linear function** of valuations:

$$b_1(v_1) = a_1 + c_1 v_1,$$

$$b_2(v_2) = a_2 + c_2 v_2.$$

- Suppose that the other player adopts the strategy

$$b_j(v_j) = a_j + c_j v_j,$$

- Player i solves

$$\max_{b_i} (v_i - b_i) \text{Prob}\{b_i > a_j + c_j v_j\} + \frac{(v_i - b_i)}{2} \text{Prob}\{b_i = a_j + c_j v_j\}.$$

- Since valuations are **uniformly distributed** over $[0, 1]$ and can take any **real value**, the **probability of a tie** is zero. Thus

$$\max_{b_i} (v_i - b_i) \text{Prob}\{b_i > a_j + c_j v_j\}.$$

Linear Bids (2)

- Rearrange the probability

$$\text{Prob}\{b_i > a_j + c_j v_j\} = \text{Prob}\left\{v_j < \frac{b_i - a_j}{c_j}\right\} = \frac{b_i - a_j}{c_j}.$$

- The last equality comes from the fact that the **cumulative probability** of the continuous uniform distribution is

$$F(v_j) = \frac{\frac{b_i - a_j}{c_j} - 0}{1 - 0} = \frac{b_i - a_j}{c_j}$$

Back to Maximisation

- Maximisation problem becomes

$$\max_{b_i} (v_i - b_i) \left(\frac{b_i - a_j}{c_j} \right) = \frac{v_i b_i}{c_j} - \frac{v_i a_j}{c_j} - \frac{b_i^2}{c_j} + \frac{b_i a_j}{c_j}.$$

- Take the first derivative w.r.t. b_i and equate to zero

$$\frac{v_i}{c_j} - \frac{2b_i}{c_j} + \frac{a_j}{c_j} = 0.$$

- Simplify and solve for b_i

$$b_i(v_i) = \frac{v_i + a_j}{2}.$$

- Notice that if $v_i < a_j$, where a_j is the bid by j when v_j is zero, the optimal bid for player i will be $b_i(v_i) = a_j$

Optimal Bids (1)

- Optimal bids are

$$b_i(v_i) = \begin{cases} \frac{v_i + a_j}{2} & \text{if } v_i > a_j \\ a_j & \text{if } v_i < a_j \end{cases}$$

Conditions on a_j

- $0 \leq a_j \leq 1$: since $v_i \in [0, 1]$, there exists some values such that $v_i < a_j$, and optimal bid is **not linear** (equal to a_j at the beginning and thus **flat**). **We rule this case out**
- $a_j \geq 1$. This cannot be since $c_j \geq 0$ would imply $b_j(v_j) = a_j + c_j v_j \geq v_j$ (recall $v_j \in [0, 1]$), which cannot be optimal. **We rule this case out**

Optimal Bids (2)

- $a_j \leq 0$. In this case $b_i(v_i) = \frac{v_i + a_j}{2}$. Replace the linear equation for $b_i(v_i)$

$$a_i + c_i v_i = \frac{v_i + a_j}{2} \implies v_i(1 - 2c_i) = 2a_i - a_j,$$

- The two sides of the equation are equal (and equal to zero) when $c_i = 1/2$ and $a_i = a_j/2$. Using the same reasoning, $c_j = 1/2$ and $a_j = a_i/2$
- Thus

$$a_j = \frac{a_i}{2} \implies a_i = 2a_j$$

Replace in the equation for a_i

$$2a_j = \frac{a_j}{2} \implies a_j = 0 \text{ and } a_i = 0$$

- Optimal bids will thus be $b_i = v_i/2$ and $b_j = v_j/2$, i.e. **half their valuation**