Dynamic Games of Incomplete Information

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Outline

- Introduction
- Requirements
- PBE

Assumptions

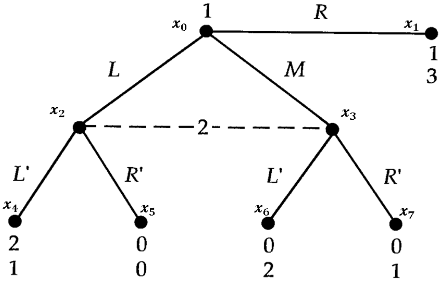
- **Dynamic**: the moves occur in *sequence*
- **Incomplete Info**: At least one player is **uncertain** about another player's payoff (the other player's **type**)

Equilibrium concept is **Perfect Bayesian Equilibrium**:

1. Bayesian equilibrium potentially includes **non-credible** threats
2. We require the Bayesian NE for the entire game to be the Bayesian NE for each **continuation game**

Continuation game starts at each info set, **singleton or not**

Example (1)



- $N = \{1, 2\}$
- Decision nodes: $X_1 = \{x_0\}$ and $X_2 = \{x_2, x_3\}$
- Info sets: $I_1 = \{x_0\}$ and $I_2 = \{x_2, x_3\}$
- Root: $r = \{x_0\}$
- Terminal nodes: $T = \{x_1, x_4, x_5, x_6, x_7\}$
- Strategies: $S_1 = \{L, M, R\}$ and $S_2 = \{L', R'\}$

Example (2)

- This game has only **one subgame**, the game as a whole
- The *NE* of the game are *subgame-perfect*
- To find the *SPNE*, solve the following **static game**

		Player 2	
		L'	R'
Player 1	L	$(1, 1)$	$(0, 0)$
	M	$(0, 2)$	$(0, 1)$
	R	$(1, 3)$	$(1, 3)$

- There are two *SPNE*, $(L, L^{'})$ and $(R, R^{'})$
- However $(R, R^{'})$ is based on a **non-credible threat** as $L^{'}$ **dominates** $R^{'}$
- Need for a **stronger equilibrium concept**

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Why Requirements?

- Four **requirements** to rule out implausible equilibria
- Introduce **beliefs** and put the latter in relation with **strategies**
- Beliefs become a fundamental element of equilibrium

1. Strategies must be **sequentially rational** given beliefs
2. Beliefs must be **reasonable** given subsequent strategies

Requirement 1

Requirement 1*:*

At **each information set** *(singleton or not), the player with the move must have a* **belief** *about which node in the information set has been reached by the play of the game. For a non-singleton information set, a belief is a* **probability distribution** *over the nodes in the information set; for a* **singleton** *information set, the player's belief puts* **probability one** *on the single decision node.*

Requirement 2

Requirement 2:

Given their beliefs, the players' strategies must be **sequentially rational**. *That is, at each information set the* **action** *taken by the player with the move (and the player's subsequent strategy)* **must be optimal given the player's belief at that information set and the other players' subsequent strategies** *(where a "subsequent strategy" is a* **complete plan of action** *covering every contingency that might arise after a given information set has been reached).*

Requirement 3

Definition:

For a given equilibrium in a given extensive-form game, an information set is **on the equilibrium path** *if it will be reached with* **positive probability** *if the game is played according to the equilibrium strategy.*

Requirement 3:

At information sets **on the equilibrium path**, *beliefs are determined by* **Bayes' rule and the players equilibrium strategies**.

Requirement 4

Requirement 4:

At information sets **off the equilibrium path** *beliefs are determined by* **Bayes' rule and the players equilibrium strategies where possible**.

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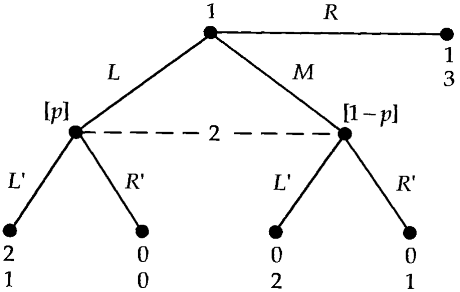
Perfect Bayesian Equilibrium

Definition: *A Perfect Bayesian Equilibrium consists of* **strategies** *and* **beliefs** *satisfying* **Requirements 1 to 4**.

Back to Example (1)



- p: Player 2's **belief** that 1 played L in the previous stage
- Compute **expected payoffs** for Player 2 from $L^{'}$ and $R^{'}$

$$\nu_2(R^{'}) = 0 \times p + 1 \times (1 - p) = 1 - p$$

$$\nu_2(L^{'}) = 1 \times p + 2 \times (1 - p) = 2 - p$$

Back to Example (2)

- Player 2's expected payoff from $L^{'}$ is **larger** than $R^{'}$ if

$$2 - p > 1 - p$$

- This is true **for every** p as $L^{'}$ **strictly dominates** $R^{'}$
- Player 1 thus chooses L in the first stage as

$$2 > 1 > 0$$

- This implies $p = 1$
- The *BPE* of this game is $(L, L^{'})$ and $p = 1$
- An equilibrium is composed by **strategies** and **beliefs**

What about Bayes' Rule?

- In our example M is **off the equilibrium path** ($p = 1$)
- Suppose that there is a *mixed-strategy equilibrium* such that

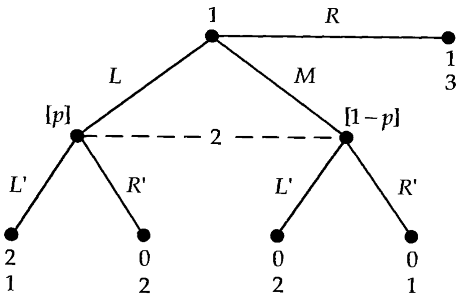
1. L is played with probability q_1
2. M is played with probability q_2
3. R is played with probability $1 - q_1 - q_2$

- Beliefs are computed according to the Bayes rule if

$$p = \frac{q_1}{q_1 + q_2}$$

Example 2 (1)

- Suppose payoffs are modified as follows:



- Compute **expected payoffs** for Player 2 from L' and R'

$$\nu_2(R') = 2 \times p + 1 \times (1 - p) = 1 + p$$
$$\nu_2(L') = 1 \times p + 2 \times (1 - p) = 2 - p$$

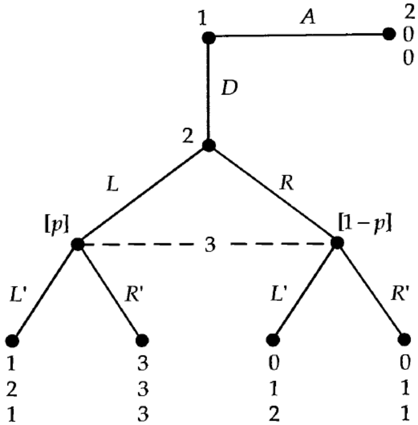
Example 2 (2)

- Player 2's expected payoff from $L^{'}$ is **larger** than $R^{'}$ if

$$2 - p > 1 + p \implies p < 1/2$$

- There are two *BPE* of this game:
  1. $(L, L^{'})$ and $p < 1/2$: If Player 2 believes that L is played with a probability lower than $1/2$, she plays $L^{'}$. Given these beliefs, Player 1 plays L
  2. $(R, R^{'})$ and $p > 1/2$: : If Player 2 believes that L is played with a probability higher than $1/2$, she plays $R^{'}$. Given these beliefs, Player 1 plays R

Why do we Need Requirement 4? (1)



- Two subgames:
  1. The game as a whole
  2. The subgame starting at Player 2's decision node

Why do we Need Requirement 4? (2)

- Solve the subgame as a static game of complete info

Player 3

Player 2	L'	R'
L	$(2, 1)$	$(3, 3)$
R	$(1, 2)$	$(1, 1)$

- The only *NE* of this subgame is (L, R')
- Player 1 in this case chooses D
- (D, L, R') and $p = 1$ satisfies Requirements 1 to 4 and it is thus a *PBE* of this game
- What about (A, L, L') and $p = 0$?

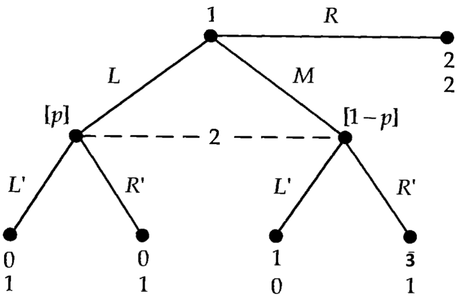
Why do we Need Requirement 4? (2)

- Nobody has a strict incentive to **deviate unilaterally** when $(A, L, L^{'})$ and $p = 0$

  1. If Player 1 plays A, 2 and 3 get zero irrespective of their action
  2. If Player 1 deviates to D she gets $1 < 2$

- This set of strategies satisfies Requirements 1 to 3, where no restriction is imposed on beliefs **off the equilibrium path**
- However, it does not satisfies Requirement 4, i.e. beliefs off the equilibrium path be determined by the Players' equilibrium strategies in the **continuation game**
- As the *NE* of the continuation game is $L, R^{'}$ it must be $p = 1$. $(A, L, L^{'})$ and $p = 0$ is **not perfect**

Another Example (1)



- Compute **expected payoffs** for Player 2 from $L^{'}$ and $R^{'}$

$$\nu_2(L^{'}) = 1 \times p + 0 \times (1 - p) = p$$

$$\nu_2(R^{'}) = 1 \times p + 1 \times (1 - p) = 1$$

Another Example (2)

- Player 2's expected payoff from $L^{'}$ is **larger** than $R^{'}$ if

$$p > 1$$

- This is impossible as $0 < p < 1$.
- Player 1 thus chooses R in the first stage as

$$3 > 1 > 0$$

- This implies $p = 0$
- The *BPE* of this game is $(R, R^{'})$ and $p = 0$
- What about $(L, L^{'})$ and $p = 1$?