TOR VERGATA
UNIVERSITÀ DEGLI STUDI DI ROMA

Sorting Algorithms

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Sorting

- Given a set of elements, the goal is to sort them in a specific order
- Examples: sort alphabetically a list of names, a list of numbers, etc...
- Fundamental algorithm used in many problems
  - E.g., if you sort, then you can do binary search
- See visualization https://visualgo.net/bn/sorting

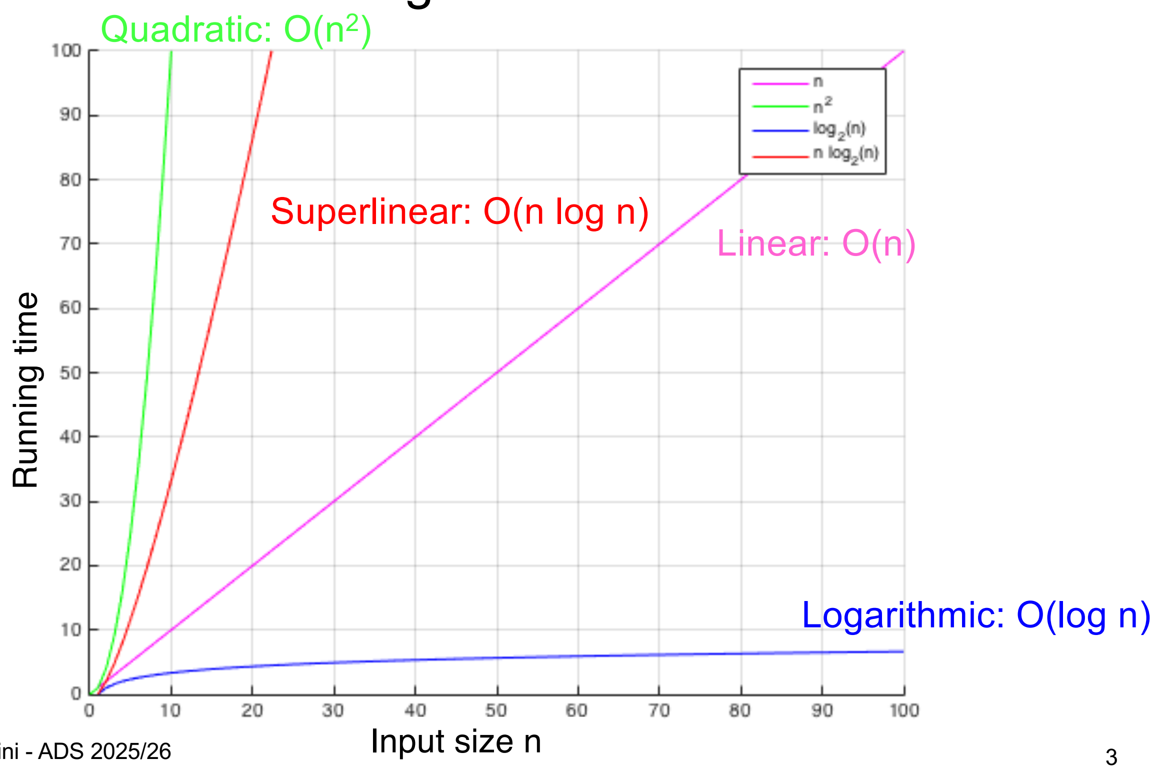
Sorting algorithms and running times

- We will study four sorting algorithms
  - To sort an array (or list) of integer numbers in non-decreasing order (i.e., from smallest to largest)
    - E.g., 1 2 2 3 3 4 5 6 6
  - All the algorithms we will consider are comparison-based, i.e., they sort by comparing elements
- Running times: $O(n^2)$, $O(n \log n)$
  - n is the number of elements to sort

n	10	100	1000	10^6	10^9
$n \log_2 n$	~ 33.3	~ 665	~ 10^4	~ 2×10^7	~ 3×10^{10}
n^2	100	10^4	10^6	10^{12}	10^{18}

Running times

- Some common big O functions



SelectionSort

- Idea: repeatedly select the smallest element (i.e., the minimum) from the unsorted portion of the list and swaps it with the first element of the unsorted portion
  - Incremental approach
- Example

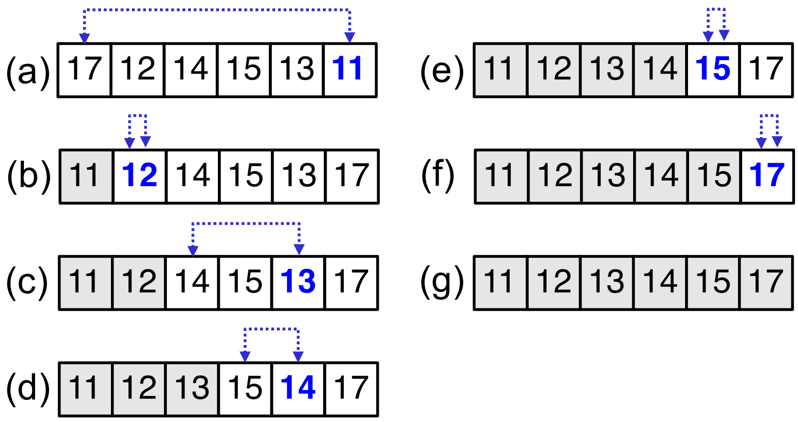


Figure (from top to bottom):
white: still unsorted
grey: already sorted
blue: minimum and swap

SelectionSort: how to find minimum

- How to find the minimum element stating from a specific position onwards
  - We refer to that position as *i*
- *smallest* is a place marker that tracks the position of the smallest value we have seen so far
- Initialize *smallest* to *i*
- Look at every value from *i*+1 onwards and update *smallest* only if you find a value smaller than the one at *smallest*

```
smallest = i
for j = i + 1 to n do
    if L[j] < L[smallest] then
        smallest = j
    end if
end for
```

SelectionSort: how to find minimum

- Example of finding minimum value from a particular position (i=1) onwards

17	12	14	15	13	11
1	2	3	4	5	6

```
smallest = i
for j = i + 1 to n do
    if L[j] < L[smallest] then
        smallest = j
    end if
end for
```

smallest = 1

j = 2

smallest = 2 because L[2] < L[smallest]

j = 3

j = 4

j = 5

when j=3,4,5 we do not update smallest

j = 6

smallest = 6 because L[6] < L[smallest] → The minimum value is L[6], that is 11

SelectionSort: algorithm

```
algorithm SelectionSort(list L)
    for i = 1 to n − 1 do
        smallest = i
        for j = i + 1 to n do
            if L[j] < L[smallest] then
                smallest = j
            end if
        end for
        if i ≠ smallest then
            swap L[i] and L[smallest]
        end if
    end for
```

Find minimum in the unsorted portion of the list

Swap minimum with the leftmost unsorted element (if needed)

Running time?

SelectionSort: running time

- Let's focus on the number of comparisons
- At step k, finding minimum takes O(n-k) time
  - Finding minimum value from position 1 onwards: n-1 comparisons
  - Finding minimum value from position 2 onwards: n-2 comparisons
  - …
  - Finding minimum value from position n-1 onwards: 1 comparison
- In total, (n-1) + (n-2) + … + 1 comparisons
  - Let's express (n-1) + (n-2) + … + 1 as $\sum_{i=1}^{n-1} i$

SelectionSort: running time

- At step k, finding minimum takes O(n-k) time
- In total, the running time is:

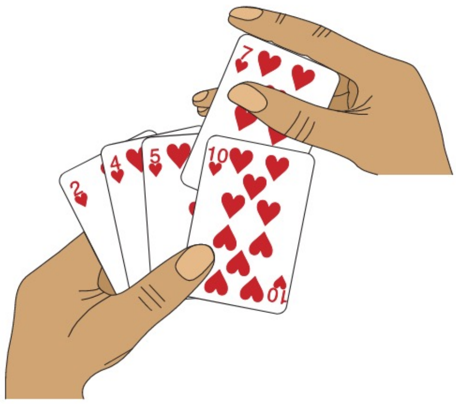
$$\sum_{k=1}^{n-1} O(n-k) = O\left(\sum_{i=1}^{n-1} i\right) = O(n^2)$$

To solve: set i=n-k and use the arithmetic series formula to express the sum by its closed form value

$$\sum_{k=1}^{n-1} i = \frac{n(n-1)}{2} = O(n^2)$$

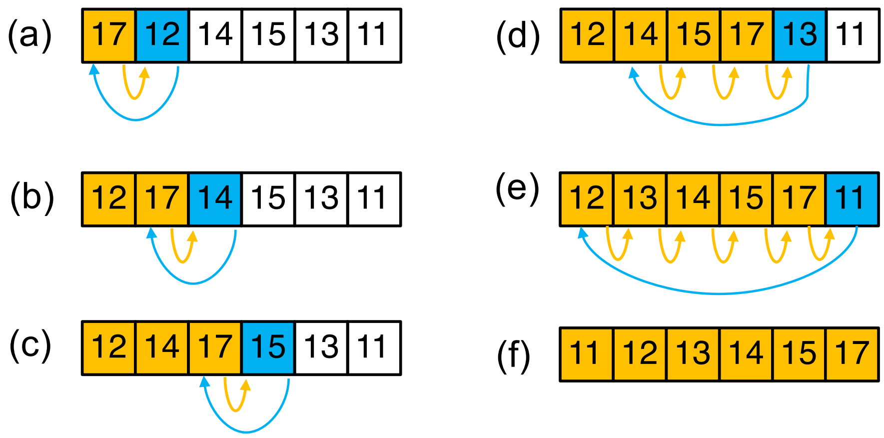
InsertionSort

- Insertion sort works similarly to how you might sort a hand of playing cards



InsertionSort

- Idea: at step k, extend the sorted portion from k-1 to k elements by inserting the k-th element into the correct position among the first k-1 elements
  - Incremental approach
- Example



InsertionSort: algorithm

$$
\begin{array}{l}
\textbf{algorithm } \texttt{InsertionSort}(\mathit{list}\ L) \\
\quad \textbf{for } i = 2 \text{ to } n \textbf{ do} \\
\quad\quad \mathit{value} = L[i] \\
\quad\quad j = i - 1 \\
\quad\quad \triangleright \text{ Insert } L[i] \text{ into the sorted subarray } L[1 : i-1] \\
\quad\quad \textbf{while } j > 0 \text{ and } L[j] > \mathit{value} \textbf{ do} \\
\quad\quad\quad L[j+1] = L[j] \\
\quad\quad\quad j = j - 1 \\
\quad\quad \textbf{end while} \\
\quad\quad L[j+1] = \mathit{value} \\
\quad \textbf{end for}
\end{array}
$$

Running time?

Programming: while loop

- A while loop repeatedly executes a block of instructions as long as a condition is true

```
while condition do
  <block of instructions>
end while
```

- Execution
  - Check the condition
  - If it is true, execute the block of instructions
  - After execution, check the condition again
  - If it becomes false, the loop terminates

Programming: while loop

- The while loop in InsertionSort shifts elements to the right until the correct position for the value is found

$$
\begin{array}{l}
\triangleright \text{ Insert } L[i] \text{ into the sorted subarray } L[1:i-1] \\
\textbf{while } j > 0 \text{ and } L[j] > \mathit{value} \textbf{ do} \\
\quad L[j+1] = L[j] \\
\quad j = j - 1 \\
\textbf{end while}
\end{array}
$$

- What the loop does
  - Compare the key with the previous element
  - If the previous element is larger, shift it one position to the right
  - Move one position left (j = j - 1) and check again
  - Stop when at least one of the conditions becomes false, that is
    - a smaller or equal element is found, or
    - the beginning of the list is reached

InsertionSort: running time

- Inserting k-th element among the first k-1 elements takes time O(k) in the worst case
- In total, the running time is

$$O\Big(\sum_{k=1}^{n-1} k\Big) = O(n^2)$$

To solve: use again the arithmetic series formula

$$\sum_{k=1}^{n-1} i = \frac{n(n-1)}{2} = O(n^2)$$

BubbleSort

- Scan the list several times
- At each scan, compare adjacent pairs of elements, and swap them if they are not in the correct order
- If no elements were swapped during a scan, you can stop because the list is sorted

- Why BubbleSort? Larger elements "bubble" to the end of the list

BubbleSort

- Example

(a)

17	12	14	15	13	11

12 17
14 17
15 17
13 17
11 17

(b)

12	14	15	13	11	17

12 14
14 15
13 15
11 15

(c)

12	14	13	11	15	17

(c)

12	14	13	11	15	17

12 14
13 14
11 14

(d)

12	13	11	14	15	17

12 13
11 13

(e)

12	11	13	14	15	17

11 12

(f)

11	12	13	14	15	17

BubbleSort: running time

- Scan the list several times
  – At each scan, compare adjacent pairs of elements, and swap them if they are not in the correct order
  – If no elements were swapped during a scan, you can stop because the list is sorted
- Each scan requires O(n) time
- We have at most O(n) scans
- Therefore, running time is $O(n^2)$
- Is the algorithm correct? Yes
  – After k-th scan, the k largest elements are at the end of the list and they are correctly sorted

Can we do better?

- The three sorting algorithms described so far are simple to understand but relatively slow
  – Running time: $O(n^2)$
- Let's analyze a faster sorting algorithm

MergeSort

- Based on a powerful algorithm design technique known as divide and conquer
- Also known as divide et impera since it was used by Romans to conquer and rule other peoples
- BinarySearch is also based on divide and conquer
  - Divide: split the sorted list into two halves by comparing the target value with the middle element
  - Conquer: narrow down the search to one of the halves based on the comparison

Divide and conquer

- Top-down approach:

1. **Divide** the original problem into two or more subproblems that are similar to the original problem but of smaller size
2. **Conquer** the subproblems by solving them *recursively*. If the size of the subproblems is small enough, solve them directly without recursion (base case)
3. **Combine** the solutions of the subproblems to form a solution to the original problem

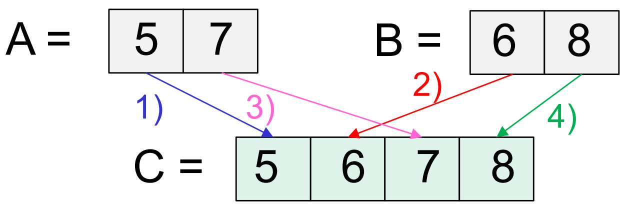
Divide and conquer applied to MergeSort

- Let’s apply divide and conquer to MergeSort

1. Divide: split the list to be sorted in two (roughly equal) halves
2. Conquer: recursively solve the two subproblems
   - If the list has one element, it is sorted
   - Otherwise, return to step 1 to sort it
3. Combine: merge the two sorted lists

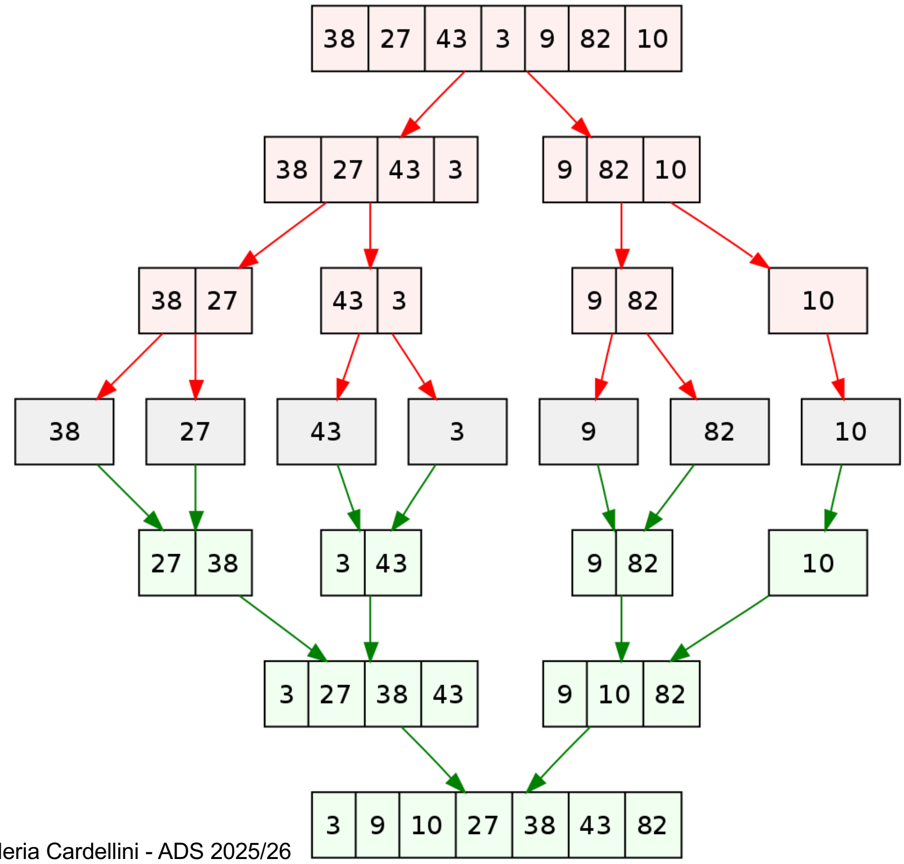
Merging two sorted lists

- Two sorted lists A e B can be merged as follows:
  - Choose the minimum from A and B, copy it to output list C and remove it from A or B. Repeat until either A or B becomes empty
  - Take all remaning elements from the non-empty list and copy them to the end of C
  - Example:



- Running time for merging: O(n), where n is the total number of elements

MergeSort: example



MergeSort: with words

- Divide the array $L[l : r]$ to be sorted into two subarrays, each of half the size. To do so, compute the midpoint m of $L[l : r]$ (by taking the average of l and r), and split $L[l : r]$ into subarrays $L[l : m]$ and $L[m+1 : r]$
- Conquer by sorting each of the two subarrays $L[l : m]$ and $L[m+1 : r]$ *recursively* using MergeSort
- Combine by merging the two sorted subarrays $L[l : m]$ and $L[m+1 : r]$ back into $L[l : r]$, producing the sorted output

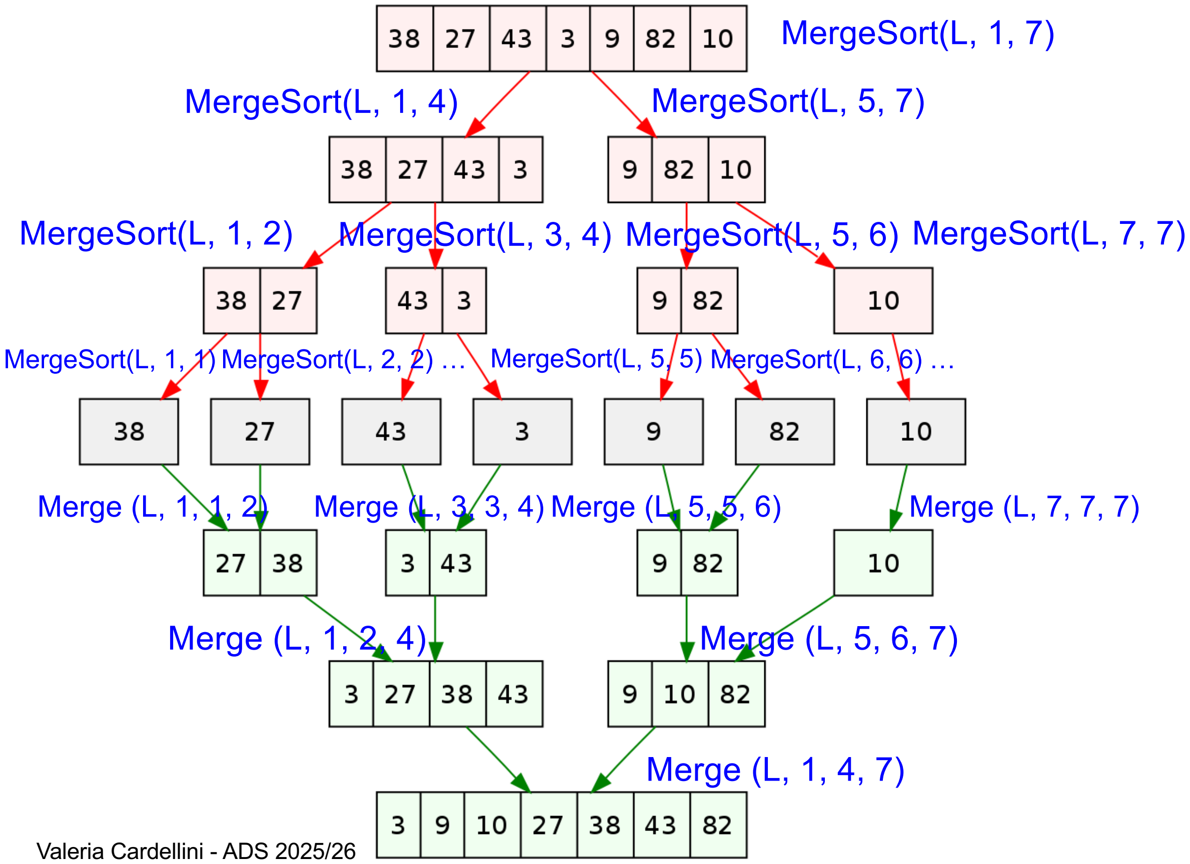
MergeSort: pseudocode

algorithm `MergeSort`(*list L, integer l, integer r*)
  if $l < r$ **then**
    $m = \lfloor (l+r)/2 \rfloor$ ▷ find midpoint of $L[l:r]$
    MergeSort(L, l, m) ▷ call MergeSort for first half
    MergeSort(L, $m+1$, r) ▷ call MergeSort for second half
    $\boxed{\text{Merge}(L,\ l,\ m,\ r)}$ ▷ merge the two sorted halves
  end if

Merge: pseudocode

algorithm `Merge`(*list L, integer l, integer m, integer r*)
  $n_1 = m - l + 1$ ▷ lenght of $L[l{:}m]$
  $n_2 = r - m$ ▷ lenght of $L[m{+}1{:}r]$
  Let $A[1 \dots n_1 + 1]$ and $B[1 \dots n_2 + 1]$ be new arrays
  for $i = 1$ to n_1 **do** ▷ copy $L[l{:}m]$ into $A[1{:}n_1]$
    $A[i] = L[l + i - 1]$
  end for
  for $j = 1$ to n_2 **do** ▷ copy $L[m{+}1{:}r]$ into $B[1{:}n_2]$
    $B[j] = L[m + j]$
  end for
  $A[n_1 + 1] = \infty$
  $B[n_2 + 1] = \infty$
  $i = 1$ ▷ i indexes the smallest remaining element in A
  $j = 1$ ▷ j indexes the smallest remaining element in B
  for $k = l$ to r **do**
    ▷ at each step copy the smallest unmerged element back into $L[l{:}r]$
    if $A[i] \leq B[j]$ **then**
      $L[k] = A[i]$
      i = i+1
    else
      $L[k] = B[j]$
      $j = j + 1$
    end if
  end for

MergeSort and its call tree: example



MergeSort: running time

- Let T(n) be the number of comparisons done by MergeSort to sort n elements:
  - T(n/2) comparisons to sort the first half (n/2 elements)
  - T(n/2) comparisons to sort the second half (n/2 elements)
  - cn comparisons to merge the two sorted subarrays, for some constant c
- T(n) = 2T(n/2) + cn

MergeSort: running time

- Let $T(n)$ be the number of comparisons done by MergeSort to sort n elements:

  $T(n) = 2T(n/2) + cn$

  $T(1) = 1$
- $T(1) = 1$ (if the list has only one element, nothing to do)
- It is a recurrence. How do we solve it?

MergeSort: running time

- We have to solve

  $$T(n) = 2T(n/2) + cn$$
- If this is true, then we can also say

  $$T(n/2) = 2T(n/4) + cn/2$$
- Putting this back into the recurrence, we obtain:

  $$T(n) = 2T(n/2) + cn = 4T(n/4) + 2cn/2 + cn$$
- i.e.,

  $$T(n) = 4T(n/4) + 2cn$$

MergeSort: running time

- We have
  $T(n) = 2T(n/2) + cn = 4T(n/4) + 2cn$
- We can also say
  $T(n/4) = 2T(n/8) + cn/4$
- Putting this back into the recurrence, we obtain:
  $T(n) = 4T(n/4) + 2cn = 8T(n/8) + 4cn/4 + 2cn$
- i.e.,
  $T(n) = 8T(n/8) + 3cn$

MergeSort: running time

- We can go on like this:
  $T(n) = 8T(n/8) + 3cn = 16T(n/16) + 4cn = ...$
- In general, for k = 1, 2, 3,...
  $T(n) = 2^k\, T(n/2^k) + kcn$
- When do we stop? Cannot go on forever…

MergeSort: running time

- We have:
  $T(n) = 2^k T(n/2^k) + kcn$
- When do we stop? Cannot go on forever…
- We can stop when k is such that $n/2^k = 1$
- i.e., $n = 2^k$, i.e., $k = \log_2 n$. For this value of k:
  $T(n) = 2^k T(n/2^k) + kcn = nT(1) + c\, n \log_2 n$
- Namely:
  $T(n) = n + c\, n \log_2 n = O(n \log n)$

MergeSort: summary

- The number of comparisons T(n) done by MergeSort to sort n elements is described by the following recurrence:
  $T(n) = 2T(n/2) + cn$
- The solution of this recurrence is
  $T(n) = O(n \log n)$
- MergeSort sorts n elements in time O(n log n)

Take-away

- We described four sorting algorithms:
  – SelectionSort, InsertionSort, BubbleSort: $O(n^2)$
  – MergeSort: $O(n \log n)$

n	10	100	1000	10^6	10^9
$n \log_2 n$	~ 33	~ 664	~ 10^4	~ 2×10^7	~ 3×10^{10} 3×10^{10}
n^2	100	10^4	10^6	10^{12}	10^{18}

- If we have to sort 1 billion ($n=10^9$) of elements, and each step takes 10 nanosec. (10^{-8} sec.):
  – n^2 steps take: $10^{18}\times10^{-8}$ sec $=10^{10}$ sec ~ 317 years!
  – $n \log_2 n$ steps take: $\log_2(3\times10^{10})\times10^{-8}$ sec ~ 5.6 sec!

Exercise

- Apply the four algorithms to sort the following list in non-decreasing order: 16, 9, 48, 11, 21, 34, 10, 9
- To check your solution, you can also sort the list using https://visualgo.net/bn/sorting by first creating the list of numbers and then applying the different algorithms

References

- T.H. Cormen, C.E. Leiserson, R.L. Rivest, C. Stein. Introduction to Algorithms, 4th ed., MIT Press, 2022
- C. Demetrescu, I. Finocchi, G. F. Italiano. Algoritmi e Strutture Dati 3ª edizione, Mc-Graw Hill, 2025 (in Italian)