

Returns to Scale

Returns to scale describe the relationship between an increase in the use of inputs and the resulting variation in output. There are different types of returns to scale:

- **Increasing returns to scale (IRS)**: As the use of inputs increases, the output increases in a more than linear way.
- **Constant returns to scale (CRS)**: As the use of inputs increases, the output increases in a linear way.
- **Decreasing returns to scale (DRS)**: As the use of inputs increases, the output increases in a less than linear way.

To verify returns to scale, we can multiply the entire function by a constant factor $\lambda > 0$, and the function will exhibit:

- IRS if $f(\lambda K, \lambda L) > \lambda f(K, L)$, because increasing inputs by a constant factor λ results in a more than proportional increase in output.
- CRS if $f(\lambda K, \lambda L) = \lambda f(K, L)$, because increasing inputs by a constant factor λ results in a proportional increase in output.
- DRS if $f(\lambda K, \lambda L) < \lambda f(K, L)$, because increasing inputs by a constant factor λ results in a less than proportional increase in output.

Determining Returns to Scale with the Cobb-Douglas Production Function

The Cobb-Douglas production function is widely used to model different types of returns to scale in economics. It is given by:

$$f(K, L) = AK^\alpha L^\beta$$

where:

- K and L are inputs for capital and labor, respectively.
- α and β are the output elasticities of capital and labor.
- A is a constant representing total factor productivity.

To analyze the returns to scale, we scale both inputs by a factor $\lambda > 0$ and observe the effect on output:

$$f(\lambda K, \lambda L) = A(\lambda K)^\alpha (\lambda L)^\beta = A\lambda^\alpha K^\alpha \lambda^\beta L^\beta = A\lambda^{\alpha+\beta} K^\alpha L^\beta$$

We compare the scaled output to the original function:

$$\frac{f(\lambda K, \lambda L)}{f(K, L)} = \frac{A\lambda^{\alpha+\beta} K^\alpha L^\beta}{AK^\alpha L^\beta} = \lambda^{\alpha+\beta}$$

The type of returns to scale can be determined as follows:

- **Increasing Returns to Scale (IRS)** if $\alpha + \beta > 1$
- **Constant Returns to Scale (CRS)** if $\alpha + \beta = 1$
- **Decreasing Returns to Scale (DRS)** if $\alpha + \beta < 1$

Exercise 1

Given the following production functions, establish the returns to scale that each of the functions exhibits, providing a clear definition of the return to scale type:

1. $f(K, L) = 2(L + K)$
2. $f(K, L) = L^{1/2}K^{2/3}$
3. $f(K, L) = 2(LK)^{1/2}$
4. $f(K, L) = L + K^2$
5. $f(K, L) = L^3K^5$

Solution:

Let's solve the exercise using the two strategies we have learned:

1. $f(K, L) = 2(L + K) \implies f(\lambda K, \lambda L) = 2(\lambda L + \lambda K) = \lambda[2(L + K)]$ (CRS)
2. $f(K, L) = L^{1/2}K^{2/3} \implies$ Scaling exponent $= \frac{1}{2} + \frac{2}{3} = \frac{7}{6} > 1$ (IRS)
3. $f(K, L) = 2(LK)^{1/2} \implies$ Scaling exponent $= \frac{1}{2} + \frac{1}{2} = 1$ (CRS)
4. $f(K, L) = L + K^2 \implies f(\lambda K, \lambda L) = \lambda L + (\lambda K)^2 = \lambda L + \lambda^2 K^2$ (DRS)
5. $f(K, L) = L^3K^5 \implies$ Scaling exponent $= 3 + 5 = 8 > 1$ (IRS)

Returns to Scale and Average Cost

Returns to scale indicate how production increases with an increase in production factors. Understanding how the average cost of a company varies also helps us understand what happens as its production increases. Specifically:

- **Constant Average Cost:** If the average cost remains constant as production increases, the average of costs for all units produced remains stable. This implies that costs also remain stable because the production factors increase proportionally.
 - Constant returns to scale
- **Increasing Average Cost:** If the average cost increases as production increases, the average of costs for all units produced increases. This implies that costs increase because the production factors increase less than proportionally, and continuing to produce becomes less costly.
 - Decreasing returns to scale
- **Decreasing Average Cost:** If the average cost decreases as production increases, the average of costs for all units produced decreases. This implies that costs decrease because the production factors increase more than proportionally, and continuing to produce becomes more costly.
 - Increasing returns to scale

Exercise

Given the following cost functions, calculate the returns to scale by examining the behavior of the average cost:

1. $CT(Q) = 5Q$
2. $CT(Q) = 3Q^{1.3}$
3. $CT(Q) = 8Q^{0.4}$

Profit Maximisation

We define profit as the difference between total revenues (everything a producer can produce at the price they can sell) and total costs (all the factors of production used multiplied by their prices). The profit function can be expressed as follows:

$$\pi = pY - (wL + rK) = pY - wL - rK$$

Constraint: A producer's constraints concern their production capacity Y , which is dependent on their production function $f(L, K)$. Thus, the problem is to maximize profit under the constraint given by the production function:

$$\text{Maximize } \pi = pY - wL - rK \quad \text{subject to } Y = f(L, K)$$

This can be directly incorporated into the function to be maximized:

$$\max_{L, K} (p[f(L, K)] - wL - rK)$$

Short-term Maximization: Conventionally, labor (L) is considered a factor of production that can be easily modified in a firm, while capital (K) is not. Therefore, in the short term, we keep the capital constant and optimize only in terms of labor:

$$\text{Maximize } \pi = p[f(L, K)] - wL - rK \quad \text{subject to } K = \bar{K}$$

To optimize this function, we compute its first derivative with respect to L , set it to zero, and solve for L .

Exercise

Given the following production function and data, calculate the labor demand function that solves the short-run profit maximization problem:

$$f(K, L) = L^{1/2}K^{1/2}, \quad p = 10, \quad K = 100, \quad w = 5, \quad r = 5$$

Solution:

$$\max_L (pf(L, K) - wL - rK) = \max_L (10 \cdot L^{1/2} \cdot 100^{1/2} - 5L - 500)$$

Let us compute the optimal amount of labor to maximize the profit:

$$\max_L (100L^{1/2} - 5L - 500)$$

Setting the first derivative with respect to L to zero, we solve for L :

$$50L^{-1/2} - 5 = 0 \quad \Rightarrow \quad L = 100$$

We then calculate the optimal level of output and the relative profit:

$$Q = f(K, L) = 100^{1/2} \cdot 100^{1/2} = 1000, \quad \pi(Q) = pQ - CT(Q) = 10 \cdot 1000 - 1000 = 0$$