Microeconomics
Cost Minimization

June 24, 2024

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The Production Function and Isoquants

The production function is the set of points, the geometric locus, that associates any available combination of inputs with the maximum obtainable level of output. Thus, it is a function where the quantity produced Y depends on the factors of production, such as capital K and labor L.

$$Y = f(K, L)$$

The isoquant is a function that represents all combinations (K, L) of inputs that provide an output-efficient manner a certain level of output, q. The isoquant therefore describes all combinations of capital and labor, all production techniques, that allow producing output q in an output-efficient manner (all combinations that have as output-efficient $Y = q$).

$$q = Y = f(K, L)$$

Thus, we know that the isoquant is the geometric locus (the set of points) where the production function is maximized. And as we have always said so far, if we want to maximize a function, the most immediate method (always provided that there are the necessary conditions to do so) is to calculate the first derivative and set it to zero. Thus, we know that along the isoquant the following condition is verified:

$$dY = 0 \to dK \cdot \frac{\partial Y}{\partial K} + dL \cdot \frac{\partial Y}{\partial L} = 0 \to f_K dK + f_L dL = 0$$

where $f_K = MP_K$ is the marginal productivity of capital, and $f_L = MP_L$ is the marginal productivity of labor. Thus, we obtain:

$$f_K dK + f_L dL = 0 \to MP_K \cdot dK + MP_L \cdot dL = 0 \to \frac{MP_L}{MP_K} = -\frac{dK}{dL}$$

Finally, we define the Marginal Rate of Technical Substitution as follows:

$$\frac{MP_L}{MP_K} = \left| \frac{dK}{dL} \right| = MRTS$$

Thus, we know that on the isoquant the MRTS, which by definition is the ratio of marginal productivities, is equal to the absolute value of the ratio of total derivatives.

Isocosts

The firm's total cost function represents the minimum cost of producing any quantity Q. We can easily divide the firm's total cost into fixed costs, those costs that the entrepreneur must incur regardless of the quantity produced, which will thus be represented mathematically by a constant, and variable costs, those costs that vary depending on the quantity produced, which will thus be represented mathematically by a function of quantity.

$$TC(Q) = FC + VC(Q)$$

The isocost is the geometric locus of combinations of labor and capital factor productive techniques all characterized by the same cost for the entrepreneur. In the case of the two production factors already introduced in the production function, L and K, defining their remuneration respectively w the wage, the salary, of workers and r the interest rate of capital, we can write:

$$\bar{c} = TC(L, K) = wL + rK \rightarrow K = -\frac{w}{r}L + \frac{\bar{c}}{r}$$

The Cost Minimization Problem in the Long Run

Now let's solve the cost minimization problem following a parallel with consumer theory and relative profit maximization. We can set up a problem as follows:

- **Utility Maximization**: In this type of problem, we must maximize the utility that a consumer can derive given their utility function (and thus their MRS) and given an income and the market prices of the two goods the consumer can choose between (and thus their budget constraint and the consequent relative prices). To calculate the optimal consumption bundle, we must calculate the tangency point between the budget constraint and the related indifference curve, that is the maximum point. To do this, we construct the following system:

$$\begin{cases} MRS = \frac{p_1}{p_2} \\ R = p_1 x_1 + p_2 x_2 \end{cases}$$

- **Cost Minimization**: In this type of problem, we must minimize the expense that an entrepreneur must incur given their total cost function and the remuneration of labor and capital (and thus the ratio between remuneration, the slope of all isocosts) and given their technology, their production function, and a level of output (and thus their MRTS and a specific isoquant). To calculate the optimal total cost, we must calculate the tangency point between the isoquant and isocost, that is the minimum point. To do this, we construct the following system:

$$\begin{cases} MRTS = \frac{w}{r} \\ q = f(K, L) \end{cases}$$

Exercise 1

Given the following production function:

$$f(L, K) = L^{\frac{1}{4}} K^{\frac{1}{4}}$$

1. Calculate the isoquant corresponding to the level $q = 200$.

2. Solve the cost minimization problem with the classic total cost formula ($TC = wL + rK$), for the isocost calculated in the first point and for the following data $w = 16$ and $r = 1$.

Solution

1. $K^{\frac{1}{4}}L^{\frac{1}{4}} = 200 \rightarrow K = 200^4 L^{-1}$

2. Solve the minimization problem:

$$MRTS = \frac{\partial Y}{\partial L} \Big/ \frac{\partial Y}{\partial K} = \frac{MP_L}{MP_K} = \frac{\frac{1}{4}L^{-\frac{3}{4}}K^{\frac{1}{4}}}{\frac{1}{4}L^{\frac{1}{4}}K^{-\frac{3}{4}}} = \frac{K}{L}$$

$$\begin{cases} MRTS = \frac{w}{r} \\ q = f(L, K) \end{cases}$$

$$\begin{cases} \frac{K}{L} = 16 \rightarrow K = 16L \\ 200 = K^{\frac{1}{4}}L^{\frac{1}{4}} \end{cases}$$

$$200 = (16L)^{\frac{1}{4}}L^{\frac{1}{4}}$$

$$200 = 16^{\frac{1}{4}}L^{\frac{1}{2}}$$

$$L = 10000$$

$$K = 160000$$

The Cost Minimization Problem in the Short Run

The cost minimization problem in the short run consists of choosing the quantities of variable inputs that minimize the total costs necessary to produce a level of output chosen by the producer, under the constraint that the quantities of fixed factors do not change. In the short run, capital is fixed, and we will denote it as $\bar{K}$. To calculate the optimal quantity of labor for a specific quantity of good to be produced, just solve the equation in a single unknown that we obtain by substituting capital within the isoquant:

$$\bar{q} = f(\bar{K}, L) \rightarrow L^* = ...$$

Exercise 2

Given the isoquant

$$\bar{q} = 100 = f(L, K) = L^{\frac{1}{4}}K^{\frac{3}{4}}$$

for $w = 1$ and $r = 3$, find which combination of inputs allows the entrepreneur to achieve the identified level of production efficiently in the short run (with $\bar{K} = 1000$) and in the long run.

Solution

1. **Short Run**

$$100 = L^{\frac{1}{4}}\bar{K}^{\frac{3}{4}}$$

$$100 = L^{\frac{1}{4}}1000^{\frac{3}{4}}$$

$$L^{\frac{1}{4}} = \frac{100}{1000^{\frac{3}{4}}} \rightarrow L = \frac{1}{10}$$

2. **Long Run**

$$MRTS = \frac{\partial Y}{\partial L}/\frac{\partial Y}{\partial K} = \frac{MP_L}{MP_K} = \frac{\frac{1}{4}L^{-\frac{3}{4}}K^{\frac{3}{4}}}{\frac{3}{4}L^{\frac{1}{4}}K^{-\frac{1}{4}}} = \frac{1}{3}\frac{K}{L}$$

$$\begin{cases} MRTS = \frac{w}{r} \\ \bar{q} = f(L,K) \end{cases}$$

$$\begin{cases} \frac{K}{3L} = \frac{1}{3} \\ 100 = L^{\frac{1}{4}}K^{\frac{3}{4}} \end{cases}$$

$$\begin{cases} K = L \\ 100 = L^{\frac{1}{4}}L^{\frac{3}{4}} \end{cases}$$

$$\begin{cases} K = 100 \\ L = 100 \end{cases}$$