Exercise Set 3: The Firm's Choices

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1 Isoquants and Marginal Rate of Technical Substitution

Given the following production function

$$Y = f(L, K) = 4L^2K^2$$

compute:

1. the equation of a generic isoquant;
2. the Marginal Rate of Technical Substitution.

1.1 Solution

An isoquant is the level curve of the production function for a given output level $\overline{q}$. Therefore, we fix:

$$\overline{q} = f(L, K) = 4L^2K^2$$
$$L^2 = \frac{\overline{q}}{4K^2} \longrightarrow L = \frac{\sqrt{\overline{q}}}{2K}$$

assuming $K > 0$.

The MRTS is the ratio between the marginal productivity of labor and the marginal productivity of capital.

The MRTS is obtained by defining the marginal change in output as a function of the two inputs and setting it equal to zero:

$$dY = \frac{\partial f(L,K)}{\partial K}dK + \frac{\partial f(L,K)}{\partial L}dL = 0$$
$$\frac{\partial f(L,K)}{\partial K}dK = -\frac{\partial f(L,K)}{\partial L}dL$$
$$\frac{dK}{dL} = -\frac{\frac{\partial f(L,K)}{\partial L}}{\frac{\partial f(L,K)}{\partial K}}$$
$$MRTS = \left|\frac{dK}{dL}\right| = \frac{\frac{\partial f(L,K)}{\partial L}}{\frac{\partial f(L,K)}{\partial K}}$$

In this exercise:

$$\frac{\partial f(L,K)}{\partial L} = 8LK^2, \qquad \frac{\partial f(L,K)}{\partial K} = 8L^2K.$$

Therefore:

$$MRTS = \frac{8LK^2}{8L^2K} = \frac{K}{L}.$$

2 Isocost Curves

Given the following cost function:

$$TC = 20L + 10K$$

compute:

1. the equation of a generic isocost line;
2. the equations of the isocosts corresponding to the levels $\bar{c} = 1$ and $\bar{c} = 5$, and provide a graphical representation.

2.1 Solution

In general, the isocost equation defines the combinations of production factors that generate a given level of cost:

$$\bar{c} = wL + rK \longrightarrow K = -\frac{w}{r}L + \frac{\bar{c}}{r}$$

To compute the equations of specific isocosts, simply substitute the corresponding values:

- for $\bar{c} = 1$:
$$1 = 20L + 10K \longrightarrow K = -2L + \frac{1}{10}$$
- for $\bar{c} = 5$:
$$5 = 20L + 10K \longrightarrow K = -2L + \frac{5}{10} = -2L + \frac{1}{2}$$

3 Costs and Productivity

Given the following production and cost functions:

$$Y = f(K, L) = K^{\frac{2}{5}}L^{\frac{3}{5}}$$
$$C(Q) = 2Q(Q + 20) + 100 = 2Q^2 + 40Q + 100$$

compute:

1. Variable cost
$$VC(Q) = 2Q^2 + 40Q$$
2. Fixed cost
$$FC(Q) = 100$$
3. Average cost
$$AC(Q) = \frac{C(Q)}{Q} = \frac{2Q^2 + 40Q + 100}{Q} = 2Q + 40 + \frac{100}{Q}$$
4. Average variable cost
$$AVC(Q) = \frac{VC(Q)}{Q} = \frac{2Q^2 + 40Q}{Q} = 2Q + 40$$
5. Average fixed cost
$$AFC(Q) = \frac{FC(Q)}{Q} = \frac{100}{Q}$$

6. Marginal cost

$$MC(Q) = \frac{\partial C(Q)}{\partial Q} = 4Q + 40$$

7. Average product of labor

$$APL = \frac{Y}{L} = \frac{K^{\frac{2}{5}}L^{\frac{3}{5}}}{L} = K^{\frac{2}{5}}L^{-\frac{2}{5}} = \left(\frac{K}{L}\right)^{\frac{2}{5}}$$

8. Average product of capital

$$APK = \frac{Y}{K} = \frac{K^{\frac{2}{5}}L^{\frac{3}{5}}}{K} = K^{-\frac{3}{5}}L^{\frac{3}{5}} = \left(\frac{L}{K}\right)^{\frac{3}{5}}$$

9. Marginal product of labor

$$MPL = \frac{\partial Y}{\partial L} = \frac{3}{5}K^{\frac{2}{5}}L^{-\frac{2}{5}} = \frac{3}{5}\left(\frac{K}{L}\right)^{\frac{2}{5}}$$

10. Marginal product of capital

$$MPK = \frac{\partial Y}{\partial K} = \frac{2}{5}K^{-\frac{3}{5}}L^{\frac{3}{5}} = \frac{2}{5}\left(\frac{L}{K}\right)^{\frac{3}{5}}$$

4 The Long-Run Cost Minimization Problem

Given the following production function:

$$Y = K^{\frac{1}{4}}L^{\frac{1}{4}}$$

Solve the cost minimization problem for the isoquant corresponding to the output level $q = 200$, knowing that:

$$TC = wL + rK,$$

where $w = 16$ and $r = 1$.

4.1 Solution

To find the isoquant equation, we set:

$$K^{\frac{1}{4}}L^{\frac{1}{4}} = 200$$

Raising both sides to the fourth power:

$$KL = 200^4 \longrightarrow K = \frac{200^4}{L}.$$

To solve the minimization problem, we must solve the following system:

$$\begin{cases} MRTS = \frac{w}{r} \\ \overline{q} = f(L, K) \end{cases}$$

Let us compute the MRTS:

$$\begin{aligned}
MRTS &= \frac{\frac{\partial f(L,K)}{\partial L}}{\frac{\partial f(L,K)}{\partial K}} \\
&= \frac{\frac{1}{4}K^{\frac{1}{4}}L^{-\frac{3}{4}}}{\frac{1}{4}K^{-\frac{3}{4}}L^{\frac{1}{4}}} \\
&= \frac{K}{L}.
\end{aligned}$$

Since:

$$\frac{w}{r} = \frac{16}{1} = 16,$$

we obtain:

$$\frac{K}{L} = 16 \longrightarrow K = 16L.$$

Substituting into the isoquant:

$$\begin{aligned}
K &= \frac{200^4}{L} \\
16L &= \frac{200^4}{L} \\
16L^2 &= 200^4 \\
L^2 &= \frac{200^4}{16} \\
L &= \frac{200^2}{4} = 10000.
\end{aligned}$$

Therefore:

$$K = 16L = 16 \cdot 10000 = 160000.$$

Hence, the optimal combination of inputs is:

$$\boxed{L^* = 10000, \qquad K^* = 160000.}$$