Returns to Scale and Profit Maximization

Microeconomics Exercises

Contents

Preliminary notions on returns to scale

A production function $f(K, L)$ is said to exhibit **returns to scale**:
- **Increasing** when, increasing all production factors by the same proportion, output increases by a *more than proportional* amount.
- **Constant** when, increasing all production factors by the same proportion, output increases in the *same proportion.*
- **Decreasing** when, increasing all production factors by the same proportion, output increases by a *less than proportional* amount.

General criterion

To verify the returns to scale of a function $f(K, L)$, it is sufficient to multiply each factor by a constant $\lambda > 0$ and compare $f(\lambda K, \lambda L)$ with $\lambda f(K, L)$:

$$\begin{cases} f(\lambda K, \lambda L) > \lambda\, f(K, L) & \Rightarrow \text{increasing returns} \\ f(\lambda K, \lambda L) = \lambda\, f(K, L) & \Rightarrow \text{constant returns} \\ f(\lambda K, \lambda L) < \lambda\, f(K, L) & \Rightarrow \text{decreasing returns} \end{cases}$$

Cobb–Douglas case

A Cobb–Douglas production function has the general form:

$$f(K, L) = K^{\alpha} L^{\beta}$$

Applying the previous criterion:

$$f(\lambda K, \lambda L) = (\lambda K)^{\alpha} (\lambda L)^{\beta} = \lambda^{\alpha+\beta} K^{\alpha} L^{\beta} = \lambda^{\alpha+\beta} f(K, L)$$

The comparison with $\lambda\, f(K, L)$ therefore reduces to comparing $\lambda^{\alpha+\beta}$ with λ^1, from which:

$$\begin{cases} \alpha + \beta > 1 & \Rightarrow \text{increasing returns} \\ \alpha + \beta = 1 & \Rightarrow \text{constant returns} \\ \alpha + \beta < 1 & \Rightarrow \text{decreasing returns} \end{cases}$$

Exercise 1 — Identifying returns to scale

Problem. For each of the following production functions, determine the type of returns to scale. Accompany each answer with a brief description of the economic meaning of the result obtained.

1. $f(K, L) = 2(L + K)$
2. $f(K, L) = L^{\frac{1}{2}}\, K^{\frac{2}{6}}$
3. $f(K, L) = 2(LK)^{\frac{1}{2}}$
4. $f(K, L) = L + K^2$
5. $f(K, L) = L^3 K^5$

Solution

1. $f(K, L) = 2(L + K)$
Let us compute $f(\lambda K, \lambda L)$:

$$f(\lambda K, \lambda L) = 2(\lambda L + \lambda K) = 2\lambda(L + K) = \lambda \cdot 2(L + K) = \lambda f(K, L)$$

Since $f(\lambda K, \lambda L) = \lambda f(K, L)$, the function exhibits **constant returns to scale**.

2. $f(K, L) = L^{\frac{1}{2}} K^{\frac{2}{6}}$
This is a Cobb–Douglas function; let us verify the sum of the exponents:

$$\frac{1}{2} + \frac{2}{6} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} < 1$$

The function therefore exhibits **decreasing returns to scale**.

3. $f(K, L) = 2(LK)^{\frac{1}{2}} = 2\, L^{\frac{1}{2}} K^{\frac{1}{2}}$
Sum of the exponents:

$$\frac{1}{2} + \frac{1}{2} = 1$$

The function exhibits **constant returns to scale**.

4. $f(K, L) = L + K^2$
This function is *not* Cobb–Douglas; therefore we use the general criterion:

$$f(\lambda K, \lambda L) = \lambda L + (\lambda K)^2 = \lambda L + \lambda^2 K^2$$

Compare with $\lambda f(K, L) = \lambda L + \lambda K^2$:

$$\lambda L + \lambda^2 K^2 \quad \text{vs} \quad \lambda L + \lambda K^2$$

Since $\lambda^2 > \lambda$ for every $\lambda > 1$, we have $f(\lambda K, \lambda L) > \lambda f(K, L)$: the function exhibits **increasing returns to scale**.

5. $f(K, L) = L^3 K^5$
Sum of the exponents:

$$3 + 5 = 8 > 1$$

The function exhibits **increasing returns to scale**.

Relationship between returns to scale and average cost

The **average cost** of a firm, defined as $\text{AC}(Q) = TC(Q)/Q$, is closely related to the returns to scale of its production technology:

- AC **constant** as Q increases $\Rightarrow$ inputs increase proportionally $\Rightarrow$ **constant returns**.
- AC **increasing** as Q increases $\Rightarrow$ inputs grow less than proportionally $\Rightarrow$ **decreasing returns**.
- AC **decreasing** as Q increases $\Rightarrow$ inputs grow more than proportionally $\Rightarrow$ **increasing returns**.

Exercise 2 — Returns to scale through average cost

Problem. Given the following total cost functions, derive the firm's returns to scale by analyzing the behavior of average cost as output Q changes.

1. $TC(Q) = 5Q$
2. $TC(Q) = 3Q^{\frac{1}{3}}$
3. $TC(Q) = 8Q^4$

Solution

1. $TC(Q) = 5Q$

$$\mathrm{AC}(Q) = \frac{TC(Q)}{Q} = \frac{5Q}{Q} = 5 \qquad \Rightarrow \qquad \frac{\partial\,\mathrm{AC}}{\partial Q} = 0$$

Average cost is constant: **constant returns to scale**.

2. $TC(Q) = 3Q^{\frac{1}{3}}$

$$\mathrm{AC}(Q) = \frac{3Q^{\frac{1}{3}}}{Q} = 3Q^{-\frac{2}{3}} \qquad \Rightarrow \qquad \frac{\partial\,\mathrm{AC}}{\partial Q} = -2Q^{-\frac{5}{3}} < 0$$

Average cost is decreasing: **increasing returns to scale**.

3. $TC(Q) = 8Q^4$

$$\mathrm{AC}(Q) = \frac{8Q^4}{Q} = 8Q^3 \qquad \Rightarrow \qquad \frac{\partial\,\mathrm{AC}}{\partial Q} = 24Q^2 > 0$$

Average cost is increasing: **decreasing returns to scale**.

Profit maximization in the short run

A firm's **profit** is the difference between total revenue and total costs:

$$\pi = p \cdot Y - (wL + rK) = p \cdot Y - wL - rK$$

where p is the output price, w the wage rate, and r the cost of capital.

Since output is constrained by the production function $Y = f(L, K)$, the profit maximization problem can be written as:

$$\max_{K,L}\ \pi = p\, f(L, K) - wL - rK$$

Short run. In the short run, capital K is fixed; therefore the problem reduces to an optimization problem in one variable:

$$\max_{L}\ \pi = p\, f(L, \bar{K}) - wL - r\bar{K}$$

The first-order condition is:

$$\frac{\partial \pi}{\partial L} = p \cdot \frac{\partial f}{\partial L} - w = 0 \qquad \Longrightarrow \qquad p \cdot \mathrm{MP}_L = w$$

that is, the value of the marginal product of labor must equal its cost.

Exercise 3 — Optimal labor demand

Problem. A firm operates with the following Cobb–Douglas production function and parameters. Derive the optimal quantity of labor that maximizes profit in the short run, together with the corresponding output level and profit.

$$f(K,L) = L^{\frac{1}{2}} K^{\frac{1}{2}} \qquad p = 10, \quad \bar{K} = 100, \quad w = 5, \quad r = 5$$

Solution

Substituting $\bar{K} = 100$ into the production function:

$$f(L,100) = L^{\frac{1}{2}} \cdot 100^{\frac{1}{2}} = 10\, L^{\frac{1}{2}}$$

The problem becomes:

$$\max_L \ \pi = 10 \cdot 10\, L^{\frac{1}{2}} - 5L - 5 \cdot 100 = 100\, L^{\frac{1}{2}} - 5L - 500$$

First-order condition:

$$\frac{\partial \pi}{\partial L} = \frac{100}{2\, L^{\frac{1}{2}}} - 5 = \frac{50}{L^{\frac{1}{2}}} - 5 = 0$$

$$L^{\frac{1}{2}} = \frac{50}{5} = 10 \qquad \Rightarrow \qquad \boxed{L^* = 100}$$

Optimal output:

$$Q^* = f(100,100) = 100^{\frac{1}{2}} \cdot 100^{\frac{1}{2}} = 100$$

Optimal profit:

$$\pi^* = p\, Q^* - wL^* - r\bar{K} = 10 \cdot 100 - 5 \cdot 100 - 5 \cdot 100 = 1000 - 500 - 500 = \boxed{0}$$

Zero profit indicates that the firm exactly covers all of its (economic) costs, a situation that is typical of a perfectly competitive market in the long run.

Perfect competition in the short and long run

In a perfectly competitive market, every firm is a *price-taker*: it accepts the price p determined by the market and chooses the level of output Q that maximizes profit. The unconstrained problem is:

$$\max_Q \ \pi = TR - TC = pQ - TC(Q)$$

Taking the derivative with respect to Q and setting it equal to zero:

$$\frac{\partial \pi}{\partial Q} = p - \frac{\partial TC(Q)}{\partial Q} = p - \mathrm{MC}(Q) = 0 \qquad \Longrightarrow \qquad \boxed{p = \mathrm{MC}(Q)}$$

The **short-run optimality condition** therefore requires price to equal marginal cost. Short-run profit is:

$$\pi_{SR} = p\, Q_{SR} - TC(Q_{SR})$$

Long-run equilibrium

In the long run, free entry and exit drive economic profit to zero. Imposing $\pi = 0$:

$$\pi = Q\big(p - \mathrm{AC}(Q)\big) = 0 \qquad \Longrightarrow \qquad p = \mathrm{AC}(Q)$$

Combining this condition with the short-run condition ($p = \mathrm{MC}$), we obtain that in the long run the firm produces at the **minimum average cost**:

$$\boxed{p = \mathrm{MC}(Q_{LR}) = \mathrm{AC}(Q_{LR}) = \mathrm{AC}_{\min}}$$

Exercise 4 — Perfect competition: supply and market equilibrium

Problem. In a perfectly competitive market there are 10 identical firms, each with the following total cost function:

$$TC(Q_i) = Q_i^2$$

Market demand is described by:

$$Q^d = 100 - 20p$$

Determine:

1. The short-run supply function of the individual firm.
2. The aggregate short-run supply function of the industry.
3. The equilibrium price and quantity in the market.
4. The individual production level and profit earned by each firm in the short run.

Solution

Point 1 — Individual firm supply.
Applying the optimality condition $p = \text{MC}(Q_i)$:

$$\text{MC}(Q_i) = \frac{\partial TC}{\partial Q_i} = 2Q_i \qquad \Rightarrow \qquad p = 2Q_i \qquad \Rightarrow \qquad \boxed{Q_i^s = \frac{p}{2}}$$

Point 2 — Aggregate industry supply.
With $n = 10$ identical firms, total supply is obtained by summing individual supplies:

$$Q^s = 10 \cdot \frac{p}{2} = \boxed{5p}$$

Point 3 — Market equilibrium.
Set demand equal to supply:

$$Q^d = Q^s \qquad \Rightarrow \qquad 100 - 20p = 5p \qquad \Rightarrow \qquad 25p = 100 \qquad \Rightarrow \qquad \boxed{p^* = 4}$$

Substituting the price into aggregate supply:

$$Q^* = 5 \cdot 4 = \boxed{20}$$

The market equilibrium is therefore $\{Q^* = 20,\ p^* = 4\}$.

Point 4 — Individual production and profit.
Since firms are identical and the market is perfectly competitive, each firm produces the same share of total output:

$$Q_i^* = \frac{Q^*}{n} = \frac{20}{10} = \boxed{2}$$

The individual short-run profit is:

$$\pi_i = p^* \cdot Q_i^* - TC(Q_i^*) = 4 \cdot 2 - (2)^2 = 8 - 4 = \boxed{4}$$

Each firm earns a positive profit equal to 4. In the long run, these profits will attract new firms into the market until profits are driven to zero.