Real numbers

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(slides by Prof. K. Colanieri)

Mathematics I

University of Rome Tor Vergata

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Natural numbers

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On this sets the operation of sum is defined but subtraction cannot always be performed.

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In this set we can sum and subtract numbers. We can also multiply numbers but division between number cannot always be performed

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The set $\mathbb{Q}$ is large enough to make sum, substraction, multiplication and division.

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- k-th power ($k \in \mathbb{N}$):

$$\left(\frac{m}{n}\right)^k = \underbrace{\frac{m}{n} \cdot \frac{m}{n} \cdots \frac{m}{n}}_{k \text{ times}} = \frac{m^k}{n^k}, \quad \text{Ex: } \left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}}$$

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and
$$\left(\frac{m}{n}\right)^{-k} = \left[\left(\frac{m}{n}\right)^{-1}\right]^k = \left[\frac{n}{m}\right]^k = \frac{n^k}{m^k}, \quad \text{Ex: } \left(\frac{1}{2}\right)^{-10} = 2^{10}$$

Operations on $\mathbb{Q}$, cont'd

Given $q \in \mathbb{Q}$, it is possible to show that:

$$q^m \cdot q^n = q^{m+n}$$

for $n,m \in \mathbb{Z}$.

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Given $q \in \mathbb{Q}$, $q \neq 0$, we have $q^0 = 1$. Indeed, for an arbitrary $k \in \mathbb{N}$, we can write:

$$q^0 = q^{k-k} = q^k \cdot q^{-k} = q^k \cdot \frac{1}{q^k} = 1$$

Decimal representation of $\mathbb{Q}$

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- $\frac{7}{12} = 0.5833333\ldots$

The decimal representation of a rational number is either **finite**, as in $\frac{3}{10}$, $-\frac{5}{2}$, or **infinite with a period**, as in $\frac{1}{3}$, $\frac{1}{22}$, $\frac{7}{12}$.

Decimal representation of $\mathbb{Q}$, cont'd

Theorem

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1. *The decimal representation of q is made by a* **finite number of digits**
2. *The decimal representation of q is made by an* **infinite number of digits but it is periodic.** *In this case the period contains at most $m - 1$ digits*

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These numbers are called **irrational numbers**.

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- The Euler's number $e = 2.718281828459045235360287471352662497\dots$

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To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \middle| m \in \mathbb{Z}, n \in \mathbb{Z} \backslash \{0\} \right\} \subset \mathbb{R}$$

Which are the numbers of $\mathbb{R}$ that are not in $\mathbb{Q}$? These numbers are called "irrational numbers" and are those whose decimal representation is not finite, nor periodic.

Examples of irrational numbers:

- $\sqrt{2}$, but even $\sqrt{3}$, $\sqrt{5}$ and more generally all the square roots of numbers which are not perfect square (a perfect square is number q such that $q = n^2$, with $n \in \mathbb{N}$, e.g. $4 = 2^2, 9 = 3^2, 16 = 4^2, 25 = 5^2, 36 = 6^2, \dots$)
- The Euler's number $e = 2.718281828459045235360287471352662497\dots$
- The Pi number $\pi = 3.14159265358979323846264338327950288 4\dots$

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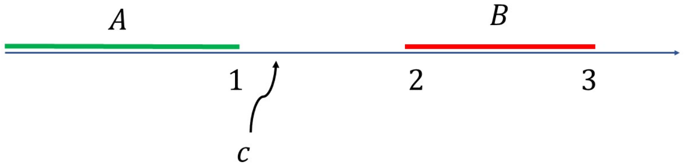
The number c is called the separating point of A and B.

Illustrative example

Consider sets $A = \{x \in \mathbb{R} : x \leq 1\}$ and $B = \{x \in \mathbb{R} : 2 \leq x \leq 3\}$



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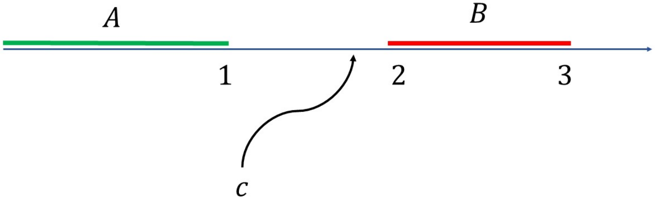

For instance this is another possible value for c.

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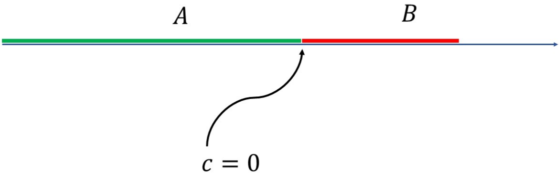
Consider sets $A = \{x \in \mathbb{R} : x \leq 0\}$ and $B = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$



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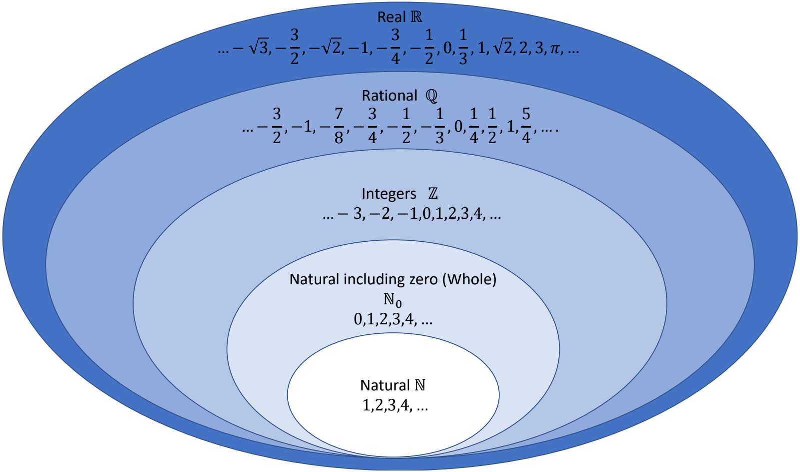
The axiom of completeness is not valid for natural numbers $\mathbb{N}$, integer numbers $\mathbb{Z}$, rational numbers $\mathbb{Q}$.

In summary



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$\ldots -\sqrt{3}, -\frac{3}{2}, -\sqrt{2}, -1, -\frac{3}{4}, -\frac{1}{2}, 0, \frac{1}{3}, 1, \sqrt{2}, 2, 3, \pi, \ldots$
Rational $\mathbb{Q}$
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Integers $\mathbb{Z}$
$\ldots -3, -2, -1, 0, 1, 2, 3, 4, \ldots$
Natural including zero (Whole)
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The set B includes the European cities that are capitals, so that **Milan** $\in A$, but **Milan** $\notin B$.

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- If $A = \{0.2, 0.5, 0.7\}$, then we can write $\forall a \in A,\ 0 < a < 1$

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Remark: In the second example, there exists **ONLY ONE** element in A such that $0 < a \leq \frac{1}{2}$. In this case we can write $\exists! a \in A$. Writing $\exists a \in A$ is correct as well, however $\exists! a \in A$ is more informative

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Union of sets: the $\cup$ operator

Definition

Let A and B be two sets. We denote by $A \cup B$ the set containing all the elements of A and all the elements of B. In symbols:

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- $A = \{1, 3, -5\},\ B = \{1, 2\},\ C = \{1, 55\} \Rightarrow A \cup B \cup C = \{1, 3, -5, 2, 55\}$

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The empty set

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Remark

For any set A, $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$

Subsets

Let A and B be two sets. We write $A \subseteq B$ if all the elements of A are also elements of B.

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- $A = \{\triangle, \bigcirc, \square\}$, $B = \{\triangle, \bigcirc, \square\}$. This time we CANNOT write $A \subset B$ because B has no elements which are not in A. For the same reason, we CANNOT write $B \subset A$

Subsets, cont'd

Remark

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In the first example, it is correct writing either $A \subseteq B$ or $A \subset B$. However, $A \subset B$ is more informative, because it says us not only that all elements of A are in B, but also that B has some elements which are not in A. Similarly, in the second example, it is correct writing either $B \subseteq A$ or $B \subset A$, but the second is more informative.

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The two relations $A \subseteq B$, $A \subset B$, can also be read from right to left:

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Subtraction between sets

Definition

Let A and B be two sets. The "A minus B" set, denoted by $A \backslash B$, is the set containing the elements in A which are not in B. In symbols:

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- $A = \{\triangle, \diamondsuit, \square\}$, $B = \{\triangle, \bigcirc\} \Rightarrow A \backslash B = \{\diamondsuit \square\}$
- $A = \{\text{All natural numbers}\}$, $B = \{\text{All natural numbers larger than } 10\}$

$$A \backslash B = \{\text{All natural numbers lower or equal to } 10\} = \{1, \ldots, 10\}$$

The complement set

Definition

Let S be the universal set and B a subset of S. The complement set of B is the "S minus B" set, namely the set of elements of S that are not contained in B. In symbols:

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- $S = \{\text{All cities in Europe}\}$, $B = \{x \in S | x \text{ is a capital city}\}$

$$B^c = \{x \in S | x \text{ is } \textit{not} \text{ a capital city}\}$$

Some subsets in $\mathbb{R}$: The intervals

Definition

A real interval with extremes $a, b \in \mathbb{R}$ such that $a \leq b$, is the set of all real numbers between a and b.
We say that a real interval is

- **open** if extremes a and b are not included and we denote it by (a, b)
- **closed** if extremes a and b are included and we denote it by $[a, b]$
- **not open nor closed** one of the extreme is included and the other is not, that is $[a, b)$ or $(a, b]$
- **bounded** if both a and b are finite numbers
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Important: Numeric sets with just one element are denoted with the curly parentheses, for instance $\{2\}$ is the set that contains only the number 2.

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- If $x = 0$, then $x \cdot y = 0$

The arrow in $\Rightarrow$ gives the direction of the implication.

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Example:

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- If $x^2 + y^2 = 0$, then $x = 0$ and $y = 0$

In this case assertion P ($x = 0$ and $y = 0$) and assertion Q ($x^2 + y^2 = 0$) are equivalent.

Sufficient and necessary conditions

Suppose that

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We say that P **is a sufficient condition for** Q.

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In summary the following are equivalent:

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- P implies Q

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- P if and only if Q

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In summary the following are equivalent:

- $P \Leftrightarrow Q$
- P is equivalent to Q
- P if and only if Q
- P is a sufficient and necessary condition for Q

The structure of a theorem

A **theorem** is made of

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Distance between a point in $\mathbb{R}$ and the origin

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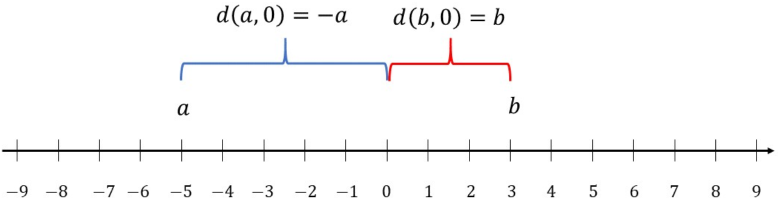
Notice that the length must be positive (or equal to zero).



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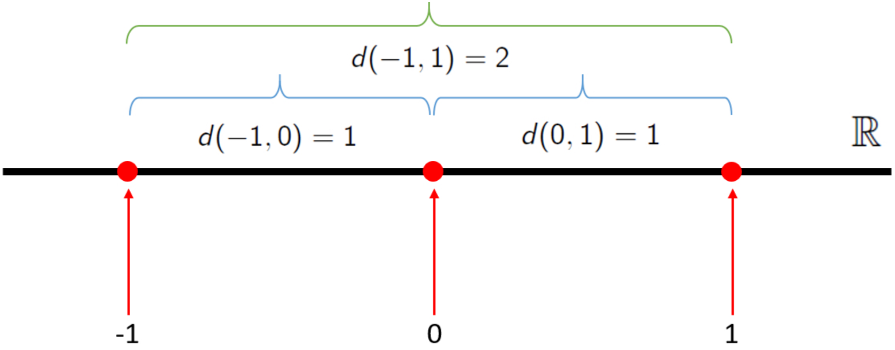

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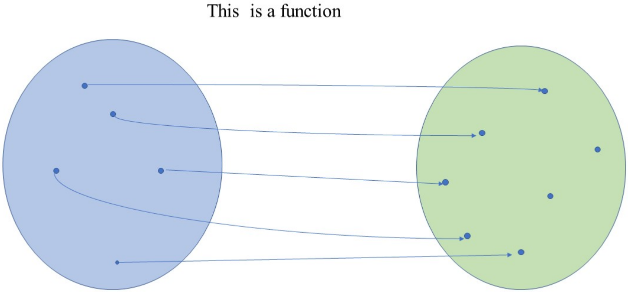

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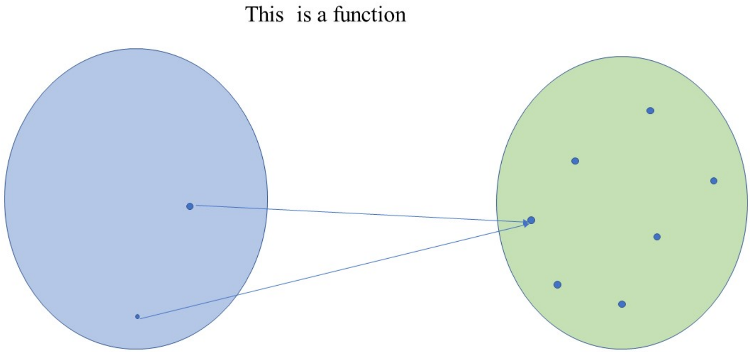
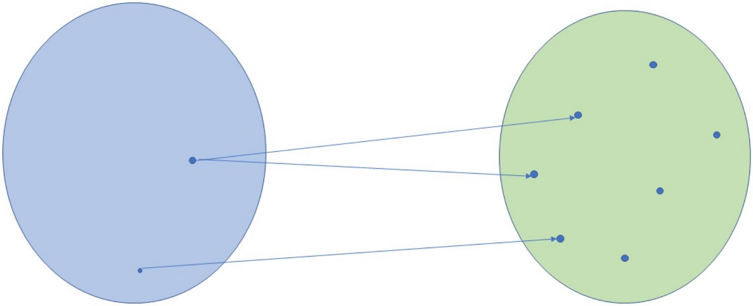


Functions

This is NOT a function



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The set D of all real values for which the law f makes sense is called the *domain* of the function f:

$$D = \{x \in \mathbb{R} : \ f(x) \text{ is well defined } \}$$

The set of all images $f(x)$, for all $x \in D$ is called the *range* of the function

$$R = \{y \in \mathbb{R} : \ y = f(x), \ \forall x \in D\}$$

Definition (Graph)

Let $f : D \to \mathbb{R}$ be a real function of a real variable. The graph of f is the set

$$G = \{(x, f(x)) : \ x \in D\}$$

The plot of the function f is the representation of the graph on a Cartesian plane.