

# Real numbers

Prof. C. Lhotka  
(slides by Prof. K. Colanieri)

Mathematics I

University of Rome Tor Vergata

Week 1, 15-21 Sept 2024

- ① Real Numbers
- ② Axiomatization of Real numbers
- ③ Set theory
- ④ Basics on Logic
- ⑤ Distance in  $\mathbb{R}$
- ⑥ Functions

# Natural numbers

We indicate with  $\mathbb{N}$  the set of natural numbers:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

# Natural numbers

We indicate with  $\mathbb{N}$  the set of natural numbers:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

We indicate with  $\mathbb{N}_0$  the set of natural numbers with zero:

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$$

# Natural numbers

We indicate with  $\mathbb{N}$  the set of natural numbers:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

We indicate with  $\mathbb{N}_0$  the set of natural numbers with zero:

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$$

Of course,  $\mathbb{N} \subset \mathbb{N}_0$ .

# Natural numbers

We indicate with  $\mathbb{N}$  the set of natural numbers:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

We indicate with  $\mathbb{N}_0$  the set of natural numbers with zero:

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$$

Of course,  $\mathbb{N} \subset \mathbb{N}_0$ .

On this sets the operation of sum is defined but subtraction cannot always be performed.

# Integer numbers

We indicate with  $\mathbb{Z}$  the set of integer numbers:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

# Integer numbers

We indicate with  $\mathbb{Z}$  the set of integer numbers:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

Thus,  $\mathbb{Z}$  is the union of  $\mathbb{N}_0$  and the set of negative numbers.

Note that  $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z}$ .



# Integer numbers

We indicate with  $\mathbb{Z}$  the set of integer numbers:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

Thus,  $\mathbb{Z}$  is the union of  $\mathbb{N}_0$  and the set of negative numbers.

Note that  $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z}$ .

In this set we can sum and subtract numbers. We can also multiply numbers but division between number cannot always be performed

# Rational numbers

We indicate with  $\mathbb{Q}$  the set of rational numbers. Rational numbers are obtained by dividing an integer number by another integer number different from zero. In symbols:

$$\mathbb{Q} = \left\{ \frac{m}{n} \text{ such that } m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

# Rational numbers

We indicate with  $\mathbb{Q}$  the set of rational numbers. Rational numbers are obtained by dividing an integer number by another integer number different from zero. In symbols:

$$\mathbb{Q} = \left\{ \frac{m}{n} \text{ such that } m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

**Examples:**  $\frac{1}{2}$ ,  $\frac{-1}{2}$ ,  $\frac{25}{12}$ ,  $\frac{2}{-3}$ , etc.

# Rational numbers

We indicate with  $\mathbb{Q}$  the set of rational numbers. Rational numbers are obtained by dividing an integer number by another integer number different from zero. In symbols:

$$\mathbb{Q} = \left\{ \frac{m}{n} \text{ such that } m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

**Examples:**  $\frac{1}{2}$ ,  $\frac{-1}{2}$ ,  $\frac{25}{12}$ ,  $\frac{2}{-3}$ , etc.

Note that, if we set  $n = 1$ , we obtain the set  $\mathbb{Z}$ . Thus,  $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q}$ .

# Rational numbers

We indicate with  $\mathbb{Q}$  the set of rational numbers. Rational numbers are obtained by dividing an integer number by another integer number different from zero. In symbols:

$$\mathbb{Q} = \left\{ \frac{m}{n} \text{ such that } m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

**Examples:**  $\frac{1}{2}$ ,  $\frac{-1}{2}$ ,  $\frac{25}{12}$ ,  $\frac{2}{-3}$ , etc.

Note that, if we set  $n = 1$ , we obtain the set  $\mathbb{Z}$ . Thus,  $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q}$ .

The set  $\mathbb{Q}$  is large enough to make sum, subtraction, multiplication and division.

# Operations on $\mathbb{Q}$

We list here some basic operations on  $\mathbb{Q}$ . These are a very elementary rules that should already be known to you from your school studies.

# Operations on $\mathbb{Q}$

We list here some basic operations on  $\mathbb{Q}$ . These are a very elementary rules that should already be known to you from your school studies.

- Sum:

$$\frac{n}{m} + \frac{k}{q} = \frac{nq + km}{mq}, \quad \text{Ex: } \frac{1}{3} + \frac{7}{4} = \frac{1 \cdot 4 + 7 \cdot 3}{3 \cdot 4} = \frac{25}{12}$$

# Operations on $\mathbb{Q}$

We list here some basic operations on  $\mathbb{Q}$ . These are a very elementary rules that should already be known to you from your school studies.

- Sum:

$$\frac{n}{m} + \frac{k}{q} = \frac{nq + km}{mq}, \quad \text{Ex: } \frac{1}{3} + \frac{7}{4} = \frac{1 \cdot 4 + 7 \cdot 3}{3 \cdot 4} = \frac{25}{12}$$

- Product:

$$\frac{n}{m} \cdot \frac{k}{q} = \frac{n \cdot k}{m \cdot q}, \quad \text{Ex: } \frac{1}{3} \cdot \frac{-7}{4} = -\frac{7}{12}$$



# Operations on $\mathbb{Q}$

We list here some basic operations on  $\mathbb{Q}$ . These are a very elementary rules that should already be known to you from your school studies.

- Sum:

$$\frac{n}{m} + \frac{k}{q} = \frac{nq + km}{mq}, \quad \text{Ex: } \frac{1}{3} + \frac{7}{4} = \frac{1 \cdot 4 + 7 \cdot 3}{3 \cdot 4} = \frac{25}{12}$$

- Product:

$$\frac{n}{m} \cdot \frac{k}{q} = \frac{n \cdot k}{m \cdot q}, \quad \text{Ex: } \frac{1}{3} \cdot \frac{-7}{4} = -\frac{7}{12}$$

- Inverse:

$$\frac{1}{\frac{n}{m}} = \frac{m}{n}, \quad \text{Ex: } \frac{1}{\frac{2}{-3}} = -\frac{3}{2}$$

Note that  $\frac{1}{\frac{n}{m}}$  is also denoted by  $(\frac{n}{m})^{-1}$ , and therefore  $(\frac{n}{m})^{-1} = \frac{1}{\frac{n}{m}} = \frac{m}{n}$ .

# Operations on $\mathbb{Q}$

We list here some basic operations on  $\mathbb{Q}$ . These are a very elementary rules that should already be known to you from your school studies.

- Sum:

$$\frac{n}{m} + \frac{k}{q} = \frac{nq + km}{mq}, \quad \text{Ex: } \frac{1}{3} + \frac{7}{4} = \frac{1 \cdot 4 + 7 \cdot 3}{3 \cdot 4} = \frac{25}{12}$$

- Product:

$$\frac{n}{m} \cdot \frac{k}{q} = \frac{n \cdot k}{m \cdot q}, \quad \text{Ex: } \frac{1}{3} \cdot \frac{-7}{4} = -\frac{7}{12}$$

- Inverse:

$$\frac{1}{\frac{n}{m}} = \frac{m}{n}, \quad \text{Ex: } \frac{1}{\frac{2}{-3}} = -\frac{3}{2}$$

Note that  $\frac{1}{\frac{n}{m}}$  is also denoted by  $(\frac{n}{m})^{-1}$ , and therefore  $(\frac{n}{m})^{-1} = \frac{1}{\frac{n}{m}} = \frac{m}{n}$ .

- $k$ -th power ( $k \in \mathbb{N}$ ):

$$\left(\frac{m}{n}\right)^k = \underbrace{\frac{m}{n} \cdot \frac{m}{n} \cdots \frac{m}{n}}_{k \text{ times}} = \frac{m^k}{n^k}, \quad \text{Ex: } \left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}}$$

# Operations on $\mathbb{Q}$

We list here some basic operations on  $\mathbb{Q}$ . These are a very elementary rules that should already be known to you from your school studies.

- Sum:

$$\frac{n}{m} + \frac{k}{q} = \frac{nq + km}{mq}, \quad \text{Ex: } \frac{1}{3} + \frac{7}{4} = \frac{1 \cdot 4 + 7 \cdot 3}{3 \cdot 4} = \frac{25}{12}$$

- Product:

$$\frac{n}{m} \cdot \frac{k}{q} = \frac{n \cdot k}{m \cdot q}, \quad \text{Ex: } \frac{1}{3} \cdot \frac{-7}{4} = -\frac{7}{12}$$

- Inverse:

$$\frac{1}{\frac{n}{m}} = \frac{m}{n}, \quad \text{Ex: } \frac{1}{\frac{2}{-3}} = -\frac{3}{2}$$

Note that  $\frac{1}{\frac{n}{m}}$  is also denoted by  $(\frac{n}{m})^{-1}$ , and therefore  $(\frac{n}{m})^{-1} = \frac{1}{\frac{n}{m}} = \frac{m}{n}$ .

- $k$ -th power ( $k \in \mathbb{N}$ ):

$$\left(\frac{m}{n}\right)^k = \underbrace{\frac{m}{n} \cdot \frac{m}{n} \cdots \frac{m}{n}}_{k \text{ times}} = \frac{m^k}{n^k}, \quad \text{Ex: } \left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}}$$

and

$$\left(\frac{m}{n}\right)^{-k} = \left[\left(\frac{m}{n}\right)^{-1}\right]^k = \left[\frac{n}{m}\right]^k = \frac{n^k}{m^k}, \quad \text{Ex: } \left(\frac{1}{2}\right)^{-10} = 2^{10}$$

# Operations on $\mathbb{Q}$ , cont'd

Given  $q \in \mathbb{Q}$ , it is possible to show that:

$$q^m \cdot q^n = q^{m+n}$$

for  $n, m \in \mathbb{Z}$ .

# Operations on $\mathbb{Q}$ , cont'd

Given  $q \in \mathbb{Q}$ , it is possible to show that:

$$q^m \cdot q^n = q^{m+n}$$

for  $n, m \in \mathbb{Z}$ .

## Examples

- $\left(\frac{2}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^2 = \left(\frac{2}{5}\right)^{3+2} = \left(\frac{2}{5}\right)^5 = \frac{2^5}{5^5}$

# Operations on $\mathbb{Q}$ , cont'd

Given  $q \in \mathbb{Q}$ , it is possible to show that:

$$q^m \cdot q^n = q^{m+n}$$

for  $n, m \in \mathbb{Z}$ .

## Examples

- $\left(\frac{2}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^2 = \left(\frac{2}{5}\right)^{3+2} = \left(\frac{2}{5}\right)^5 = \frac{2^5}{5^5}$
- $\left(\frac{2}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^{-2} = \left(\frac{2}{5}\right)^{3-2} = \left(\frac{2}{5}\right)^1 = \frac{2}{5}$

# Operations on $\mathbb{Q}$ , cont'd

Given  $q \in \mathbb{Q}$ , it is possible to show that:

$$q^m \cdot q^n = q^{m+n}$$

for  $n, m \in \mathbb{Z}$ .

## Examples

- $\left(\frac{2}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^2 = \left(\frac{2}{5}\right)^{3+2} = \left(\frac{2}{5}\right)^5 = \frac{2^5}{5^5}$
- $\left(\frac{2}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^{-2} = \left(\frac{2}{5}\right)^{3-2} = \left(\frac{2}{5}\right)^1 = \frac{2}{5}$
- $\left(\frac{2}{5}\right)^2 \cdot \left(\frac{2}{5}\right)^{-3} = \left(\frac{2}{5}\right)^{2-3} = \left(\frac{2}{5}\right)^{-1} = \frac{5}{2}$

# Operations on $\mathbb{Q}$ , cont'd

Given  $q \in \mathbb{Q}$ , it is possible to show that:

$$q^m \cdot q^n = q^{m+n}$$

for  $n, m \in \mathbb{Z}$ .

## Examples

- $\left(\frac{2}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^2 = \left(\frac{2}{5}\right)^{3+2} = \left(\frac{2}{5}\right)^5 = \frac{2^5}{5^5}$
- $\left(\frac{2}{5}\right)^3 \cdot \left(\frac{2}{5}\right)^{-2} = \left(\frac{2}{5}\right)^{3-2} = \left(\frac{2}{5}\right)^1 = \frac{2}{5}$
- $\left(\frac{2}{5}\right)^2 \cdot \left(\frac{2}{5}\right)^{-3} = \left(\frac{2}{5}\right)^{2-3} = \left(\frac{2}{5}\right)^{-1} = \frac{5}{2}$
- $\left(-\frac{2}{5}\right)^2 \cdot \left(-\frac{2}{5}\right)^{-3} = \left(-\frac{2}{5}\right)^{2-3} = \left(-\frac{2}{5}\right)^{-1} = -\frac{5}{2}$



# Operations on $\mathbb{Q}$ , cont'd

Given  $q \in \mathbb{Q}$ , it is possible to show that:

$$q^m \cdot q^n = q^{m+n}$$

for  $n, m \in \mathbb{Z}$ .

## Examples

- $(\frac{2}{5})^3 \cdot (\frac{2}{5})^2 = (\frac{2}{5})^{3+2} = (\frac{2}{5})^5 = \frac{2^5}{5^5}$
- $(\frac{2}{5})^3 \cdot (\frac{2}{5})^{-2} = (\frac{2}{5})^{3-2} = (\frac{2}{5})^1 = \frac{2}{5}$
- $(\frac{2}{5})^2 \cdot (\frac{2}{5})^{-3} = (\frac{2}{5})^{2-3} = (\frac{2}{5})^{-1} = \frac{5}{2}$
- $(-\frac{2}{5})^2 \cdot (-\frac{2}{5})^{-3} = (-\frac{2}{5})^{2-3} = (-\frac{2}{5})^{-1} = -\frac{5}{2}$

Given  $q \in \mathbb{Q}$ ,  $q \neq 0$ , we have  $q^0 = 1$ . Indeed, for an arbitrary  $k \in \mathbb{N}$ , we can write:

$$q^0 = q^{k-k} = q^k \cdot q^{-k} = q^k \cdot \frac{1}{q^k} = 1$$

# Decimal representation of $\mathbb{Q}$

So far we have expressed the elements of  $\mathbb{Q}$  as fractions. They can also be expressed in decimal notation.

# Decimal representation of $\mathbb{Q}$

So far we have expressed the elements of  $\mathbb{Q}$  as fractions. They can also be expressed in decimal notation.

## Examples

# Decimal representation of $\mathbb{Q}$

So far we have expressed the elements of  $\mathbb{Q}$  as fractions. They can also be expressed in decimal notation.

## Examples

- $\frac{3}{10} = 0.3$

# Decimal representation of $\mathbb{Q}$

So far we have expressed the elements of  $\mathbb{Q}$  as fractions. They can also be expressed in decimal notation.

## Examples

- $\frac{3}{10} = 0.3$
- $-\frac{5}{2} = -2.5$

# Decimal representation of $\mathbb{Q}$

So far we have expressed the elements of  $\mathbb{Q}$  as fractions. They can also be expressed in decimal notation.

## Examples

- $\frac{3}{10} = 0.3$
- $-\frac{5}{2} = -2.5$
- $\frac{1}{3} = 0.33333\dots$

# Decimal representation of $\mathbb{Q}$

So far we have expressed the elements of  $\mathbb{Q}$  as fractions. They can also be expressed in decimal notation.

## Examples

- $\frac{3}{10} = 0.3$
- $-\frac{5}{2} = -2.5$
- $\frac{1}{3} = 0.33333\dots$
- $\frac{1}{22} = 0.0454545\dots$

# Decimal representation of $\mathbb{Q}$

So far we have expressed the elements of  $\mathbb{Q}$  as fractions. They can also be expressed in decimal notation.

## Examples

- $\frac{3}{10} = 0.3$
- $-\frac{5}{2} = -2.5$
- $\frac{1}{3} = 0.33333\dots$
- $\frac{1}{22} = 0.0454545\dots$
- $\frac{7}{12} = 0.5833333\dots$

The decimal representation of a rational number is either **finite**, as in  $\frac{3}{10}$ ,  $-\frac{5}{2}$ , or **infinite with a period**, as in  $\frac{1}{3}$ ,  $\frac{1}{22}$ ,  $\frac{7}{12}$ .



# Decimal representation of $\mathbb{Q}$ , cont'd

## Theorem

*Let  $q = \frac{n}{m}$  be a rational number. Then there are two **mutually exclusive** possibilities:*

# Decimal representation of $\mathbb{Q}$ , cont'd

## Theorem

Let  $q = \frac{n}{m}$  be a rational number. Then there are two **mutually exclusive possibilities**:

- 1 The decimal representation of  $q$  is made by a **finite number of digits**

# Decimal representation of $\mathbb{Q}$ , cont'd

## Theorem

Let  $q = \frac{n}{m}$  be a rational number. Then there are two **mutually exclusive possibilities**:

- 1 The decimal representation of  $q$  is made by a **finite number of digits**
- 2 The decimal representation of  $q$  is made by an **infinite number of digits but it is periodic**. In this case the period contains at most  $m - 1$  digits

# Decimal representation of $\mathbb{Q}$ , cont'd

Fraction	Decimal representation	Length of period
$\frac{9}{11}$	$0.\textcolor{blue}{8}\textcolor{red}{1}\textcolor{blue}{8}\textcolor{red}{1}\dots = 0.\overline{81}$	2

# Decimal representation of $\mathbb{Q}$ , cont'd

Fraction	Decimal representation	Length of period
$\frac{9}{11}$	$0.\textcolor{blue}{8}1\textcolor{red}{8}1 \dots = 0.\overline{81}$	2
$\frac{1}{7}$	$0.142857\textcolor{red}{1}42857 \dots = 0.\overline{142857}$	6

# Decimal representation of $\mathbb{Q}$ , cont'd

Fraction	Decimal representation	Length of period
$\frac{9}{11}$	$0.\textcolor{blue}{8}1\textcolor{red}{8}1\textcolor{blue}{8}1 \dots = 0.\overline{81}$	2
$\frac{1}{7}$	$0.142857\textcolor{red}{1}42857 \dots = 0.\overline{142857}$	6
$\frac{1}{81}$	$0.0123456790\textcolor{red}{1}2345679 = 0.\overline{012345679}$	9

# Decimal representation of $\mathbb{Q}$ , cont'd

Fraction	Decimal representation	Length of period
$\frac{9}{11}$	$0.\textcolor{blue}{8}1\textcolor{red}{8}1\textcolor{blue}{8}1 \dots = 0.\overline{81}$	2
$\frac{1}{7}$	$0.142857\textcolor{red}{1}42857 \dots = 0.\overline{142857}$	6
$\frac{1}{81}$	$0.0123456790\textcolor{red}{1}2345679 = 0.\overline{012345679}$	9
$\frac{1}{29}$	$0.\overline{0344827586206896551724137931}$	28

# Incompleteness of $\mathbb{Q}$

The set  $\mathbb{Q}$  is insufficient for many purposes. For instance, assume we want to solve the following equation:

$$x^2 = 2$$

We know that the solutions are  $x = \pm\sqrt{2}$ .



# Incompleteness of $\mathbb{Q}$

The set  $\mathbb{Q}$  is insufficient for many purposes. For instance, assume we want to solve the following equation:

$$x^2 = 2$$

We know that the solutions are  $x = \pm\sqrt{2}$ . What about the decimal representation of  $\sqrt{2}$ ?

# Incompleteness of $\mathbb{Q}$

The set  $\mathbb{Q}$  is insufficient for many purposes. For instance, assume we want to solve the following equation:

$$x^2 = 2$$

We know that the solutions are  $x = \pm\sqrt{2}$ . What about the decimal representation of  $\sqrt{2}$ ?

$$\begin{aligned}\sqrt{2} = & 1.41421356237309504880168872420 \\ & 9698078569671875376948073176679 \\ & 7379907324784621070388503875343 \\ & 276415727350138462309122970249248360\dots\end{aligned}$$

# Incompleteness of $\mathbb{Q}$

The set  $\mathbb{Q}$  is insufficient for many purposes. For instance, assume we want to solve the following equation:

$$x^2 = 2$$

We know that the solutions are  $x = \pm\sqrt{2}$ . What about the decimal representation of  $\sqrt{2}$ ?

$$\begin{aligned}\sqrt{2} = & 1.41421356237309504880168872420 \\ & 9698078569671875376948073176679 \\ & 7379907324784621070388503875343 \\ & 276415727350138462309122970249248360\dots\end{aligned}$$

**There is no period!**

# Incompleteness of $\mathbb{Q}$

The set  $\mathbb{Q}$  is insufficient for many purposes. For instance, assume we want to solve the following equation:

$$x^2 = 2$$

We know that the solutions are  $x = \pm\sqrt{2}$ . What about the decimal representation of  $\sqrt{2}$ ?

$$\begin{aligned}\sqrt{2} = & 1.41421356237309504880168872420 \\ & 9698078569671875376948073176679 \\ & 7379907324784621070388503875343 \\ & 276415727350138462309122970249248360\dots\end{aligned}$$

**There is no period!**

This means that there exist numbers, such as  $\sqrt{2}$ , which are not rational, meaning that they are NOT contained in  $\mathbb{Q}$ .

# Incompleteness of $\mathbb{Q}$

The set  $\mathbb{Q}$  is insufficient for many purposes. For instance, assume we want to solve the following equation:

$$x^2 = 2$$

We know that the solutions are  $x = \pm\sqrt{2}$ . What about the decimal representation of  $\sqrt{2}$ ?

$$\begin{aligned}\sqrt{2} = & 1.41421356237309504880168872420 \\ & 9698078569671875376948073176679 \\ & 7379907324784621070388503875343 \\ & 276415727350138462309122970249248360\dots\end{aligned}$$

**There is no period!**

This means that there exist numbers, such as  $\sqrt{2}$ , which are not rational, meaning that they are NOT contained in  $\mathbb{Q}$ .

These numbers are called **irrational numbers**.

# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset$$

# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset$$

# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset$$



# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}$$

# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}$$

**Which are the numbers of  $\mathbb{R}$  that are not in  $\mathbb{Q}$ ?**

# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}$$

**Which are the numbers of  $\mathbb{R}$  that are not in  $\mathbb{Q}$ ?** These numbers are called “irrational numbers” and are those whose decimal representation is not finite, nor periodic.

# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}$$

**Which are the numbers of  $\mathbb{R}$  that are not in  $\mathbb{Q}$ ?** These numbers are called “irrational numbers” and are those whose decimal representation is not finite, nor periodic.

Examples of irrational numbers:

- $\sqrt{2}$ ,

# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}$$

**Which are the numbers of  $\mathbb{R}$  that are not in  $\mathbb{Q}$ ?** These numbers are called “irrational numbers” and are those whose decimal representation is not finite, nor periodic.

Examples of irrational numbers:

- $\sqrt{2}$ , but even  $\sqrt{3}$ ,  $\sqrt{5}$

# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}$$

**Which are the numbers of  $\mathbb{R}$  that are not in  $\mathbb{Q}$ ?** These numbers are called “irrational numbers” and are those whose decimal representation is not finite, nor periodic.

Examples of irrational numbers:

- $\sqrt{2}$ , but even  $\sqrt{3}$ ,  $\sqrt{5}$  and more generally all the square roots of numbers which are not perfect square

# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}$$

**Which are the numbers of  $\mathbb{R}$  that are not in  $\mathbb{Q}$ ?** These numbers are called “irrational numbers” and are those whose decimal representation is not finite, nor periodic.

Examples of irrational numbers:

- $\sqrt{2}$ , but even  $\sqrt{3}$ ,  $\sqrt{5}$  and more generally all the square roots of numbers which are not perfect square (a perfect square is number  $q$  such that  $q = n^2$ , with  $n \in \mathbb{N}$ , e.g.  $4 = 2^2, 9 = 3^2, 16 = 4^2, 25 = 5^2, 36 = 6^2, \dots$ )

# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}$$

**Which are the numbers of  $\mathbb{R}$  that are not in  $\mathbb{Q}$ ?** These numbers are called “irrational numbers” and are those whose decimal representation is not finite, nor periodic.

Examples of irrational numbers:

- $\sqrt{2}$ , but even  $\sqrt{3}$ ,  $\sqrt{5}$  and more generally all the square roots of numbers which are not perfect square (a perfect square is number  $q$  such that  $q = n^2$ , with  $n \in \mathbb{N}$ , e.g.  $4 = 2^2$ ,  $9 = 3^2$ ,  $16 = 4^2$ ,  $25 = 5^2$ ,  $36 = 6^2$ , ...)
- The Euler's number  $e = 2.718281828459045235360287471352662497 \dots$



# Irrational numbers

To summarize, numerical sets are:

$$\mathbb{N} = \{1, 2, 3, \dots\} \subset \mathbb{Z} = \{\dots, -1, 0, 1, \dots\} \subset \mathbb{Q} = \left\{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\} \right\} \subset \mathbb{R}$$

**Which are the numbers of  $\mathbb{R}$  that are not in  $\mathbb{Q}$ ?** These numbers are called “irrational numbers” and are those whose decimal representation is not finite, nor periodic.

Examples of irrational numbers:

- $\sqrt{2}$ , but even  $\sqrt{3}$ ,  $\sqrt{5}$  and more generally all the square roots of numbers which are not perfect square (a perfect square is number  $q$  such that  $q = n^2$ , with  $n \in \mathbb{N}$ , e.g.  $4 = 2^2$ ,  $9 = 3^2$ ,  $16 = 4^2$ ,  $25 = 5^2$ ,  $36 = 6^2$ , ...)
- The Euler's number  $e = 2.718281828459045235360287471352662497 \dots$
- The Pi number  $\pi = 3.141592653589793238462643383279502884 \dots$

# Real Numbers: Axioms

**Operations:** The operations of sum (+) and multiplication ( $\cdot$ ) between pairs of real numbers are defined and have the following properties

# Real Numbers: Axioms

**Operations:** The operations of sum (+) and multiplication (·) between pairs of real numbers are defined and have the following properties

- Associative property:  $(a + b) + c = a + (b + c)$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ , for all  $a, b, c \in \mathbb{R}$

# Real Numbers: Axioms

**Operations:** The operations of sum (+) and multiplication (·) between pairs of real numbers are defined and have the following properties

- Associative property:  $(a + b) + c = a + (b + c)$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ , for all  $a, b, c \in \mathbb{R}$
- Commutative property:  $a + b = b + a$ ,  $a \cdot b = b \cdot a$ , for all  $a, b \in \mathbb{R}$

# Real Numbers: Axioms

**Operations:** The operations of sum (+) and multiplication (·) between pairs of real numbers are defined and have the following properties

- Associative property:  $(a + b) + c = a + (b + c)$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ , for all  $a, b, c \in \mathbb{R}$
- Commutative property:  $a + b = b + a$ ,  $a \cdot b = b \cdot a$ , for all  $a, b \in \mathbb{R}$
- Distributive property:  $a \cdot (b + c) = a \cdot b + a \cdot c$  for all  $a, b, c \in \mathbb{R}$

# Real Numbers: Axioms

**Operations:** The operations of sum (+) and multiplication (·) between pairs of real numbers are defined and have the following properties

- Associative property:  $(a + b) + c = a + (b + c)$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ , for all  $a, b, c \in \mathbb{R}$
- Commutative property:  $a + b = b + a$ ,  $a \cdot b = b \cdot a$ , for all  $a, b \in \mathbb{R}$
- Distributive property:  $a \cdot (b + c) = a \cdot b + a \cdot c$  for all  $a, b, c \in \mathbb{R}$
- Existence of the neutral element: for any  $a \in \mathbb{R}$  there are two distinct numbers, namely 0 and 1 such that  $a + 0 = a$  and  $a \cdot 1 = a$ .

# Real Numbers: Axioms

**Operations:** The operations of sum (+) and multiplication (·) between pairs of real numbers are defined and have the following properties

- Associative property:  $(a + b) + c = a + (b + c)$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ , for all  $a, b, c \in \mathbb{R}$
- Commutative property:  $a + b = b + a$ ,  $a \cdot b = b \cdot a$ , for all  $a, b \in \mathbb{R}$
- Distributive property:  $a \cdot (b + c) = a \cdot b + a \cdot c$  for all  $a, b, c \in \mathbb{R}$
- Existence of the neutral element: for any  $a \in \mathbb{R}$  there are two distinct numbers, namely 0 and 1 such that  $a + 0 = a$  and  $a \cdot 1 = a$ . These numbers are called the neutral number of sum and multiplication, respectively.

# Real Numbers: Axioms

**Operations:** The operations of sum (+) and multiplication (·) between pairs of real numbers are defined and have the following properties

- Associative property:  $(a + b) + c = a + (b + c)$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ , for all  $a, b, c \in \mathbb{R}$
- Commutative property:  $a + b = b + a$ ,  $a \cdot b = b \cdot a$ , for all  $a, b \in \mathbb{R}$
- Distributive property:  $a \cdot (b + c) = a \cdot b + a \cdot c$  for all  $a, b, c \in \mathbb{R}$
- Existence of the neutral element: for any  $a \in \mathbb{R}$  there are two distinct numbers, namely 0 and 1 such that  $a + 0 = a$  and  $a \cdot 1 = a$ . These numbers are called the neutral number of sum and multiplication, respectively.
- Existence of the opposite: for any  $a \in \mathbb{R}$  there is a real number  $-a$  such that  $a + (-a) = 0$ .  $-a$  is called the opposite of  $a$



# Real Numbers: Axioms

**Operations:** The operations of sum (+) and multiplication (·) between pairs of real numbers are defined and have the following properties

- Associative property:  $(a + b) + c = a + (b + c)$ ,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ , for all  $a, b, c \in \mathbb{R}$
- Commutative property:  $a + b = b + a$ ,  $a \cdot b = b \cdot a$ , for all  $a, b \in \mathbb{R}$
- Distributive property:  $a \cdot (b + c) = a \cdot b + a \cdot c$  for all  $a, b, c \in \mathbb{R}$
- Existence of the neutral element: for any  $a \in \mathbb{R}$  there are two distinct numbers, namely 0 and 1 such that  $a + 0 = a$  and  $a \cdot 1 = a$ . These numbers are called the neutral number of sum and multiplication, respectively.
- Existence of the opposite: for any  $a \in \mathbb{R}$  there is a real number  $-a$  such that  $a + (-a) = 0$ .  $-a$  is called the opposite of  $a$
- Existence of the inverse: for any  $a \in \mathbb{R}$  there is a real number  $\frac{1}{a}$  such that  $a \cdot \frac{1}{a} = 1$ .  $\frac{1}{a}$  is called the inverse of  $a$

# Real Numbers: Axioms

**Order Relation** There is a relation *minor or equal to*,  $\leq$ , between pairs of real numbers with the properties

# Real Numbers: Axioms

**Order Relation** There is a relation *minor or equal to*,  $\leq$ , between pairs of real numbers with the properties

- For any pair of real numbers  $a, b$  either  $a \leq b$  or  $b \leq a$ .

# Real Numbers: Axioms

**Order Relation** There is a relation *minor or equal to*,  $\leq$ , between pairs of real numbers with the properties

- For any pair of real numbers  $a, b$  either  $a \leq b$  or  $b \leq a$ .
- If  $a \leq b$  **and**  $b \leq a$  then necessarily  $a = b$ .

# Real Numbers: Axioms

**Order Relation** There is a relation *minor or equal to*,  $\leq$ , between pairs of real numbers with the properties

- For any pair of real numbers  $a, b$  either  $a \leq b$  or  $b \leq a$ .
- If  $a \leq b$  **and**  $b \leq a$  then necessarily  $a = b$ .
- If  $a \leq b$ , then also  $a + c \leq b + c$ , for any real number  $c$ .

# Real Numbers: Axioms

**Order Relation** There is a relation *minor or equal to*,  $\leq$ , between pairs of real numbers with the properties

- For any pair of real numbers  $a, b$  either  $a \leq b$  or  $b \leq a$ .
- If  $a \leq b$  **and**  $b \leq a$  then necessarily  $a = b$ .
- If  $a \leq b$ , then also  $a + c \leq b + c$ , for any real number  $c$ .
- If  $0 \leq a$  and  $0 \leq b$  then  $0 \leq a + b$  and  $0 \leq a \cdot b$ .

# Real Numbers: Axioms

**Order Relation** There is a relation *minor or equal to*,  $\leq$ , between pairs of real numbers with the properties

- For any pair of real numbers  $a, b$  either  $a \leq b$  or  $b \leq a$ .
- If  $a \leq b$  **and**  $b \leq a$  then necessarily  $a = b$ .
- If  $a \leq b$ , then also  $a + c \leq b + c$ , for any real number  $c$ .
- If  $0 \leq a$  and  $0 \leq b$  then  $0 \leq a + b$  and  $0 \leq a \cdot b$ .

**Completeness:** Let  $A, B$  two non-empty subsets of real numbers such that any number in  $A$  is smaller than or equal to any other number in  $B$ . Then there is a real number  $c$  such that  $a \leq c \leq b$ , for any  $a \in A$  and any  $b \in B$ .

# Real Numbers: Axioms

**Order Relation** There is a relation *minor or equal to*,  $\leq$ , between pairs of real numbers with the properties

- For any pair of real numbers  $a, b$  either  $a \leq b$  or  $b \leq a$ .
- If  $a \leq b$  **and**  $b \leq a$  then necessarily  $a = b$ .
- If  $a \leq b$ , then also  $a + c \leq b + c$ , for any real number  $c$ .
- If  $0 \leq a$  and  $0 \leq b$  then  $0 \leq a + b$  and  $0 \leq a \cdot b$ .

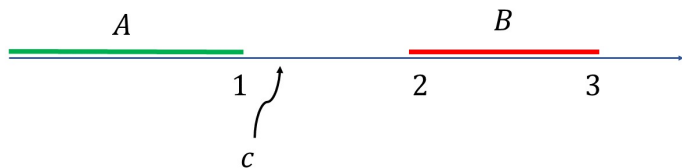
**Completeness:** Let  $A, B$  two non-empty subsets of real numbers such that any number in  $A$  is smaller than or equal to any other number in  $B$ . Then there is a real number  $c$  such that  $a \leq c \leq b$ , for any  $a \in A$  and any  $b \in B$ .

The number  $c$  is called the separating point of  $A$  and  $B$ .



# Illustrative example

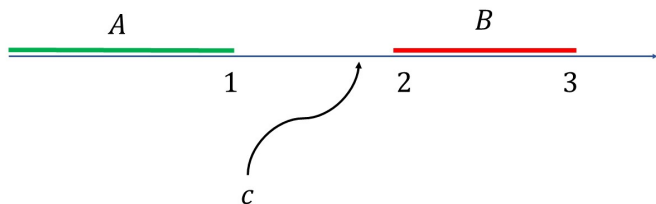
Consider sets  $A = \{x \in \mathbb{R} : x \leq 1\}$  and  $B = \{x \in \mathbb{R} : 2 \leq x \leq 3\}$



In this example the separating point is not unique.

# Illustrative example

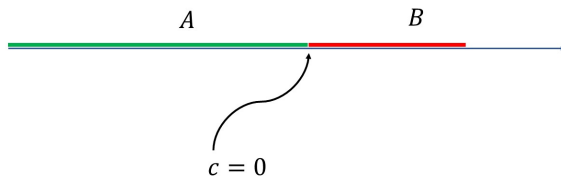
Consider sets  $A = \{x \in \mathbb{R} : x \leq 1\}$  and  $B = \{x \in \mathbb{R} : 2 \leq x \leq 3\}$



For instance this is another possible value for  $c$ .

# Illustrative example

Consider sets  $A = \{x \in \mathbb{R} : x \leq 0\}$  and  $B = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$



In this case the separating point is unique.

## $\mathbb{Q}$ is not $\mathbb{R}$

It is clear that the set  $\mathbb{Q}$  satisfies all axioms of operations and ordering. Does it satisfy the axiom of completeness?

## $\mathbb{Q}$ is not $\mathbb{R}$

It is clear that the set  $\mathbb{Q}$  satisfies all axioms of operations and ordering. Does it satisfy the axiom of completeness?

The answer is no, and the following example shows that there are two non-empty subsets of **rational** numbers,  $A, B$ , such that any number in  $A$  is smaller than or equal to any other number in  $B$ , whose separating point  $c$  is not rational!

# $\mathbb{Q}$ is not $\mathbb{R}$

It is clear that the set  $\mathbb{Q}$  satisfies all axioms of operations and ordering. Does it satisfy the axiom of completeness?

The answer is no, and the following example shows that there are two non-empty subsets of **rational** numbers,  $A, B$ , such that any number in  $A$  is smaller than or equal to any other number in  $B$ , whose separating point  $c$  is not rational!

Consider the sets:

$$A = \{x \in \mathbb{Q} : x < 0 \text{ or } x \geq 0 \text{ and } x^2 < 2\}$$

$$B = \{x \in \mathbb{Q} : x \geq 0 \text{ and } x^2 \geq 2\}$$

## $\mathbb{Q}$ is not $\mathbb{R}$

It is clear that the set  $\mathbb{Q}$  satisfies all axioms of operations and ordering. Does it satisfy the axiom of completeness?

The answer is no, and the following example shows that there are two non-empty subsets of **rational** numbers,  $A, B$ , such that any number in  $A$  is smaller than or equal to any other number in  $B$ , whose separating point  $c$  is not rational!

Consider the sets:

$$A = \{x \in \mathbb{Q} : x < 0 \text{ or } x \geq 0 \text{ and } x^2 < 2\}$$

$$B = \{x \in \mathbb{Q} : x \geq 0 \text{ and } x^2 \geq 2\}$$

These sets are disjoint, i.e.  $A \cap B = \emptyset$  and the separation point is  $c = \sqrt{2}$ . Hence the axiom of completeness is not satisfied since **the separation point is not rational**.

## $\mathbb{Q}$ is not $\mathbb{R}$

It is clear that the set  $\mathbb{Q}$  satisfies all axioms of operations and ordering. Does it satisfy the axiom of completeness?

The answer is no, and the following example shows that there are two non-empty subsets of **rational** numbers,  $A, B$ , such that any number in  $A$  is smaller than or equal to any other number in  $B$ , whose separating point  $c$  is not rational!

Consider the sets:

$$A = \{x \in \mathbb{Q} : x < 0 \text{ or } x \geq 0 \text{ and } x^2 < 2\}$$

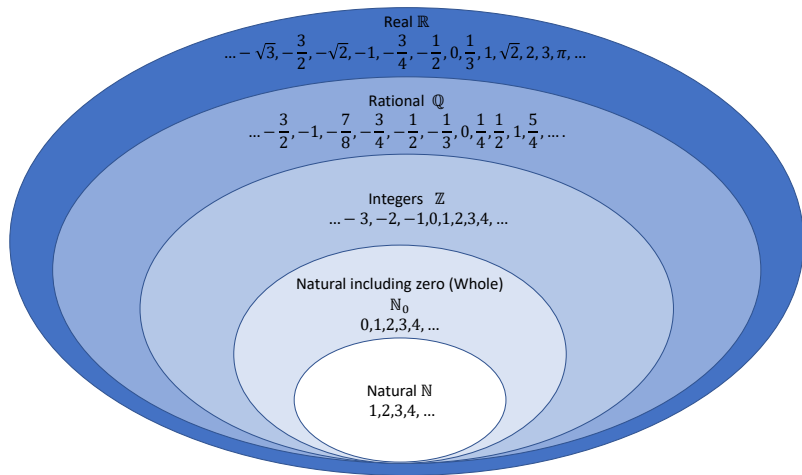
$$B = \{x \in \mathbb{Q} : x \geq 0 \text{ and } x^2 \geq 2\}$$

These sets are disjoint, i.e.  $A \cap B = \emptyset$  and the separation point is  $c = \sqrt{2}$ . Hence the axiom of completeness is not satisfied since **the separation point is not rational**.

**The axiom of completeness is not valid for natural numbers  $\mathbb{N}$ , integer numbers  $\mathbb{Z}$ , rational numbers  $\mathbb{Q}$ .**



# In summary



# Extensional and intensional definitions

## Observation

A set may contain infinite objects, so it may be complicated listing all of them. For this reason, sometimes we use *intensional* definitions instead of *extensional* definitions

# Extensional and intensional definitions

## Observation

A set may contain infinite objects, so it may be complicated listing all of them. For this reason, sometimes we use *intensional* definitions instead of *extensional* definitions

## Extensional definition

All objects are explicitly listed:

- $A = \{\triangle, \bigcirc, \square\}$

# Extensional and intensional definitions

## Observation

A set may contain infinite objects, so it may be complicated listing all of them. For this reason, sometimes we use *intensional* definitions instead of *extensional* definitions

## Extensional definition

All objects are explicitly listed:

- $A = \{\triangle, \bigcirc, \square\}$
- $B = \{1, 7, 99, 1.23, -2\}$

# Extensional and intensional definitions

## Observation

A set may contain infinite objects, so it may be complicated listing all of them. For this reason, sometimes we use *intensional* definitions instead of *extensional* definitions

## Extensional definition

All objects are explicitly listed:

- $A = \{\triangle, \bigcirc, \square\}$
- $B = \{1, 7, 99, 1.23, -2\}$
- $C = \{\text{New York, London, Sydney}\}$

# Extensional and intensional definitions

## Observation

A set may contain infinite objects, so it may be complicated listing all of them. For this reason, sometimes we use *intensional* definitions instead of *extensional* definitions

### Extensional definition

All objects are explicitly listed:

- $A = \{\triangle, \bigcirc, \square\}$
- $B = \{1, 7, 99, 1.23, -2\}$
- $C = \{\text{New York, London, Sydney}\}$

### Intensional definition

We state the property which *unambiguously* defines the objects in the set:

# Extensional and intensional definitions

## Observation

A set may contain infinite objects, so it may be complicated listing all of them. For this reason, sometimes we use *intensional* definitions instead of *extensional* definitions

### Extensional definition

All objects are explicitly listed:

- $A = \{\triangle, \bigcirc, \square\}$
- $B = \{1, 7, 99, 1.23, -2\}$
- $C = \{\text{New York, London, Sydney}\}$

### Intensional definition

We state the property which *unambiguously* defines the objects in the set:

- $A = \{\text{All cities in Europe}\}$

# Extensional and intensional definitions

## Observation

A set may contain infinite objects, so it may be complicated listing all of them. For this reason, sometimes we use *intensional* definitions instead of *extensional* definitions

### Extensional definition

All objects are explicitly listed:

- $A = \{\triangle, \bigcirc, \square\}$
- $B = \{1, 7, 99, 1.23, -2\}$
- $C = \{\text{New York, London, Sydney}\}$

### Intensional definition

We state the property which *unambiguously* defines the objects in the set:

- $A = \{\text{All cities in Europe}\}$
- $B = \{\text{All natural numbers larger than 100}\}$



# Extensional and intensional definitions

## Observation

A set may contain infinite objects, so it may be complicated listing all of them. For this reason, sometimes we use *intensional* definitions instead of *extensional* definitions

### Extensional definition

All objects are explicitly listed:

- $A = \{\triangle, \bigcirc, \square\}$
- $B = \{1, 7, 99, 1.23, -2\}$
- $C = \{\text{New York, London, Sydney}\}$

### Intensional definition

We state the property which *unambiguously* defines the objects in the set:

- $A = \{\text{All cities in Europe}\}$
- $B = \{\text{All natural numbers larger than 100}\}$
- $C = \{\text{All people in this class whose surname has 'A' as a first letter}\}$

# The $\in$ , $\notin$ quantifiers

To say that the object  $a$  is contained in the set  $A$ , we write  $a \in A$

# The $\in$ , $\notin$ quantifiers

To say that the object  $a$  is contained in the set  $A$ , we write  $a \in A$

To say that the object  $a$  is *not* contained in the set  $A$ , we write  $a \notin A$

# The $\in$ , $\notin$ quantifiers

To say that the object  $a$  is contained in the set  $A$ , we write  $a \in A$

To say that the object  $a$  is *not* contained in the set  $A$ , we write  $a \notin A$

## Examples

- If  $A = \{\triangle, \bigcirc, \square\}$ , then  $\triangle \in A$ ,  $\bigcirc \in A$ ,  $\square \in A$  but  $\diamond \notin A$

# The $\in$ , $\notin$ quantifiers

To say that the object  $a$  is contained in the set  $A$ , we write  $a \in A$

To say that the object  $a$  is *not* contained in the set  $A$ , we write  $a \notin A$

## Examples

- If  $A = \{\triangle, \bigcirc, \square\}$ , then  $\triangle \in A$ ,  $\bigcirc \in A$ ,  $\square \in A$  but  $\diamond \notin A$
- If  $A = \{1, 7, 99, 1.23, -2\}$ , then  $1 \in A$ ,  $7 \in A$ , but  $5 \notin A$

# The $\in$ , $\notin$ quantifiers

To say that the object  $a$  is contained in the set  $A$ , we write  $a \in A$

To say that the object  $a$  is *not* contained in the set  $A$ , we write  $a \notin A$

## Examples

- If  $A = \{\triangle, \bigcirc, \square\}$ , then  $\triangle \in A$ ,  $\bigcirc \in A$ ,  $\square \in A$  but  $\diamond \notin A$
- If  $A = \{1, 7, 99, 1.23, -2\}$ , then  $1 \in A$ ,  $7 \in A$ , but  $5 \notin A$
- If  $A = \{\text{All cities in Europe}\}$ , then  $\text{Paris} \in A$ , but  $\text{New York} \notin A$

# The $\in$ , $\notin$ quantifiers

To say that the object  $a$  is contained in the set  $A$ , we write  $a \in A$

To say that the object  $a$  is *not* contained in the set  $A$ , we write  $a \notin A$

## Examples

- If  $A = \{\triangle, \bigcirc, \square\}$ , then  $\triangle \in A$ ,  $\bigcirc \in A$ ,  $\square \in A$  but  $\diamond \notin A$
- If  $A = \{1, 7, 99, 1.23, -2\}$ , then  $1 \in A$ ,  $7 \in A$ , but  $5 \notin A$
- If  $A = \{\text{All cities in Europe}\}$ , then  $\text{Paris} \in A$ , but  $\text{New York} \notin A$
- If  $A = \{\text{All natural numbers larger than } 100\}$ , then  $101 \in A$ , but  $100 \notin A$

# The $\in$ , $\notin$ quantifiers

To say that the object  $a$  is contained in the set  $A$ , we write  $a \in A$

To say that the object  $a$  is *not* contained in the set  $A$ , we write  $a \notin A$

## Examples

- If  $A = \{\triangle, \bigcirc, \square\}$ , then  $\triangle \in A$ ,  $\bigcirc \in A$ ,  $\square \in A$  but  $\diamond \notin A$
- If  $A = \{1, 7, 99, 1.23, -2\}$ , then  $1 \in A$ ,  $7 \in A$ , but  $5 \notin A$
- If  $A = \{\text{All cities in Europe}\}$ , then  $\text{Paris} \in A$ , but  $\text{New York} \notin A$
- If  $A = \{\text{All natural numbers larger than } 100\}$ , then  $101 \in A$ , but  $100 \notin A$

**Remark:** The  $\in$  symbol is often used in intensional definitions:

$$A = \{\text{All cities in Europe}\}, \quad B = \{x \in A \mid x \text{ is a capital}\}$$



# The $\in$ , $\notin$ quantifiers

To say that the object  $a$  is contained in the set  $A$ , we write  $a \in A$

To say that the object  $a$  is *not* contained in the set  $A$ , we write  $a \notin A$

## Examples

- If  $A = \{\triangle, \bigcirc, \square\}$ , then  $\triangle \in A$ ,  $\bigcirc \in A$ ,  $\square \in A$  but  $\diamond \notin A$
- If  $A = \{1, 7, 99, 1.23, -2\}$ , then  $1 \in A$ ,  $7 \in A$ , but  $5 \notin A$
- If  $A = \{\text{All cities in Europe}\}$ , then  $\text{Paris} \in A$ , but  $\text{New York} \notin A$
- If  $A = \{\text{All natural numbers larger than } 100\}$ , then  $101 \in A$ , but  $100 \notin A$

**Remark:** The  $\in$  symbol is often used in intensional definitions:

$$A = \{\text{All cities in Europe}\}, \quad B = \{x \in A \mid x \text{ is a capital}\}$$

The set  $B$  includes the European cities that are capitals, so that **Milan**  $\in A$ , but **Milan**  $\notin B$ .

# The $\forall$ quantifier

To say that a certain property holds for **ALL** objects in  $A$  we write  $\forall a \in A$

# The $\forall$ quantifier

To say that a certain property holds for **ALL** objects in  $A$  we write  $\forall a \in A$

## Examples

- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\forall a \in A, a \geq 0$

# The $\forall$ quantifier

To say that a certain property holds for **ALL** objects in  $A$  we write  $\forall a \in A$

## Examples

- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\forall a \in A, a \geq 0$
- If  $A = \{1, 77, 31, 5\}$ , then we can write  $\forall a \in A, a$  is odd

# The $\forall$ quantifier

To say that a certain property holds for **ALL** objects in  $A$  we write  $\forall a \in A$

## Examples

- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\forall a \in A, a \geq 0$
- If  $A = \{1, 77, 31, 5\}$ , then we can write  $\forall a \in A, a$  is odd
- If  $A = \{0.2, 0.5, 0.7\}$ , then we can write  $\forall a \in A, 0 < a < 1$

# The $\exists$ , $\nexists$ quantifiers

To say that a certain property holds for **AT LEAST ONE** object in  $A$  we write  
 $\exists a \in A$

# The $\exists$ , $\nexists$ quantifiers

To say that a certain property holds for **AT LEAST ONE** object in  $A$  we write  $\exists a \in A$

To say that a certain property holds for **NO** objects in  $A$  we write  $\nexists a \in A$

# The $\exists$ , $\nexists$ quantifiers

To say that a certain property holds for **AT LEAST ONE** object in  $A$  we write  $\exists a \in A$

To say that a certain property holds for **NO** objects in  $A$  we write  $\nexists a \in A$

## Examples

- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\exists a \in A$  such that  $a \leq \frac{1}{2}$ . Indeed  $0 \leq \frac{1}{2}$  and  $0.5 \leq \frac{1}{2}$



# The $\exists$ , $\nexists$ quantifiers

To say that a certain property holds for **AT LEAST ONE** object in  $A$  we write  $\exists a \in A$

To say that a certain property holds for **NO** objects in  $A$  we write  $\nexists a \in A$

## Examples

- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\exists a \in A$  such that  $a \leq \frac{1}{2}$ . Indeed  $0 \leq \frac{1}{2}$  and  $0.5 \leq \frac{1}{2}$
- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\exists a \in A$  such that  $0 < a \leq \frac{1}{2}$ . Indeed  $0 < 0.5 \leq \frac{1}{2}$

# The $\exists$ , $\nexists$ quantifiers

To say that a certain property holds for **AT LEAST ONE** object in  $A$  we write  $\exists a \in A$

To say that a certain property holds for **NO** objects in  $A$  we write  $\nexists a \in A$

## Examples

- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\exists a \in A$  such that  $a \leq \frac{1}{2}$ . Indeed  $0 \leq \frac{1}{2}$  and  $0.5 \leq \frac{1}{2}$
- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\exists a \in A$  such that  $0 < a \leq \frac{1}{2}$ . Indeed  $0 < 0.5 \leq \frac{1}{2}$
- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\nexists a \in A$  such that  $a < 0$ . Indeed all objects in  $A$  are larger or equal to zero

# The $\exists$ , $\nexists$ quantifiers

To say that a certain property holds for **AT LEAST ONE** object in  $A$  we write  $\exists a \in A$

To say that a certain property holds for **NO** objects in  $A$  we write  $\nexists a \in A$

## Examples

- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\exists a \in A$  such that  $a \leq \frac{1}{2}$ . Indeed  $0 \leq \frac{1}{2}$  and  $0.5 \leq \frac{1}{2}$
- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\exists a \in A$  such that  $0 < a \leq \frac{1}{2}$ . Indeed  $0 < 0.5 \leq \frac{1}{2}$
- If  $A = \{0, 0.5, 44.5, 2\}$ , then we can write  $\nexists a \in A$  such that  $a < 0$ . Indeed all objects in  $A$  are larger or equal to zero

**Remark:** In the second example, there exists **ONLY ONE** element in  $A$  such that  $0 < a \leq \frac{1}{2}$ . In this case we can write  $\exists! a \in A$ . Writing  $\exists a \in A$  is correct as well, however  $\exists! a \in A$  is more informative

# Summary of quantifiers

- $a \in A$  means “the element  $a$  belongs to the set  $A$ ”

# Summary of quantifiers

- $a \in A$  means “the element  $a$  belongs to the set  $A$ ”
- $\forall a \in A$  means “for all elements  $a$  in the set  $A$ ”

# Summary of quantifiers

- $a \in A$  means “the element  $a$  belongs to the set  $A$ ”
- $\forall a \in A$  means “for all elements  $a$  in the set  $A$ ”
- $\exists a \in A$  means “there exists at least one element  $a$  in the set  $A$ ”

# Summary of quantifiers

- $a \in A$  means “the element  $a$  belongs to the set  $A$ ”
- $\forall a \in A$  means “for all elements  $a$  in the set  $A$ ”
- $\exists a \in A$  means “there exists at least one element  $a$  in the set  $A$ ”
- $\nexists a \in A$  means “there exists no element in the set  $A$ ”

# Summary of quantifiers

- $a \in A$  means “the element  $a$  belongs to the set  $A$ ”
- $\forall a \in A$  means “for all elements  $a$  in the set  $A$ ”
- $\exists a \in A$  means “there exists at least one element  $a$  in the set  $A$ ”
- $\nexists a \in A$  means “there exists no element in the set  $A$ ”
- $\exists! a \in A$  means “there exists a unique element  $a$  in the set  $A$ ”



# Union of sets: the $\cup$ operator

## Definition

Let  $A$  and  $B$  be two sets. We denote by  $A \cup B$  the set containing all the elements of  $A$  and all the elements of  $B$ . In symbols:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

# Union of sets: the $\cup$ operator

## Definition

Let  $A$  and  $B$  be two sets. We denote by  $A \cup B$  the set containing all the elements of  $A$  and all the elements of  $B$ . In symbols:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

## Examples

- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \diamond, \square\} \Rightarrow A \cup B = \{\triangle, \bigcirc, \diamond, \square\}$

# Union of sets: the $\cup$ operator

## Definition

Let  $A$  and  $B$  be two sets. We denote by  $A \cup B$  the set containing all the elements of  $A$  and all the elements of  $B$ . In symbols:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

## Examples

- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \diamond, \square\} \Rightarrow A \cup B = \{\triangle, \bigcirc, \diamond, \square\}$
- $A = \{\text{Milan, Rome}\}$ ,  $B = \{\text{Sydney}\} \Rightarrow A \cup B = \{\text{Milan, Rome, Sydney}\}$

# Union of sets: the $\cup$ operator

## Definition

Let  $A$  and  $B$  be two sets. We denote by  $A \cup B$  the set containing all the elements of  $A$  and all the elements of  $B$ . In symbols:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

## Examples

- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \diamond, \square\} \Rightarrow A \cup B = \{\triangle, \bigcirc, \diamond, \square\}$
- $A = \{\text{Milan, Rome}\}$ ,  $B = \{\text{Sydney}\} \Rightarrow A \cup B = \{\text{Milan, Rome, Sydney}\}$
- $A = \{1, 100, 4.4\}$ ,  $B = \{4.4\} \Rightarrow A \cup B = \{1, 100, 4.4\}$

# Union of sets: the $\cup$ operator

## Definition

Let  $A$  and  $B$  be two sets. We denote by  $A \cup B$  the set containing all the elements of  $A$  and all the elements of  $B$ . In symbols:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

## Examples

- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \diamond, \square\} \Rightarrow A \cup B = \{\triangle, \bigcirc, \diamond, \square\}$
- $A = \{\text{Milan, Rome}\}$ ,  $B = \{\text{Sydney}\} \Rightarrow A \cup B = \{\text{Milan, Rome, Sydney}\}$
- $A = \{1, 100, 4.4\}$ ,  $B = \{4.4\} \Rightarrow A \cup B = \{1, 100, 4.4\}$
- $A = \{1, 3, -5\}$ ,  $B = \{1, 2\}$ ,  $C = \{1, 55\} \Rightarrow A \cup B \cup C = \{1, 3, -5, 2, 55\}$

# Intersection of sets: the $\cap$ operator

## Definition

Let  $A$  and  $B$  be two sets. We denote by  $A \cap B$  the set containing the elements in common between  $A$  and  $B$ . In symbols:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

# Intersection of sets: the $\cap$ operator

## Definition

Let  $A$  and  $B$  be two sets. We denote by  $A \cap B$  the set containing the elements in common between  $A$  and  $B$ . In symbols:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

## Examples

- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \diamond, \square\} \Rightarrow A \cap B = \{\triangle, \square\}$

# Intersection of sets: the $\cap$ operator

## Definition

Let  $A$  and  $B$  be two sets. We denote by  $A \cap B$  the set containing the elements in common between  $A$  and  $B$ . In symbols:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

## Examples

- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \diamond, \square\} \Rightarrow A \cap B = \{\triangle, \square\}$
- $A = \{\text{Milan, Rome}\}$ ,  $B = \{\text{Sydney}\} \Rightarrow A \cap B = \emptyset$



# Intersection of sets: the $\cap$ operator

## Definition

Let  $A$  and  $B$  be two sets. We denote by  $A \cap B$  the set containing the elements in common between  $A$  and  $B$ . In symbols:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

## Examples

- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \diamond, \square\} \Rightarrow A \cap B = \{\triangle, \square\}$
- $A = \{\text{Milan}, \text{Rome}\}$ ,  $B = \{\text{Sydney}\} \Rightarrow A \cap B = \emptyset$
- $A = \{1, 100, 4.4\}$ ,  $B = \{4.4\} \Rightarrow A \cap B = \{4.4\}$

# Intersection of sets: the $\cap$ operator

## Definition

Let  $A$  and  $B$  be two sets. We denote by  $A \cap B$  the set containing the elements in common between  $A$  and  $B$ . In symbols:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

## Examples

- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \diamond, \square\} \Rightarrow A \cap B = \{\triangle, \square\}$
- $A = \{\text{Milan}, \text{Rome}\}$ ,  $B = \{\text{Sydney}\} \Rightarrow A \cap B = \emptyset$
- $A = \{1, 100, 4.4\}$ ,  $B = \{4.4\} \Rightarrow A \cap B = \{4.4\}$
- $A = \{1, 3, -5\}$ ,  $B = \{1, 2\}$ ,  $C = \{1, 55\} \Rightarrow A \cap B \cap C = \{1\}$

# The empty set

## Definition

An empty set is a set with no elements, and is denoted by  $\emptyset$

# The empty set

## Definition

An empty set is a set with no elements, and is denoted by  $\emptyset$

## Remark

For any set  $A$ ,  $A \cup \emptyset = A$ ,  $A \cap \emptyset = \emptyset$

# Subsets

Let  $A$  and  $B$  be two sets. We write  $A \subseteq B$  if all the elements of  $A$  are also elements of  $B$ .

# Subsets

Let  $A$  and  $B$  be two sets. We write  $A \subseteq B$  if all the elements of  $A$  are also elements of  $B$ .

## Examples

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\} \Rightarrow A \subseteq B$

# Subsets

Let  $A$  and  $B$  be two sets. We write  $A \subseteq B$  if all the elements of  $A$  are also elements of  $B$ .

## Examples

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\} \Rightarrow A \subseteq B$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\} \Rightarrow B \subseteq A$

# Subsets

Let  $A$  and  $B$  be two sets. We write  $A \subseteq B$  if all the elements of  $A$  are also elements of  $B$ .

## Examples

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\} \Rightarrow A \subseteq B$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\} \Rightarrow B \subseteq A$
- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \bigcirc, \square\} \Rightarrow A \subseteq B$  and  $B \subseteq A$



# Subsets

Let  $A$  and  $B$  be two sets. We write  $A \subseteq B$  if all the elements of  $A$  are also elements of  $B$ .

## Examples

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\} \Rightarrow A \subseteq B$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\} \Rightarrow B \subseteq A$
- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \bigcirc, \square\} \Rightarrow A \subseteq B$  and  $B \subseteq A$

Let  $A$  and  $B$  be two sets. We write  $A \subset B$  if all the elements of  $A$  are also elements of  $B$  **and we know that some elements of  $B$  are not in  $A$**

# Subsets

Let  $A$  and  $B$  be two sets. We write  $A \subseteq B$  if all the elements of  $A$  are also elements of  $B$ .

## Examples

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\} \Rightarrow A \subseteq B$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\} \Rightarrow B \subseteq A$
- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \bigcirc, \square\} \Rightarrow A \subseteq B$  and  $B \subseteq A$

Let  $A$  and  $B$  be two sets. We write  $A \subset B$  if all the elements of  $A$  are also elements of  $B$  **and we know that some elements of  $B$  are not in  $A$**

## Examples

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\} \Rightarrow A \subset B$

# Subsets

Let  $A$  and  $B$  be two sets. We write  $A \subseteq B$  if all the elements of  $A$  are also elements of  $B$ .

## Examples

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\} \Rightarrow A \subseteq B$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\} \Rightarrow B \subseteq A$
- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \bigcirc, \square\} \Rightarrow A \subseteq B$  and  $B \subseteq A$

Let  $A$  and  $B$  be two sets. We write  $A \subset B$  if all the elements of  $A$  are also elements of  $B$  **and we know that some elements of  $B$  are not in  $A$**

## Examples

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\} \Rightarrow A \subset B$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\} \Rightarrow B \subset A$

# Subsets

Let  $A$  and  $B$  be two sets. We write  $A \subseteq B$  if all the elements of  $A$  are also elements of  $B$ .

## Examples

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\} \Rightarrow A \subseteq B$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\} \Rightarrow B \subseteq A$
- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \bigcirc, \square\} \Rightarrow A \subseteq B$  and  $B \subseteq A$

Let  $A$  and  $B$  be two sets. We write  $A \subset B$  if all the elements of  $A$  are also elements of  $B$  **and we know that some elements of  $B$  are not in  $A$**

## Examples

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\} \Rightarrow A \subset B$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\} \Rightarrow B \subset A$
- $A = \{\triangle, \bigcirc, \square\}$ ,  $B = \{\triangle, \bigcirc, \square\}$ . This time we CANNOT write  $A \subset B$  because  $B$  has no elements which are not in  $A$ . For the same reason, we CANNOT write  $B \subset A$

# Subsets, cont'd

## Remark

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\}$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\}$

# Subsets, cont'd

## Remark

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\}$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\}$

In the first example, it is correct writing either  $A \subseteq B$  or  $A \subset B$ . However,  $A \subset B$  is more informative, because it says us not only that all elements of  $A$  are in  $B$ , but also that  $B$  has some elements which are not in  $A$ . Similarly, in the second example, it is correct writing either  $B \subseteq A$  or  $B \subset A$ , but the second is more informative.

# Subsets, cont'd

## Remark

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\}$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\}$

In the first example, it is correct writing either  $A \subseteq B$  or  $A \subset B$ . However,  $A \subset B$  is more informative, because it says us not only that all elements of  $A$  are in  $B$ , but also that  $B$  has some elements which are not in  $A$ . Similarly, in the second example, it is correct writing either  $B \subseteq A$  or  $B \subset A$ , but the second is more informative.

The two relations  $A \subseteq B$ ,  $A \subset B$ , can also be read from right to left:

- We write  $B \supseteq A$  if  $A \subseteq B$

# Subsets, cont'd

## Remark

- $A = \{1, 100\}$ ,  $B = \{1, 100, 4.4\}$
- $A = \{\text{All cities in Europe}\}$ ,  $B = \{\text{Paris, Milan}\}$

In the first example, it is correct writing either  $A \subseteq B$  or  $A \subset B$ . However,  $A \subset B$  is more informative, because it says us not only that all elements of  $A$  are in  $B$ , but also that  $B$  has some elements which are not in  $A$ . Similarly, in the second example, it is correct writing either  $B \subseteq A$  or  $B \subset A$ , but the second is more informative.

The two relations  $A \subseteq B$ ,  $A \subset B$ , can also be read from right to left:

- We write  $B \supseteq A$  if  $A \subseteq B$
- We write  $B \supset A$  if  $A \subset B$



# Subtraction between sets

## Definition

Let  $A$  and  $B$  be two sets. The “A minus B” set, denoted by  $A \setminus B$ , is the set containing the elements in  $A$  which are not in  $B$ . In symbols:

$$A \setminus B = \{x \in A : x \notin B\}$$

# Subtraction between sets

## Definition

Let  $A$  and  $B$  be two sets. The “A minus B” set, denoted by  $A \setminus B$ , is the set containing the elements in  $A$  which are not in  $B$ . In symbols:

$$A \setminus B = \{x \in A : x \notin B\}$$

## Examples

- $A = \{1, 100, 4\}$ ,  $B = \{1\} \Rightarrow A \setminus B = \{100, 4\}$

# Subtraction between sets

## Definition

Let  $A$  and  $B$  be two sets. The “A minus B” set, denoted by  $A \setminus B$ , is the set containing the elements in  $A$  which are not in  $B$ . In symbols:

$$A \setminus B = \{x \in A : x \notin B\}$$

## Examples

- $A = \{1, 100, 4\}$ ,  $B = \{1\} \Rightarrow A \setminus B = \{100, 4\}$
- $A = \{1, 100, 4\}$ ,  $B = \{2, 3\} \Rightarrow A \setminus B = \{1, 100, 4\}$

# Subtraction between sets

## Definition

Let  $A$  and  $B$  be two sets. The “A minus B” set, denoted by  $A \setminus B$ , is the set containing the elements in  $A$  which are not in  $B$ . In symbols:

$$A \setminus B = \{x \in A : x \notin B\}$$

## Examples

- $A = \{1, 100, 4\}$ ,  $B = \{1\} \Rightarrow A \setminus B = \{100, 4\}$
- $A = \{1, 100, 4\}$ ,  $B = \{2, 3\} \Rightarrow A \setminus B = \{1, 100, 4\}$
- $A = \{\triangle, \diamond, \square\}$ ,  $B = \{\triangle, \bigcirc\} \Rightarrow A \setminus B = \{\diamond, \square\}$

# Subtraction between sets

## Definition

Let  $A$  and  $B$  be two sets. The “A minus B” set, denoted by  $A \setminus B$ , is the set containing the elements in  $A$  which are not in  $B$ . In symbols:

$$A \setminus B = \{x \in A : x \notin B\}$$

## Examples

- $A = \{1, 100, 4\}$ ,  $B = \{1\} \Rightarrow A \setminus B = \{100, 4\}$
- $A = \{1, 100, 4\}$ ,  $B = \{2, 3\} \Rightarrow A \setminus B = \{1, 100, 4\}$
- $A = \{\triangle, \diamond, \square\}$ ,  $B = \{\triangle, \bigcirc\} \Rightarrow A \setminus B = \{\diamond, \square\}$
- $A = \{\text{All natural numbers}\}$ ,  $B = \{\text{All natural numbers larger than } 10\}$

$$A \setminus B = \{\text{All natural numbers lower or equal to } 10\} = \{1, \dots, 10\}$$

# The complement set

## Definition

Let  $S$  be the universal set and  $B$  a subset of  $S$ . The complement set of  $B$  is the “ $S$  minus  $B$ ” set, namely the set of elements of  $S$  that are not contained in  $B$ .

In symbols:

$$B^c = S \setminus B = \{x \in S : x \notin B\}$$

# The complement set

## Definition

Let  $S$  be the universal set and  $B$  a subset of  $S$ . The complement set of  $B$  is the “ $S$  minus  $B$ ” set, namely the set of elements of  $S$  that are not contained in  $B$ .

In symbols:

$$B^c = S \setminus B = \{x \in S : x \notin B\}$$

## Example

- $S = \{1, 100, 4\}$ ,  $B = \{1\} \Rightarrow B^c = \{100, 4\}$

# The complement set

## Definition

Let  $S$  be the universal set and  $B$  a subset of  $S$ . The complement set of  $B$  is the “ $S$  minus  $B$ ” set, namely the set of elements of  $S$  that are not contained in  $B$ .

In symbols:

$$B^c = S \setminus B = \{x \in S : x \notin B\}$$

## Example

- $S = \{1, 100, 4\}$ ,  $B = \{1\} \Rightarrow B^c = \{100, 4\}$
- $S = \{\text{All cities in Europe}\}$ ,  $B = \{x \in S \mid x \text{ is a capital city}\}$

$$B^c = \{x \in S \mid x \text{ is *not* a capital city}\}$$



# Some subsets in $\mathbb{R}$ : The intervals

## Definition

A real interval with extremes  $a, b \in \mathbb{R}$  such that  $a \leq b$ , is the set of all real numbers between  $a$  and  $b$ .

We say that a real interval is

- **open** if extremes  $a$  and  $b$  are not included and we denote it by  $(a, b)$
- **closed** if extremes  $a$  and  $b$  are included and we denote it by  $[a, b]$
- **not open nor closed** one of the extreme is included and the other is not, that is  $[a, b)$  or  $(a, b]$
- **bounded** if both  $a$  and  $b$  are finite numbers
- **unbounded** if either  $a$ , or  $b$  or both are infinite, e.g.  $(-\infty, 1]$ ,  $(-3, +\infty)$ ,  $(-\infty, +\infty)$

# Some subsets in $\mathbb{R}$ : The intervals

## Definition

A real interval with extremes  $a, b \in \mathbb{R}$  such that  $a \leq b$ , is the set of all real numbers between  $a$  and  $b$ .

We say that a real interval is

- **open** if extremes  $a$  and  $b$  are not included and we denote it by  $(a, b)$
- **closed** if extremes  $a$  and  $b$  are included and we denote it by  $[a, b]$
- **not open nor closed** one of the extreme is included and the other is not, that is  $[a, b)$  or  $(a, b]$
- **bounded** if both  $a$  and  $b$  are finite numbers
- **unbounded** if either  $a$ , or  $b$  or both are infinite, e.g.  $(-\infty, 1]$ ,  $(-3, +\infty)$ ,  $(-\infty, +\infty)$

**Important:** Numeric sets with just one element are denoted with the curly parentheses, for instance  $\{2\}$  is the set that contains only the number 2.

# Statements

An assertion that can be either *true* or *false* is a statement.

# Statements

An assertion that can be either *true* or *false* is a statement.

Consider two statements, say statements  $P$  and statement  $Q$ .

# Statements

An assertion that can be either *true* or *false* is a statement.

Consider two statements, say statements  $P$  and statement  $Q$ .

We say

$$P \Rightarrow Q$$

to mean **If  $P$  holds true then also  $Q$  holds true** or  **$P$  implies  $Q$** .

# Statements

An assertion that can be either *true* or *false* is a statement.

Consider two statements, say statements  $P$  and statement  $Q$ .

We say

$$P \Rightarrow Q$$

to mean **If  $P$  holds true then also  $Q$  holds true** or  $P$  implies  $Q$ .

## **Example:**

- If it rains, then we open the umbrella

# Statements

An assertion that can be either *true* or *false* is a statement.

Consider two statements, say statements  $P$  and statement  $Q$ .

We say

$$P \Rightarrow Q$$

to mean **If  $P$  holds true then also  $Q$  holds true** or  $P$  implies  $Q$ .

## Example:

- If it rains, then we open the umbrella
- If  $x = 0$ , then  $x \cdot y = 0$

# Statements

An assertion that can be either *true* or *false* is a statement.

Consider two statements, say statements  $P$  and statement  $Q$ .

We say

$$P \Rightarrow Q$$

to mean **If  $P$  holds true then also  $Q$  holds true** or  **$P$  implies  $Q$** .

## Example:

- If it rains, then we open the umbrella
- If  $x = 0$ , then  $x \cdot y = 0$

The arrow in  $\Rightarrow$  gives the direction of the implication.



# Statements

If

$$P \Rightarrow Q \text{ and } Q \Rightarrow P$$

then *assertion P and assertion Q are equivalent.*

# Statements

If

$$P \Rightarrow Q \text{ and } Q \Rightarrow P$$

then *assertion P and assertion Q are equivalent*. In this case we write

$$P \Leftrightarrow Q$$

to mean **P if and only if Q**, or *P is equivalent to Q*.

# Statements

If

$$P \Rightarrow Q \text{ and } Q \Rightarrow P$$

then *assertion  $P$  and assertion  $Q$  are equivalent*. In this case we write

$$P \Leftrightarrow Q$$

to mean  **$P$  if and only if  $Q$** , or  $P$  is equivalent to  $Q$ .

## Example:

- If  $x = 0$  and  $y = 0$ , then  $x^2 + y^2 = 0$

# Statements

If

$$P \Rightarrow Q \text{ and } Q \Rightarrow P$$

then *assertion  $P$  and assertion  $Q$  are equivalent*. In this case we write

$$P \Leftrightarrow Q$$

to mean  **$P$  if and only if  $Q$** , or  $P$  is equivalent to  $Q$ .

## Example:

- If  $x = 0$  and  $y = 0$ , then  $x^2 + y^2 = 0$
- If  $x^2 + y^2 = 0$ , then  $x = 0$  and  $y = 0$

# Statements

If

$$P \Rightarrow Q \text{ and } Q \Rightarrow P$$

then *assertion  $P$  and assertion  $Q$  are equivalent*. In this case we write

$$P \Leftrightarrow Q$$

to mean  **$P$  if and only if  $Q$** , or  $P$  is equivalent to  $Q$ .

## Example:

- If  $x = 0$  and  $y = 0$ , then  $x^2 + y^2 = 0$
- If  $x^2 + y^2 = 0$ , then  $x = 0$  and  $y = 0$

In this case assertion  $P$  ( $x = 0$  and  $y = 0$ ) and assertion  $Q$  ( $x^2 + y^2 = 0$ ) are equivalent.

# Sufficient and necessary conditions

Suppose that

$$P \Rightarrow Q$$

We say that  **$P$  is a sufficient condition for  $Q$ .**

# Sufficient and necessary conditions

Suppose that

$$P \Rightarrow Q$$

We say that  **$P$  is a sufficient condition for  $Q$** . Indeed it is sufficient that  $P$  is true to get that also  $Q$  is true.

# Sufficient and necessary conditions

Suppose that

$$P \Rightarrow Q$$

We say that  **$P$  is a sufficient condition for  $Q$** . Indeed it is sufficient that  $P$  is true to get that also  $Q$  is true.

We also say that  **$Q$  is a necessary condition for  $P$** .



# Sufficient and necessary conditions

Suppose that

$$P \Rightarrow Q$$

We say that  **$P$  is a sufficient condition for  $Q$** . Indeed it is sufficient that  $P$  is true to get that also  $Q$  is true.

We also say that  **$Q$  is a necessary condition for  $P$** . Indeed, if  $P$  is true then necessarily  $Q$  is true.

# Sufficient and necessary conditions

Suppose that

$$P \Rightarrow Q$$

We say that  **$P$  is a sufficient condition for  $Q$** . Indeed it is sufficient that  $P$  is true to get that also  $Q$  is true.

We also say that  **$Q$  is a necessary condition for  $P$** . Indeed, if  $P$  is true then necessarily  $Q$  is true.

In summary the following are equivalent:

- $P \Rightarrow Q$

# Sufficient and necessary conditions

Suppose that

$$P \Rightarrow Q$$

We say that  **$P$  is a sufficient condition for  $Q$** . Indeed it is sufficient that  $P$  is true to get that also  $Q$  is true.

We also say that  **$Q$  is a necessary condition for  $P$** . Indeed, if  $P$  is true then necessarily  $Q$  is true.

In summary the following are equivalent:

- $P \Rightarrow Q$
- $P$  implies  $Q$

# Sufficient and necessary conditions

Suppose that

$$P \Rightarrow Q$$

We say that  **$P$  is a sufficient condition for  $Q$** . Indeed it is sufficient that  $P$  is true to get that also  $Q$  is true.

We also say that  **$Q$  is a necessary condition for  $P$** . Indeed, if  $P$  is true then necessarily  $Q$  is true.

In summary the following are equivalent:

- $P \Rightarrow Q$
- $P$  implies  $Q$
- If  $P$  then  $Q$

# Sufficient and necessary conditions

Suppose that

$$P \Rightarrow Q$$

We say that  **$P$  is a sufficient condition for  $Q$** . Indeed it is sufficient that  $P$  is true to get that also  $Q$  is true.

We also say that  **$Q$  is a necessary condition for  $P$** . Indeed, if  $P$  is true then necessarily  $Q$  is true.

In summary the following are equivalent:

- $P \Rightarrow Q$
- $P$  implies  $Q$
- If  $P$  then  $Q$
- $P$  is a sufficient condition for  $Q$

# Sufficient and necessary conditions

Suppose that

$$P \Rightarrow Q$$

We say that  **$P$  is a sufficient condition for  $Q$** . Indeed it is sufficient that  $P$  is true to get that also  $Q$  is true.

We also say that  **$Q$  is a necessary condition for  $P$** . Indeed, if  $P$  is true then necessarily  $Q$  is true.

In summary the following are equivalent:

- $P \Rightarrow Q$
- $P$  implies  $Q$
- If  $P$  then  $Q$
- $P$  is a sufficient condition for  $Q$
- $Q$  is a necessary condition for  $P$

# Sufficient and necessary conditions

Suppose that

$$P \Leftrightarrow Q$$

# Sufficient and necessary conditions

Suppose that

$$P \Leftrightarrow Q$$

We know that this expression means that both the implications  $P \Rightarrow Q$  and  $Q \Rightarrow P$  hold at the same time.



# Sufficient and necessary conditions

Suppose that

$$P \Leftrightarrow Q$$

We know that this expression means that both the implications  $P \Rightarrow Q$  and  $Q \Rightarrow P$  hold at the same time.

In this case  $P$  is a sufficient condition for  $Q$  (from the first implication)

# Sufficient and necessary conditions

Suppose that

$$P \Leftrightarrow Q$$

We know that this expression means that both the implications  $P \Rightarrow Q$  and  $Q \Rightarrow P$  hold at the same time.

In this case  $P$  is a sufficient condition for  $Q$  (from the first implication) and  $P$  is also a necessary condition for  $Q$  (from the second implication).

# Sufficient and necessary conditions

Suppose that

$$P \Leftrightarrow Q$$

We know that this expression means that both the implications  $P \Rightarrow Q$  and  $Q \Rightarrow P$  hold at the same time.

In this case  $P$  is a sufficient condition for  $Q$  (from the first implication) and  $P$  is also a necessary condition for  $Q$  (from the second implication). Hence we say that  $P$  is a **sufficient and necessary condition** for  $Q$ .

# Sufficient and necessary conditions

Suppose that

$$P \Leftrightarrow Q$$

We know that this expression means that both the implications  $P \Rightarrow Q$  and  $Q \Rightarrow P$  hold at the same time.

In this case  $P$  is a sufficient condition for  $Q$  (from the first implication) and  $P$  is also a necessary condition for  $Q$  (from the second implication). Hence we say that  $P$  is a **sufficient and necessary condition** for  $Q$ .

In summary the following are equivalent:

- $P \Leftrightarrow Q$

# Sufficient and necessary conditions

Suppose that

$$P \Leftrightarrow Q$$

We know that this expression means that both the implications  $P \Rightarrow Q$  and  $Q \Rightarrow P$  hold at the same time.

In this case  $P$  is a sufficient condition for  $Q$  (from the first implication) and  $P$  is also a necessary condition for  $Q$  (from the second implication). Hence we say that  $P$  is a **sufficient and necessary condition** for  $Q$ .

In summary the following are equivalent:

- $P \Leftrightarrow Q$
- $P$  is equivalent to  $Q$

# Sufficient and necessary conditions

Suppose that

$$P \Leftrightarrow Q$$

We know that this expression means that both the implications  $P \Rightarrow Q$  and  $Q \Rightarrow P$  hold at the same time.

In this case  $P$  is a sufficient condition for  $Q$  (from the first implication) and  $P$  is also a necessary condition for  $Q$  (from the second implication). Hence we say that  $P$  is a **sufficient and necessary condition** for  $Q$ .

In summary the following are equivalent:

- $P \Leftrightarrow Q$
- $P$  is equivalent to  $Q$
- $P$  if and only if  $Q$

# Sufficient and necessary conditions

Suppose that

$$P \Leftrightarrow Q$$

We know that this expression means that both the implications  $P \Rightarrow Q$  and  $Q \Rightarrow P$  hold at the same time.

In this case  $P$  is a sufficient condition for  $Q$  (from the first implication) and  $P$  is also a necessary condition for  $Q$  (from the second implication). Hence we say that  $P$  is a **sufficient and necessary condition** for  $Q$ .

In summary the following are equivalent:

- $P \Leftrightarrow Q$
- $P$  is equivalent to  $Q$
- $P$  if and only if  $Q$
- $P$  is a sufficient and necessary condition for  $Q$

# The structure of a theorem

A **theorem** is made of

- **Hypotheses**: these are the premises under which the conclusion holds



# The structure of a theorem

A **theorem** is made of

- **Hypotheses**: these are the premises under which the conclusion holds

# The structure of a theorem

A **theorem** is made of

- **Hypotheses**: these are the premises under which the conclusion holds
- Thesis: this is the conclusion which follows from the premises

# The structure of a theorem

A **theorem** is made of

- **Hypotheses**: these are the premises under which the conclusion holds
- Thesis: this is the conclusion which follows from the premises

When stating a theorem it must be clear what the hypotheses are and what the thesis is!

# The structure of a theorem

A **theorem** is made of

- **Hypotheses**: these are the premises under which the conclusion holds
- Thesis: this is the conclusion which follows from the premises

When stating a theorem it must be clear what the hypotheses are and what the thesis is!

**Example: The Pythagorean Theorem**

# The structure of a theorem

A **theorem** is made of

- **Hypotheses**: these are the premises under which the conclusion holds
- Thesis: this is the conclusion which follows from the premises

When stating a theorem it must be clear what the hypotheses are and what the thesis is!

## **Example: The Pythagorean Theorem**

Hypothesis: If a triangle is right-angled

# The structure of a theorem

A **theorem** is made of

- **Hypotheses**: these are the premises under which the conclusion holds
- Thesis: this is the conclusion which follows from the premises

When stating a theorem it must be clear what the hypotheses are and what the thesis is!

## **Example: The Pythagorean Theorem**

Hypothesis: If a triangle is right-angled

Thesis: the square of the hypotenuse is equal to the sum of the squares of the other two sides

# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \end{cases}$$



# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

## Examples

- $|1| =$

# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

## Examples

- $|1| = 1$ ,

# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

## Examples

- $|1| = 1, \quad |-1| =$

# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

## Examples

- $|1| = 1, \quad |-1| = 1,$

# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

## Examples

- $|1| = 1$ ,  $|-1| = 1$ ,  $|\frac{1}{2}| =$

# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

## Examples

- $|1| = 1, \quad |-1| = 1, \quad \left|\frac{1}{2}\right| = \frac{1}{2},$

# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

## Examples

- $|1| = 1$ ,  $|-1| = 1$ ,  $|\frac{1}{2}| = \frac{1}{2}$ ,  $|\frac{2}{127}| =$



# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

## Examples

- $|1| = 1$ ,  $|-1| = 1$ ,  $|\frac{1}{2}| = \frac{1}{2}$ ,  $|\frac{2}{127}| = \frac{2}{127}$ ,

# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

## Examples

- $|1| = 1$ ,  $|-1| = 1$ ,  $|\frac{1}{2}| = \frac{1}{2}$ ,  $|\frac{2}{127}| = \frac{2}{127}$ ,  $|\pi| =$

# The absolute value

## Definition

Let  $a \in \mathbb{R}$ . The absolute value of  $a$ , denoted by  $|a|$ , is given by:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

## Examples

- $|1| = 1$ ,  $|-1| = 1$ ,  $|\frac{1}{2}| = \frac{1}{2}$ ,  $|\frac{2}{127}| = \frac{2}{127}$ ,  $|\pi| = \pi$

# Distance between a point in $\mathbb{R}$ and the origin

## Definition

Let  $a \in \mathbb{R}$ . The distance between  $a$  and the origin is the length of the segment line between the origin and point  $a$ , hence it corresponds to  $a$  if  $a > 0$  and  $-a$  if  $a < 0$ . Put in other words

$$d(a, 0) = |a|$$

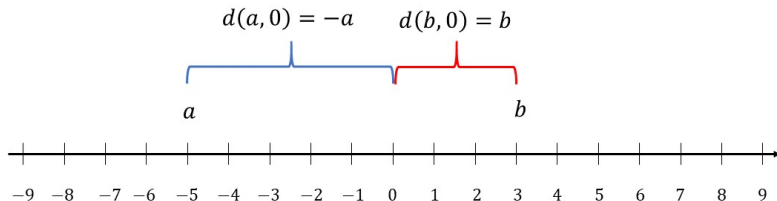
# Distance between a point in $\mathbb{R}$ and the origin

## Definition

Let  $a \in \mathbb{R}$ . The distance between  $a$  and the origin is the length of the segment line between the origin and point  $a$ , hence it corresponds to  $a$  if  $a > 0$  and  $-a$  if  $a < 0$ . Put in other words

$$d(a, 0) = |a|$$

Notice that the length must be positive (or equal to zero).



# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0, \quad \forall x_1, x_2 \in \mathbb{R}$

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0, \quad \forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point



# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| =$

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| = 1$ ,

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| = 1$ ,  $d(2, 1) = |2 - 1| =$

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| = 1$ ,  $d(2, 1) = |2 - 1| = 1$

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| = 1$ ,  $d(2, 1) = |2 - 1| = 1$
- $d(1, -1) = |1 - (-1)| =$

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| = 1$ ,  $d(2, 1) = |2 - 1| = 1$
- $d(1, -1) = |1 - (-1)| = 2$ ,

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| = 1$ ,  $d(2, 1) = |2 - 1| = 1$
- $d(1, -1) = |1 - (-1)| = 2$ ,  $d(-1, 1) = |-1 - 1| =$



# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| = 1$ ,  $d(2, 1) = |2 - 1| = 1$
- $d(1, -1) = |1 - (-1)| = 2$ ,  $d(-1, 1) = |-1 - 1| = 2$

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| = 1$ ,  $d(2, 1) = |2 - 1| = 1$
- $d(1, -1) = |1 - (-1)| = 2$ ,  $d(-1, 1) = |-1 - 1| = 2$
- $d(0, -\frac{1}{2}) = |0 - (-\frac{1}{2})| =$

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| = 1$ ,  $d(2, 1) = |2 - 1| = 1$
- $d(1, -1) = |1 - (-1)| = 2$ ,  $d(-1, 1) = |-1 - 1| = 2$
- $d(0, -\frac{1}{2}) = |0 - (-\frac{1}{2})| = \frac{1}{2}$ ,

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| = 1$ ,  $d(2, 1) = |2 - 1| = 1$
- $d(1, -1) = |1 - (-1)| = 2$ ,  $d(-1, 1) = |-1 - 1| = 2$
- $d(0, -\frac{1}{2}) = |0 - (-\frac{1}{2})| = \frac{1}{2}$ ,  $d(-\frac{1}{2}, 0) = |-\frac{1}{2} - 0| =$

# Distance between two points in $\mathbb{R}$

## Definition

Let  $x_1 \in \mathbb{R}$  and  $x_2 \in \mathbb{R}$ . The distance between  $x_1$  and  $x_2$  is given by:

$$d(x_1, x_2) = |x_1 - x_2|$$

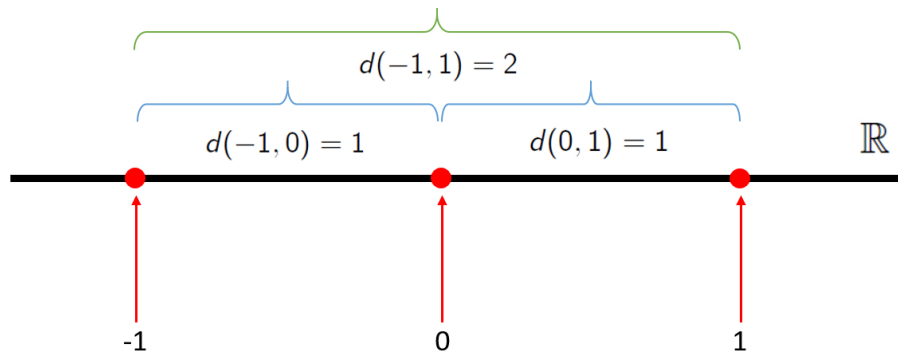
Note that, by definition:

- $d(x_1, x_2) \geq 0$ ,  $\forall x_1, x_2 \in \mathbb{R}$
- $d(x_1, x_2) = 0$  if and only if  $x_1$  and  $x_2$  are the same point
- $d(x_1, x_2) = d(x_2, x_1)$ , since  $|x_1 - x_2| = |x_2 - x_1|$  (that is the order does not matter when we measure distances).

## Examples

- $d(1, 2) = |1 - 2| = 1$ ,  $d(2, 1) = |2 - 1| = 1$
- $d(1, -1) = |1 - (-1)| = 2$ ,  $d(-1, 1) = |-1 - 1| = 2$
- $d(0, -\frac{1}{2}) = |0 - (-\frac{1}{2})| = \frac{1}{2}$ ,  $d(-\frac{1}{2}, 0) = |-\frac{1}{2} - 0| = \frac{1}{2}$

# Distance between two points in $\mathbb{R}$ : example



# Functions

## Definition (General definition)

Let  $A$  and  $B$  be two sets. A function  $f$  defined on  $A$  and with values in  $B$  is a law that associates to any element  $x \in A$  **one and only one** element  $y \in B$

# Functions

## Definition (General definition)

Let  $A$  and  $B$  be two sets. A function  $f$  defined on  $A$  and with values in  $B$  is a law that associates to any element  $x \in A$  **one and only one** element  $y \in B$

To indicate a function we use the notation

$$f : A \rightarrow B \quad \text{or} \quad y = f(x).$$



# Functions

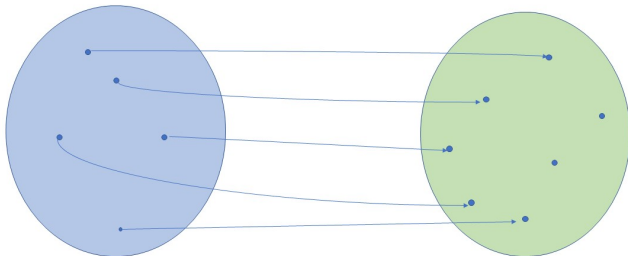
## Definition (General definition)

Let  $A$  and  $B$  be two sets. A function  $f$  defined on  $A$  and with values in  $B$  is a law that associates to any element  $x \in A$  **one and only one** element  $y \in B$

To indicate a function we use the notation

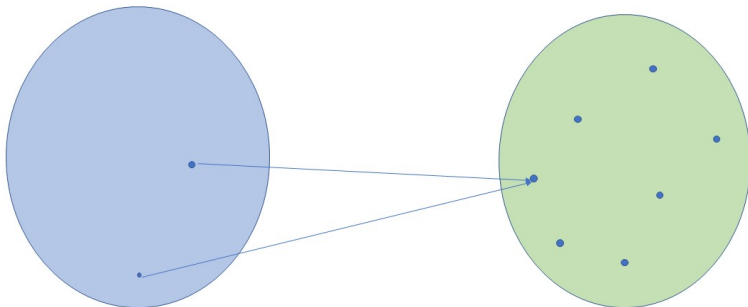
$$f : A \rightarrow B \quad \text{or} \quad y = f(x).$$

This is a function



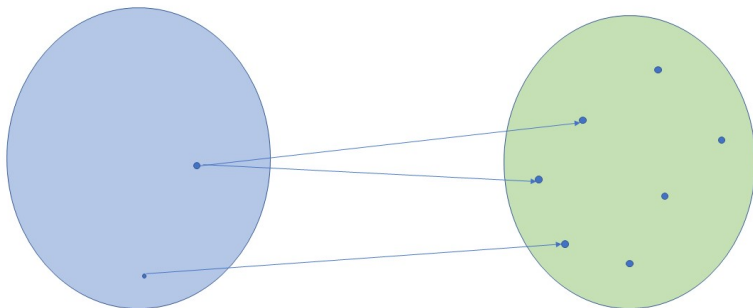
# Functions

This is a function



# Functions

This is NOT a function



# Functions

## Definition (Real function of a Real variable)

Let  $D \subseteq \mathbb{R}$  (eventually  $D = \mathbb{R}$ ).

# Functions

## Definition (Real function of a Real variable)

Let  $D \subseteq \mathbb{R}$  (eventually  $D = \mathbb{R}$ ). A real function of a real variable is a law that associates to any real number  $x \in D$  **one and only one** real number  $y \in \mathbb{R}$  such that  $y = f(x)$ .

# Functions

## Definition (Real function of a Real variable)

Let  $D \subseteq \mathbb{R}$  (eventually  $D = \mathbb{R}$ ). A real function of a real variable is a law that associates to any real number  $x \in D$  **one and only one** real number  $y \in \mathbb{R}$  such that  $y = f(x)$ .

The variable  $x$  is called the **independent variable**,

# Functions

## Definition (Real function of a Real variable)

Let  $D \subseteq \mathbb{R}$  (eventually  $D = \mathbb{R}$ ). A real function of a real variable is a law that associates to any real number  $x \in D$  **one and only one** real number  $y \in \mathbb{R}$  such that  $y = f(x)$ .

The variable  $x$  is called the **independent variable**, the variable  $y$  is called the **dependent variable**.

# Functions

## Definition (Real function of a Real variable)

Let  $D \subseteq \mathbb{R}$  (eventually  $D = \mathbb{R}$ ). A real function of a real variable is a law that associates to any real number  $x \in D$  **one and only one** real number  $y \in \mathbb{R}$  such that  $y = f(x)$ .

The variable  $x$  is called the **independent variable**, the variable  $y$  is called the **dependent variable**. In Economics  $x$  is also called the exogenous variable and  $y$  the endogenous variable.



# Functions

## Definition (Real function of a Real variable)

Let  $D \subseteq \mathbb{R}$  (eventually  $D = \mathbb{R}$ ). A real function of a real variable is a law that associates to any real number  $x \in D$  **one and only one** real number  $y \in \mathbb{R}$  such that  $y = f(x)$ .

The variable  $x$  is called the **independent variable**, the variable  $y$  is called the **dependent variable**. In Economics  $x$  is also called the exogenous variable and  $y$  the endogenous variable.

$y$  is called **the image of  $x$  through the function  $f$**

# Functions

## Definition (Domain and Range)

The set  $D$  of all real values for which the law  $f$  makes sense is called the *domain* of the function  $f$ :

# Functions

## Definition (Domain and Range)

The set  $D$  of all real values for which the law  $f$  makes sense is called the *domain* of the function  $f$ :

$$D = \{x \in \mathbb{R} : f(x) \text{ is well defined} \}$$

# Functions

## Definition (Domain and Range)

The set  $D$  of all real values for which the law  $f$  makes sense is called the *domain* of the function  $f$ :

$$D = \{x \in \mathbb{R} : f(x) \text{ is well defined} \}$$

The set of all images  $f(x)$ , for all  $x \in D$  is called the *range* of the function

# Functions

## Definition (Domain and Range)

The set  $D$  of all real values for which the law  $f$  makes sense is called the *domain* of the function  $f$ :

$$D = \{x \in \mathbb{R} : f(x) \text{ is well defined} \}$$

The set of all images  $f(x)$ , for all  $x \in D$  is called the *range* of the function

$$R = \{y \in \mathbb{R} : y = f(x), \forall x \in D\}$$

## Definition (Graph)

Let  $f : D \rightarrow \mathbb{R}$  be a real function of a real variable. The graph of  $f$  is the set

# Functions

## Definition (Domain and Range)

The set  $D$  of all real values for which the law  $f$  makes sense is called the *domain* of the function  $f$ :

$$D = \{x \in \mathbb{R} : f(x) \text{ is well defined} \}$$

The set of all images  $f(x)$ , for all  $x \in D$  is called the *range* of the function

$$R = \{y \in \mathbb{R} : y = f(x), \forall x \in D\}$$

## Definition (Graph)

Let  $f : D \rightarrow \mathbb{R}$  be a real function of a real variable. The graph of  $f$  is the set

$$G = \{(x, f(x)) : x \in D\}$$

# Functions

## Definition (Domain and Range)

The set  $D$  of all real values for which the law  $f$  makes sense is called the *domain* of the function  $f$ :

$$D = \{x \in \mathbb{R} : f(x) \text{ is well defined} \}$$

The set of all images  $f(x)$ , for all  $x \in D$  is called the *range* of the function

$$R = \{y \in \mathbb{R} : y = f(x), \forall x \in D\}$$

## Definition (Graph)

Let  $f : D \rightarrow \mathbb{R}$  be a real function of a real variable. The graph of  $f$  is the set

$$G = \{(x, f(x)) : x \in D\}$$

The plot of the function  $f$  is the representation of the graph on a Cartesian plane.