Week 2

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Mathematics I

University of Rome Tor Vergata

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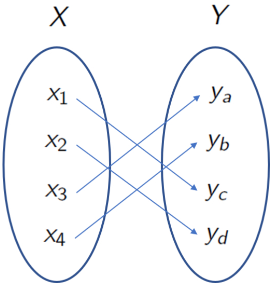
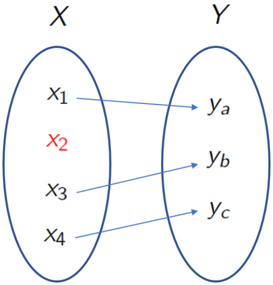
Functions: the intuition

Intuitively, a function is a rule that associates to **each** element of a set X, **only and only one** element in another set Y.

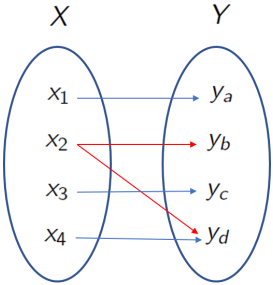


This is a function between X and Y

Functions: the intuition, cont'd



This is **not** a function between X and Y because $x_2 \in X$ is not mapped into any element in Y



This is **not** a function between X and Y because x_2 is mapped into more than one element in Y

Functions: the definition

Definition

Let $D \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$. A function is a rule that associates to each element of D one and only one element of $\mathbb{R}$. In symbols we write:

$$f : D \to \mathbb{R}$$

meaning that

$$\forall x \in D \ \Rightarrow \exists! \ y \in \mathbb{R} : \ y = f(x)$$

The set D is called the **domain** of the function.

- The variable x is called "independent variable", it can take values in D
- The variable y is called the "dependent variable", it can take values in $\mathbb{R}$.
- In economics x is called the exogenous variable and y is called the endogenous variable

The domain of a function

Definition

Let $f : D \to \mathbb{R}$ be a function. The domain of the function, $D \subseteq \mathbb{R}$, is the set of all values $x \in \mathbb{R}$ for which the expression $f(x)$ makes sense.

Three cases require computations:

1. $f(x)$ is a rational function
2. $f(x)$ is an irrational (with even index)
3. $f(x)$ is a logarithmic function

Domain of rational functions

Let $f(x) = \dfrac{P(x)}{Q(x)}$, and assume that $Q(x)$ makes sense for all $x \in \mathbb{R}$.
Then the function $f(x)$ is well defined if and only if $Q(x) \neq 0$. This means that

$$D = \{x \in \mathbb{R} : \; Q(x) \neq 0\}$$

Examples

- $f(x) = \dfrac{x+3}{x^2-1}$

$$D = \{x \in \mathbb{R} : \; x \neq \pm 1\}$$

- $f(x) = e^{\frac{x+5}{x-3}}$

$$D = \{x \in \mathbb{R} : \; x \neq 3\}$$

Domain of irrational functions

Let $f(x) = \sqrt[n]{G(x)}$, and assume that $G(x)$ makes sense for all $x \in \mathbb{R}$.
There are two possibilities:

- if n is even, then the function $f(x)$ is well defined if and only if $G(x) \geq 0$. This means that

$$D = \{x \in \mathbb{R} : \; G(x) \geq 0\}$$

- if n is odd, then the function $f(x)$ is well defined for all $x \in \mathbb{R}$

Examples

- $f(x) = \sqrt{x^2 - 5}$
  This is an irrational function with even index $(n = 2)$. Then we have

$$D = \{x \in \mathbb{R} : \; x \leq -\sqrt{5} \text{ or } x \geq \sqrt{5}\}$$

- $f(x) = \sqrt[3]{x + 2}$
  This is an irrational function with odd index $(n = 3)$. Then we have

$$D = \mathbb{R}$$

Domain of logarithmic functions

Let $f(x) = \log H(x)$, and assume that $H(x)$ makes sense for all $x \in \mathbb{R}$.
Then the function $f(x)$ is well defined if and only if $H(x) > 0$. This means that

$$D = \{x \in \mathbb{R} : \ H(x) \geq 0\}$$

Examples

- $f(x) = \log(1 - x^2)$

$$D = \{x \in \mathbb{R} : \ -1 < x < 1\}$$

- $f(x) = \log(x^2 + 2)$
  Since $x^2 + 2 > 0$ for all $x \in \mathbb{R}$ we get that

$$D = \mathbb{R}$$

Example

These three condition must be combined together if a function contains fractions, roots and logarithms.

Example

$$f(x) = \frac{x}{\log(x+2)}$$

We have that:

- $\log(x+2) \neq 0$ for the existence of the fraction
- $x + 2 > 0$ for the existence of the logarithm

Hence we have

$$\begin{cases} \log(x+2) & \neq 0 \\ x+2 & > 0 \end{cases}$$

which implies that

$$D = \{x \in \mathbb{R} : \; x > -2 \text{ and } x \neq -1\}$$

The range of a function

Intuitively, the range of a function is the set of all points of $\mathbb{R}$ that can be obtained by applying the function f to the points of D. That is, the set of all possible "dependent variables".

We can also say that the range os the set of all *images* of points of the domain through the function f

Definition

Let $f : D \to \mathbb{R}$ be a function. The range of f is the set:

$$R_f = \{y \in \mathbb{R} \mid \exists x \in D \ : \ y = f(x)\}$$

Examples

- $y = f(x) = x, \quad D = \mathbb{R}, \quad R_f = \mathbb{R}$, because by applying $f(x)$ to each $x \in D$ we obtain any point in $\mathbb{R}$.
- $y = f(x) = x^2, \quad D = \mathbb{R}, \quad R_f = \{x \in \mathbb{R} \mid x \geq 0\}$, because by applying $f(x)$ to each $x \in D$ we obtain only zero or a positive number.
- $y = f(x) = \frac{1}{x}, \quad D = \mathbb{R} \backslash \{0\}, \quad R_f = \mathbb{R} \backslash \{0\}$, because by applying $f(x)$ to each $x \in D$ we obtain any point of $\mathbb{R}$ except zero.

Odd/even functions

Definition

Let $f : D \to \mathbb{R}$ be a function. f is **even** if

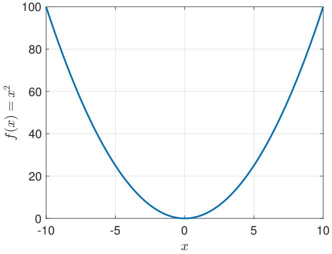
1. for any $x \in D$ then also $-x \in D$
2. $f(-x) = f(x)$

Notice that both conditions must hold.

Because of $f(x) = f(-x)$ for all $x \in \mathbb{R}$, then the plot of the graph of the function is symmetric with respect to the axis $x = 0$.

Example of an even function

$$f(x) = x^2$$



Indeed: The domain of the functions is $D = \mathbb{R}$. For all $x \in D$

- $-x \in D$
- $f(-x) = (-x)^2 = x^2 = f(x)$

Odd/even functions

Definition

Let $f : D \to \mathbb{R}$ be a function. f is **odd** if

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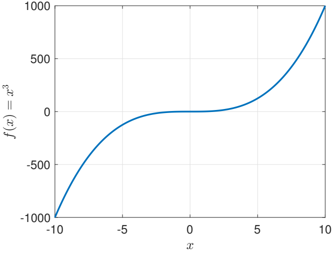
Notice that both conditions must hold.

Because of $f(-x) = f(x)$ for all $x \in \mathbb{R}$, then the plot of the graph of the function is symmetric with respect to the origin.

Example of an even function

$$f(x) = x^3$$

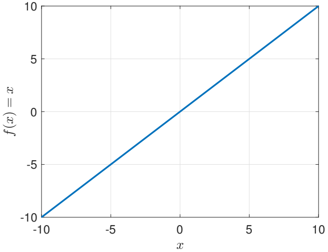


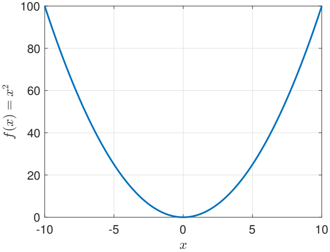
Indeed: The domain of the functions is $D = \mathbb{R}$. For all $x \in D$

- $-x \in D$
- $f(-x) = (-x)^3 = -x^3 = -f(x)$

Increasing and decreasing functions: the intuition

To get an intuition of when a function is increasing or decreasing, simply look at its graph:





- The function $f(x) = x$ is increasing in its entire domain
- The function $f(x) = x^2$ in decreasing in $(-\infty, 0)$ and increasing in $(0, +\infty)$

Increasing and decreasing functions: the definition

Definition

Let $f : D \to \mathbb{R}$ a function and let $I = (a, b) \subseteq D$ an open interval in D. The function f is strictly increasing in I if:

$$\forall x_1, x_2 \in I \ : \ x_1 < x_2 \ \Rightarrow \ f(x_1) < f(x_2)$$

The function f is increasing if:

$$\forall x_1, x_2 \in I \ : \ x_1 < x_2 \ \Rightarrow \ f(x_1) \leq f(x_2)$$

Increasing and decreasing functions: the definition

Definition

Let $f : D \to \mathbb{R}$ a function and let $I = (a, b) \subset D$ an open interval in D. The function f is strictly decreasing in I if:

$$\forall x_1, x_2 \in I \;:\; x_1 < x_2 \;\Rightarrow\; f(x_1) > f(x_2)$$

The function f is decreasing if:

$$\forall x_1, x_2 \in I \;:\; x_1 < x_2 \;\Rightarrow\; f(x_1) \geq f(x_2)$$

Increasing and decreasing functions: examples

$$f(x) = 2x + 3$$

Show that the function is strictly increasing in $\mathbb{R}$.

Note that $D = \mathbb{R}$. Consider two points x_1, $x_2 \in \mathbb{R}$, with $x_1 < x_2$. We will show that $f(x_1) < f(x_2)$ (Notice that $f(x_1) = 2x_1 + 3$ and $f(x_2) = 2x_2 + 3$)

To do this, we apply properties of real numbers:

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow 2x_1 + 3 < 2x_2 + 3$$

Thus, f is strictly increasing in $\mathbb{R}$.

Increasing and decreasing functions: examples, cont'd

$$f(x) = x^2$$

Show that the function is decreasing in $(-\infty, 0)$.

Note that $D = \mathbb{R}$. Consider two points x_1, $x_2 \in (-\infty, 0)$, with $x_1 < x_2$. We will show that $f(x_1) > f(x_2)$ (Notice that $f(x_1) = x_1^2$ and $f(x_2) = x_2^2$)

To do this, we apply properties of real numbers:

Since $x_1 < x_2$ and they are both negative, we get that $|x_1| > |x_2|$, then it holds that

$$x_1 < x_2 \Rightarrow (x_1)^2 > (x_2)^2$$

Thus, f is strictly decreasing in $(-\infty, 0)$.

A more general definition of function

Definition

Given two sets $X \subseteq \mathbb{R}$, $Y \subseteq \mathbb{R}$ a function f is a rule that associates to each element $x \in X$ one and only one element $y \in Y$. We write:

$$f : X \to Y$$

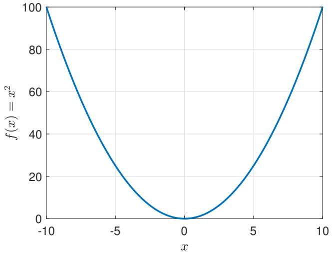
meaning that

$$\forall x \in X \ \Rightarrow \exists! \ y \in Y : \ y = f(x)$$

Notice that for the function f to be well defined we must have that $X \subseteq D$. The set D is the largest set of real numbers for which the expression $f(x)$ makes sense.

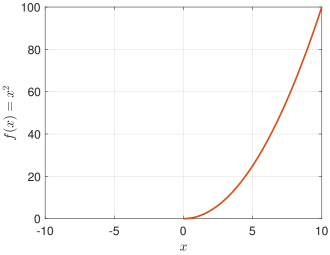
A more general definition of function, example

Let's consider the same function $f(x) = x^2$ defined in two different domains:



$$f(x) : \mathbb{R} \to \mathbb{R}$$

$$f(x) = x^2$$



$$f(x) : [0, +\infty) \to \mathbb{R}$$

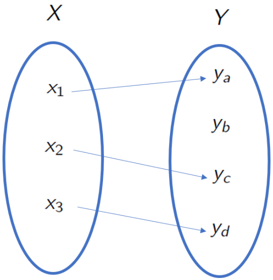
$$f(x) = x^2$$

Even if the formula $f(x) = x^2$ is the same in both cases, the two functions are completely different.

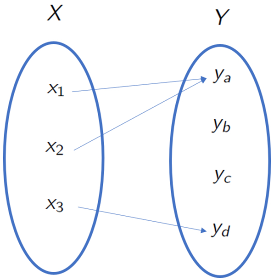
A function is not only determined by the form of $f(x)$. Sets X and Y matter!
Important: When X and Y are not specified it is implicitly assumed that $X = D$ and $Y = \mathbb{R}$.

Injective functions: the intuition

Intuitively, a function $f : X \to Y$ is injective if the images of distinct points in X correspond to distinct points in Y



This function is injective because distinct elements of X are mapped into distinct elements of Y



This function is **not** injective because x_1 and x_2 are mapped into the same element of Y

Injective functions: the definition

Definition

Let $f : X \to Y$ be a function, with $X \subseteq D$, $Y \subseteq \mathbb{R}$. f is said to be injective in X if

$$\forall x_1, x_2 \in X, \text{ if } x_1 \neq x_2, \Rightarrow f(x_1) \neq f(x_2)$$

Equivalently: f is said to be injective in X if

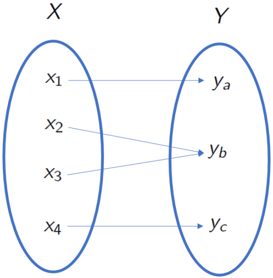
$$\text{for all } x_1, x_2 \in X \text{ such that } f(x_1) = f(x_2) \text{ then } x_1 = x_2$$

Examples

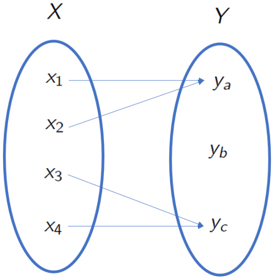
- The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not injective.
  Indeed, $f(2) = 4$ and $f(-2) = 4$. Thus, f maps $2, -2 \in \mathbb{R}$ to the same point $y = 4$.
- The function $f : [0, +\infty) \to \mathbb{R}$, $f(x) = x^2$ is injective!
  There is at most one positive number x such that $x^2 = y$, for all $y \in \mathbb{R}$.
- The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is injective.
  Indeed, f maps each $x \in \mathbb{R}$ to distinct points $y \in \mathbb{R}$.

Surjective functions: the intuition

Intuitively, a function $f : X \to Y$ is surjective if all the elements of the co-domain are "reached" by the function



This function is surjective because all elements of Y are "reached" by f. Note however that f is not injective



This function is **not** surjective because there are no elements in X that are mapped into y_b. Note also that f is not injective

Surjective functions: the definition

Definition

A function $f : X \to Y$ is said to be surjective if

$$\forall y \in Y, \quad \exists x \in X : \; y = f(x)$$

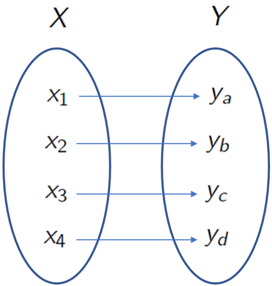
Examples

- The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not surjective.
  Indeed, $y = x^2$ is always a non-negative number, and therefore negative real numbers (which belong to the co-domain) are **not** "reached" by the function.
- The function $f : \mathbb{R} \to [0, +\infty)$, $f(x) = x^2$ is surjective!
  Every point $y \in [0, +\infty)$ can be "reached" by the function.
- The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is surjective.
  Every point of $\mathbb{R}$ can be "reached" by the function.

Bijective functions

Definition

A function $f : X \to Y$ is said to be bijective if it is both injective and surjective.



This function is injective because it associates to distinct elements in X distinct elements in Y and at the same time it is surjective because all elements of Y are "reached" by the function.

The inverse function

Why are bijective functions important? Bijective functions are **invertible**

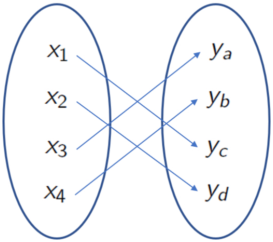
Theorem

Let $f : X \to Y$ be a function, with $X \subseteq D$ and $Y \subseteq \mathbb{R}$.

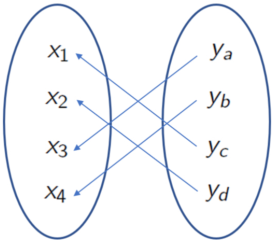
f is invertible **if and only if** *f is bijective.*

Moreover the inverse function $f^{-1} : Y \to X$ exists and it is unique.

$f : X \to Y$

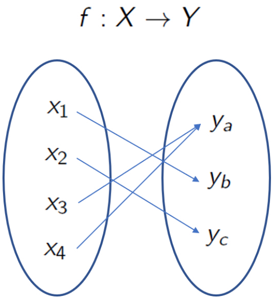


$f^{-1} : Y \to X$

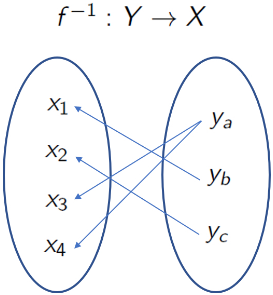


The inverse function: existence

If the function f is not bijective, the inverse function does not exist



Note that this function is not injective

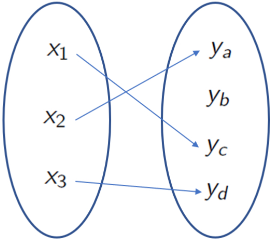


$f^{-1} : Y \to X$ **cannot** be a function because it associates to y_a more than one element in X

The inverse function: existence, cont'd

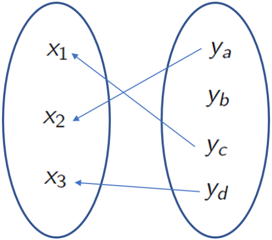
If the function f is not bijective, the inverse function does not exist

$f : X \to Y$



Note that this function is not surjective

$f^{-1} : Y \to X$



$f^{-1} : Y \to X$ **cannot** be a function because there are elements in Y that are not mapped into any element in X

The inverse function

How is the inverse function defined?

Definition

Let $f : X \to Y$ be an invertible function, with $X \subseteq D$ and $Y \subseteq \mathbb{R}$.
Then the inverse function $f^{-1} : Y \to X$ is the function that verifies

$$f(f^{-1}(y)) = y \text{ and } f^{-1}(f(x)) = x$$

Sufficient conditions for invertibility

Is there a **sufficient condition** that guarantees invertibility of a function?

Theorem

Sufficient conditions for a function to be invertible Let $f : X \to Y$ be a function, with $X \subseteq D$ and $Y \subseteq \mathbb{R}$.

If f is strictly monotonic in X (i.e. strictly increasing or strictly decreasing in X) and Y coincides with the set of all images of real numbers in X, then f is invertible.

Two conditions must hold:

1. f must be strictly monotonic in X
2. Y must coincides with the set of all images of elements in X

Notice that the theorem goes in one direction only!!

The computation of the inverse function

Problem: given an invertible function $f : X \to Y$, how do we determine the inverse of f?

Solution: given $y \in Y$, we want to find $x \in X$ such that $y = f(x)$.

For doing this we just solve the equation $y = f(x)$ with respect to x!

The inverse function: examples

- $f : \mathbb{R} \to [0, +\infty), \quad f(x) = x^2$

  The function f defined in this way is **not** injective, and therefore it is **not** invertible.

- $f : [0, +\infty) \to [0, +\infty), \quad f(x) = x^2$

  First, observe that f is injective and surjective. Therefore, the inverse exists and it is unique. To determine it, we solve the equation $y = x^2$ with respect to x. We have:

  $$y = x^2 \iff x = \sqrt{y}$$

  So the inverse is the function $f^{-1} : [0, +\infty) \to [0, +\infty)$, $f^{-1}(y) = \sqrt{y}$.

The inverse function: examples

- $f : (-\infty, 0] \to [0, +\infty), \quad f(x) = x^2$

  f is injective and surjective. Therefore, the inverse exists and it is unique. To determine it, we solve the equation $y = x^2$ with respect to x. Note that now we are looking for negative x. We have:

  $$y = x^2 \;\Leftrightarrow\; x = -\sqrt{y}$$

  So the inverse is the function $f^{-1} : [0, +\infty) \to (-\infty, 0]$, $f^{-1}(y) = -\sqrt{y}$.

Linear functions

$$f : \mathbb{R} \to \mathbb{R} \text{ such that } \mathbf{f(x)=mx+q}, \quad m, q \in \mathbb{R}$$

- Graphically, this is the equation of a straight line.
- $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$, for any $x_1 \neq x_2$, is the slope of the line
    - If $m > 0$ the function is increasing
    - If $m < 0$ the function is decreasing
    - If $m = 0$ we have a flat (horizontal) line
- The absolute value of m indicates how fast the line increases or decreases

Linear functions

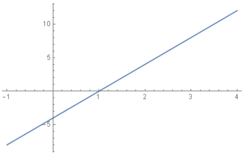


Figure: $m > 0$

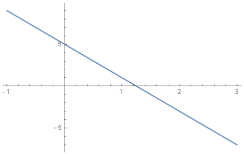


Figure: $m < 0$

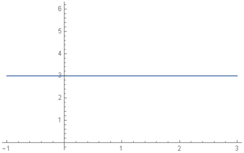


Figure: $m = 0$

How to compute the equation of a straight line

1. Point-slope equation.
   Let $P_1 = (x_1, y_1)$ be a point on the line and m the slope.
   Then the equation of the line through P_1 with slope m is

$$y - y_1 = m(x - x_1)$$

2. Two points.
   Let $P_1 = (x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on the line.
   To compute the equation of the line we first compute the slope: $m = \frac{y_2 - y_1}{x_2 - x_1}$
   and next we use the formula

$$y - y_1 = m(x - x_1)$$