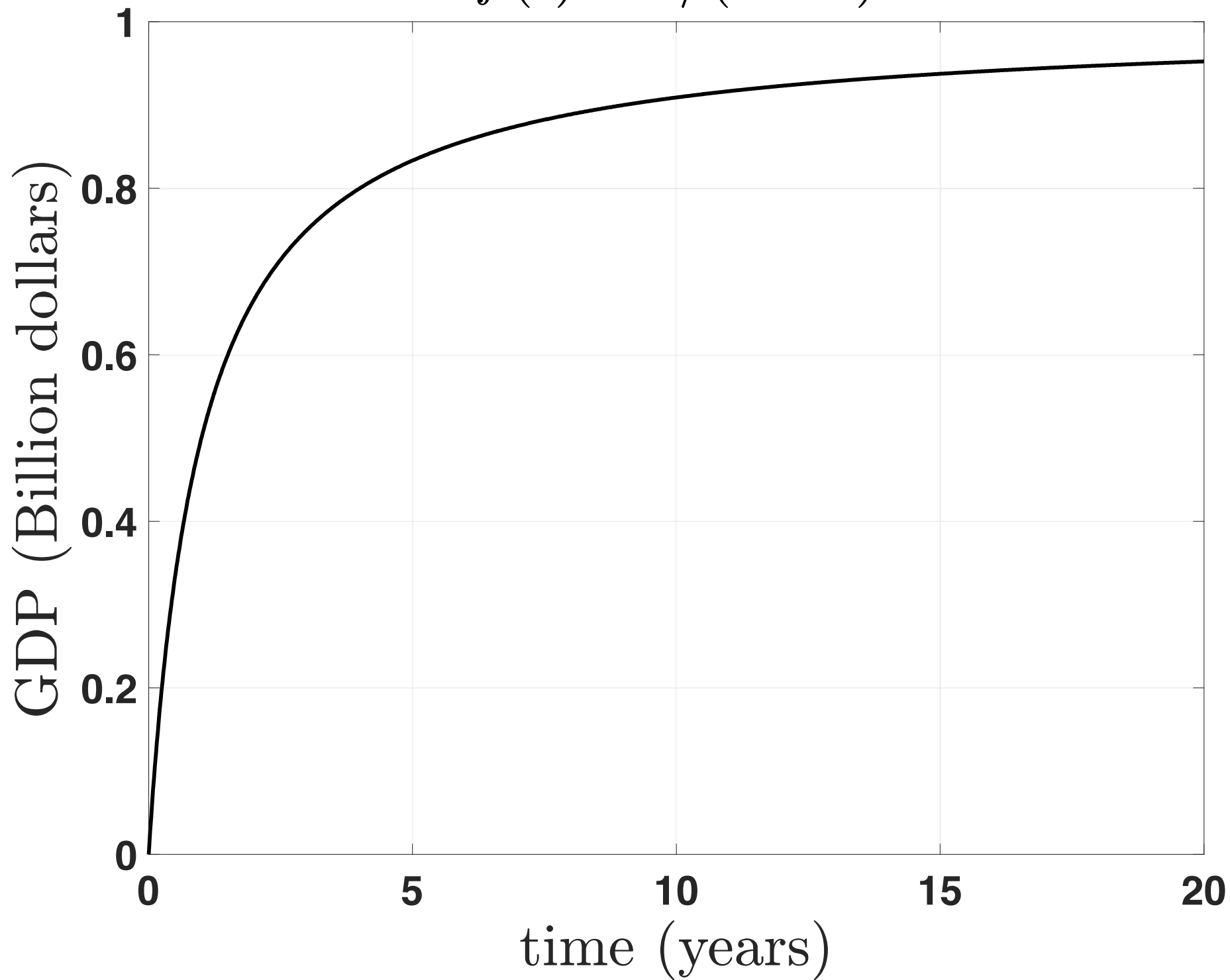


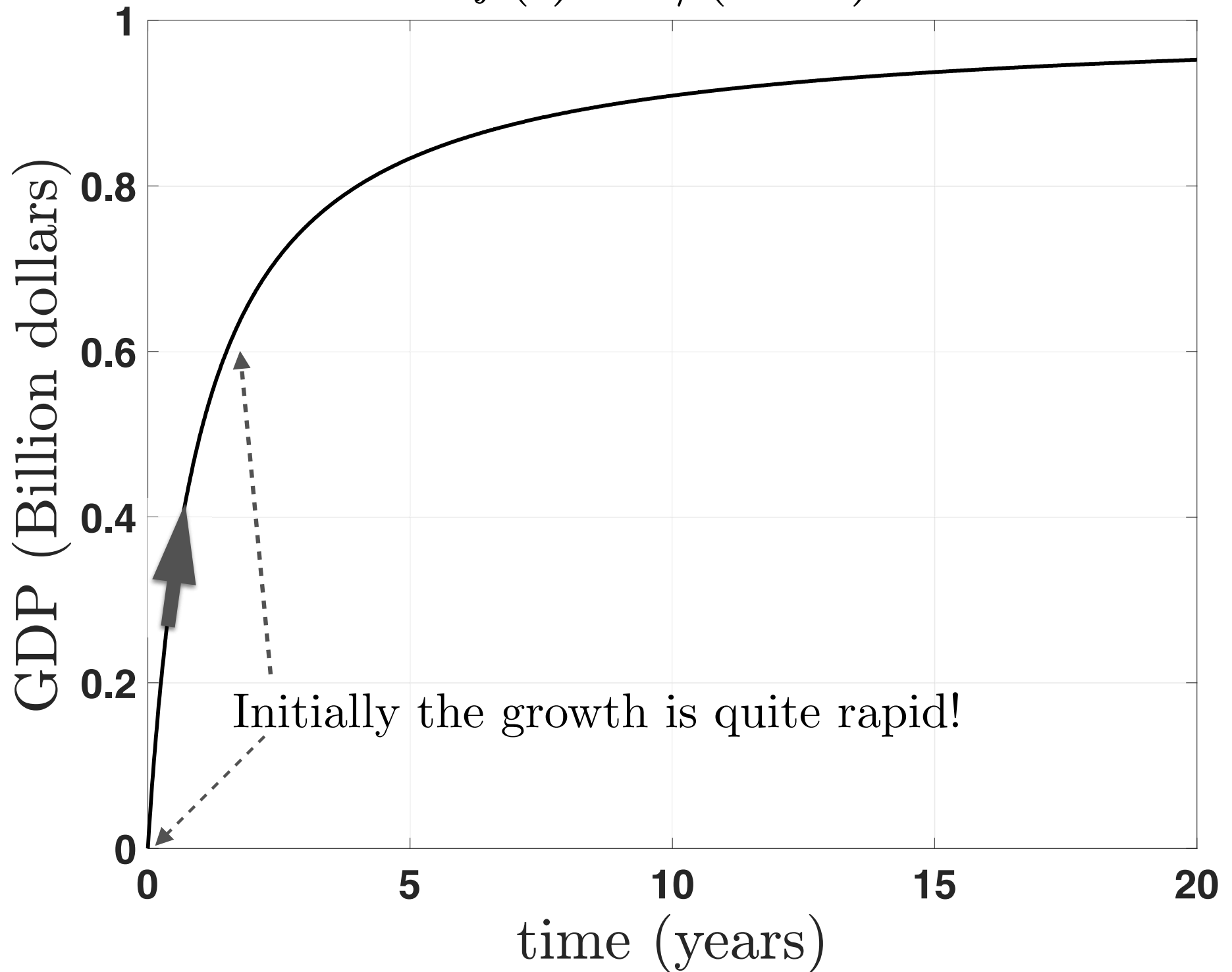
Outline

- ① Derivatives
- ② How to compute derivatives
- ③ Fundamental Theorems on Derivatives
- ④ Local Maxima and Minima

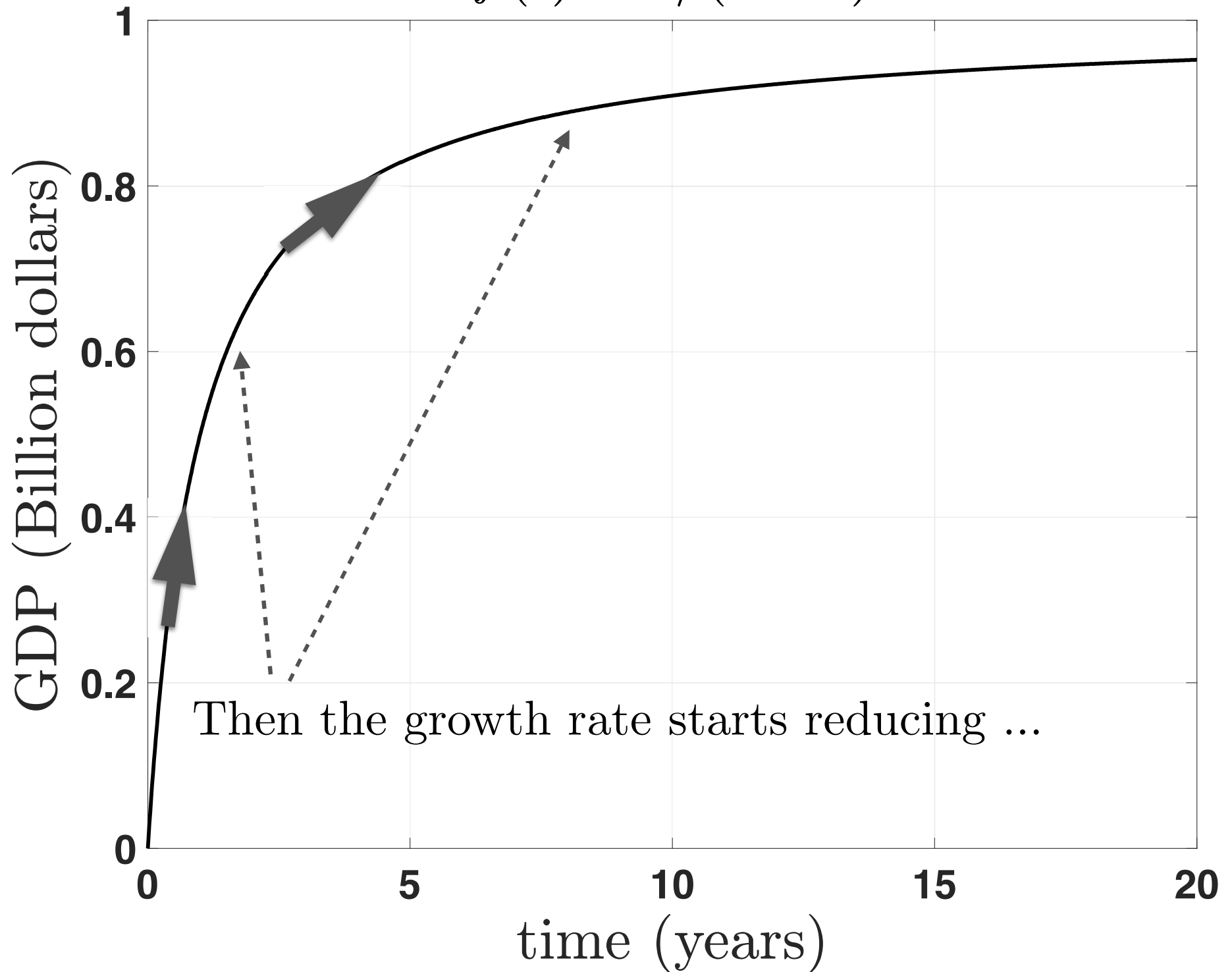
$$f(t) = t/(1+t)$$



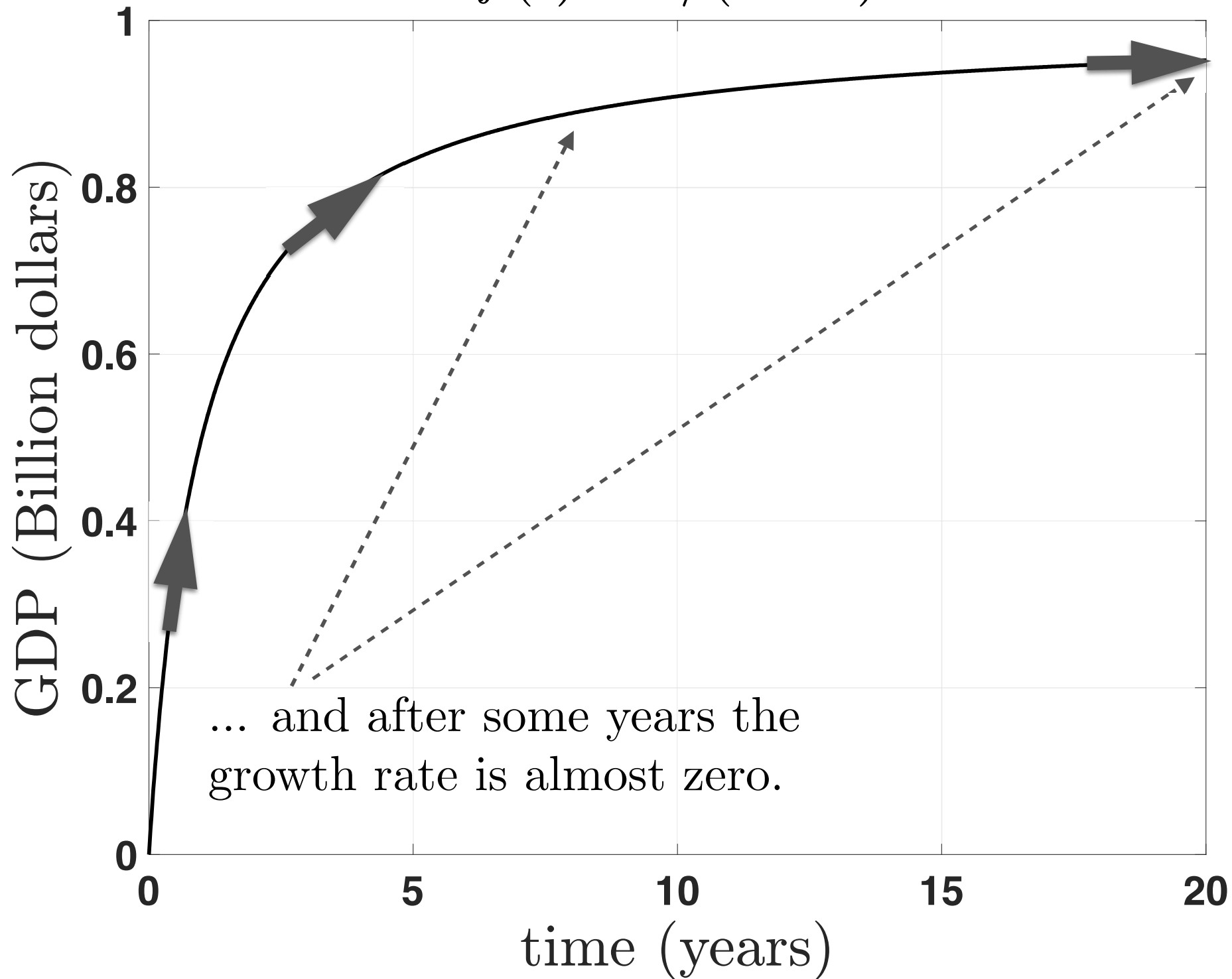
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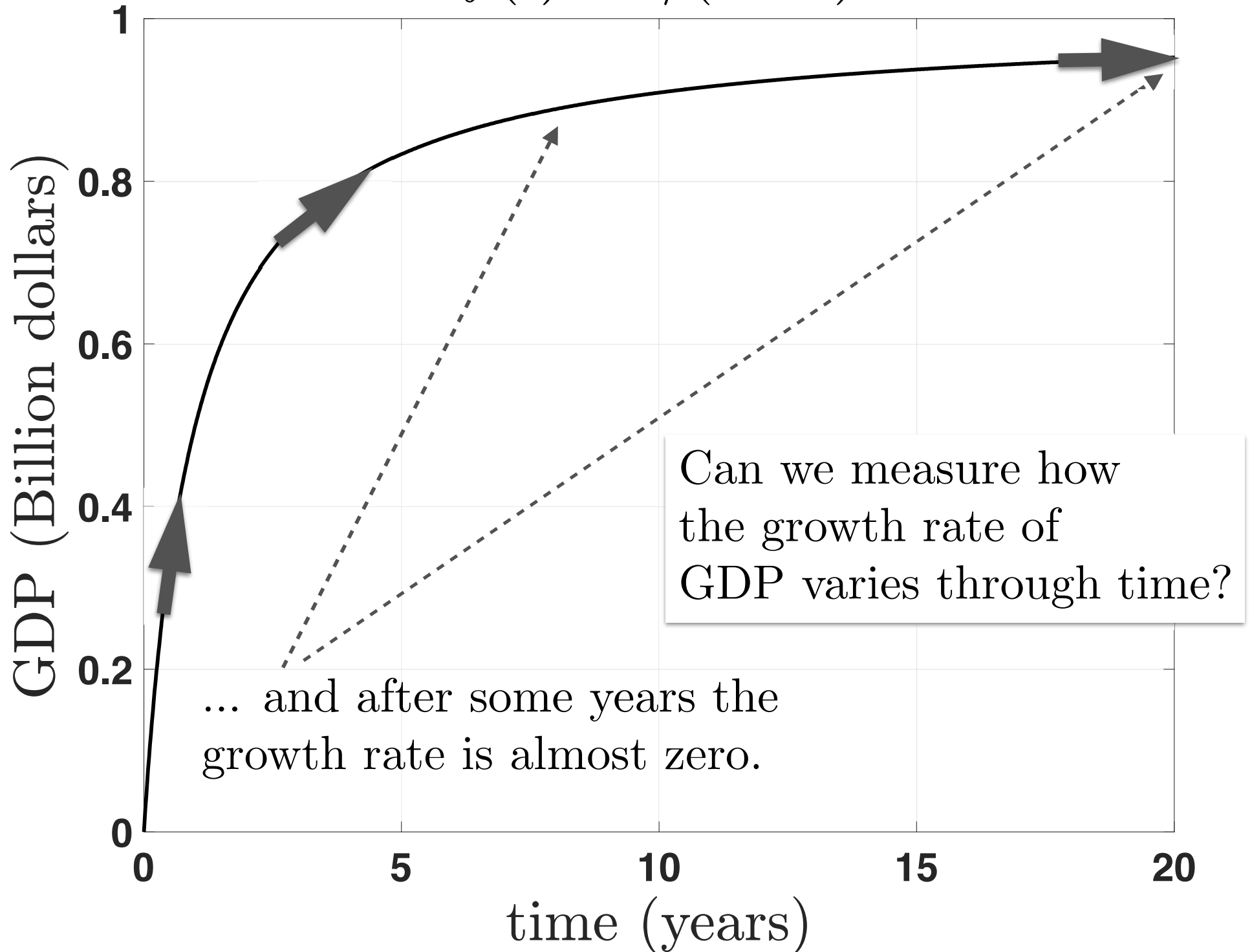
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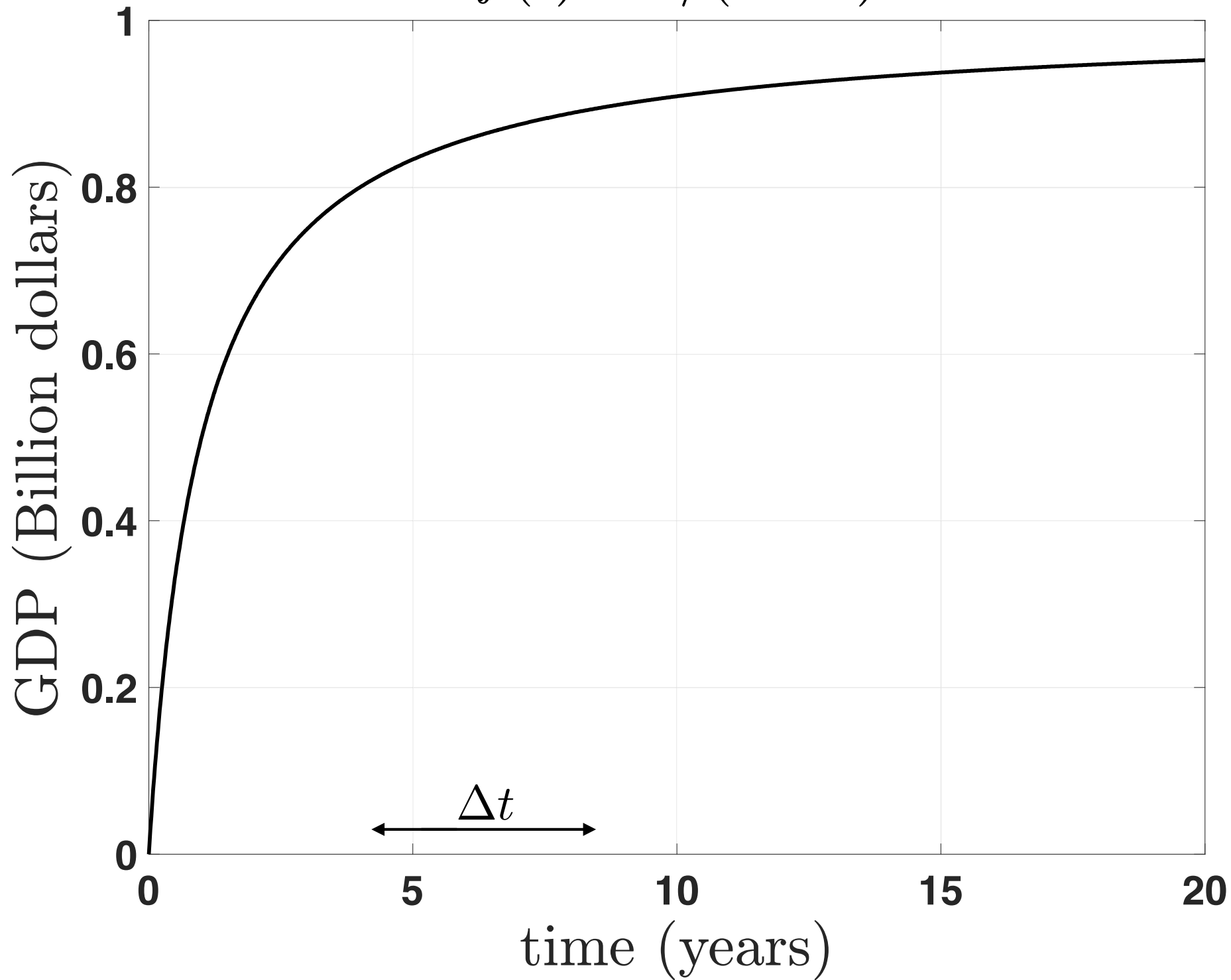
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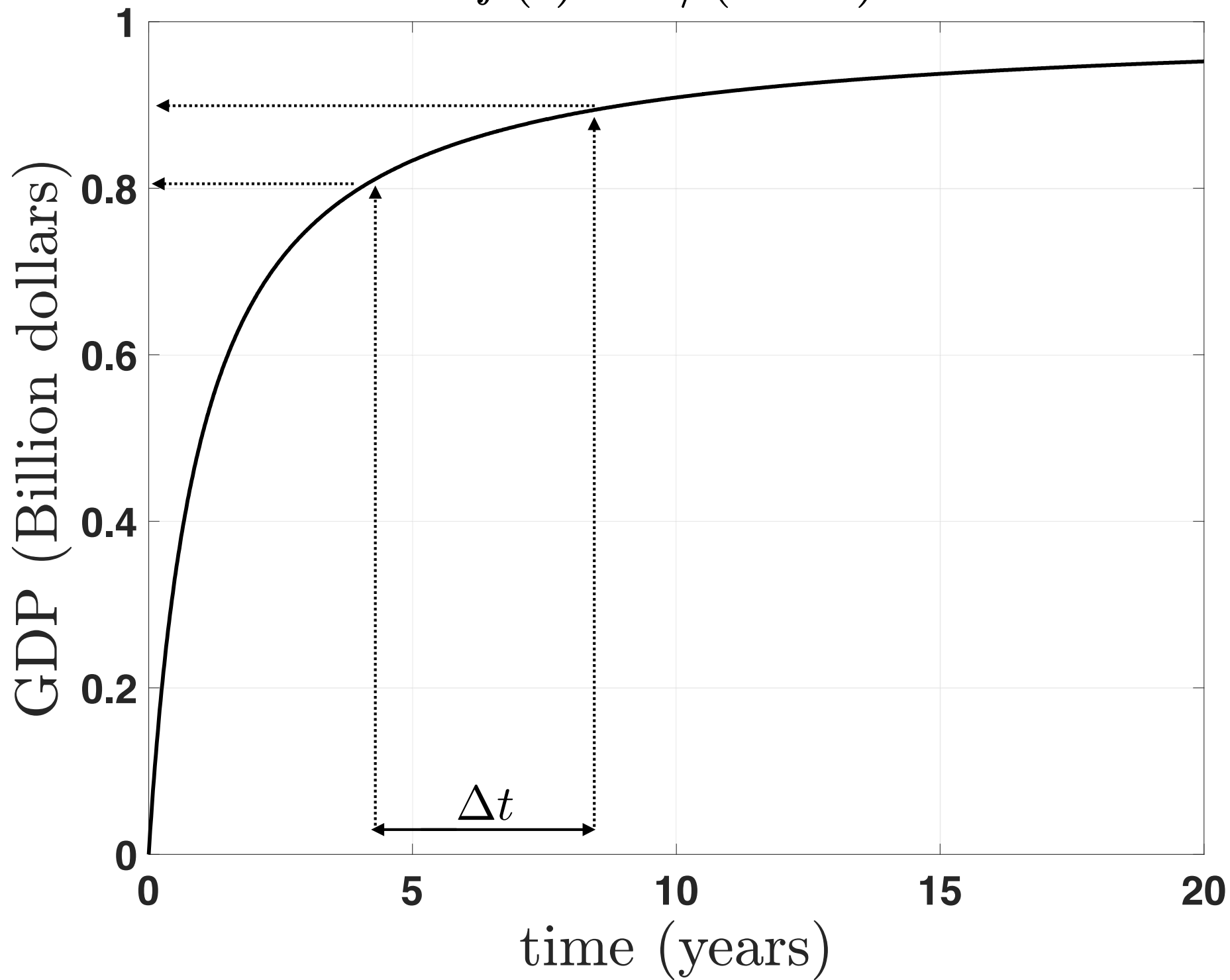
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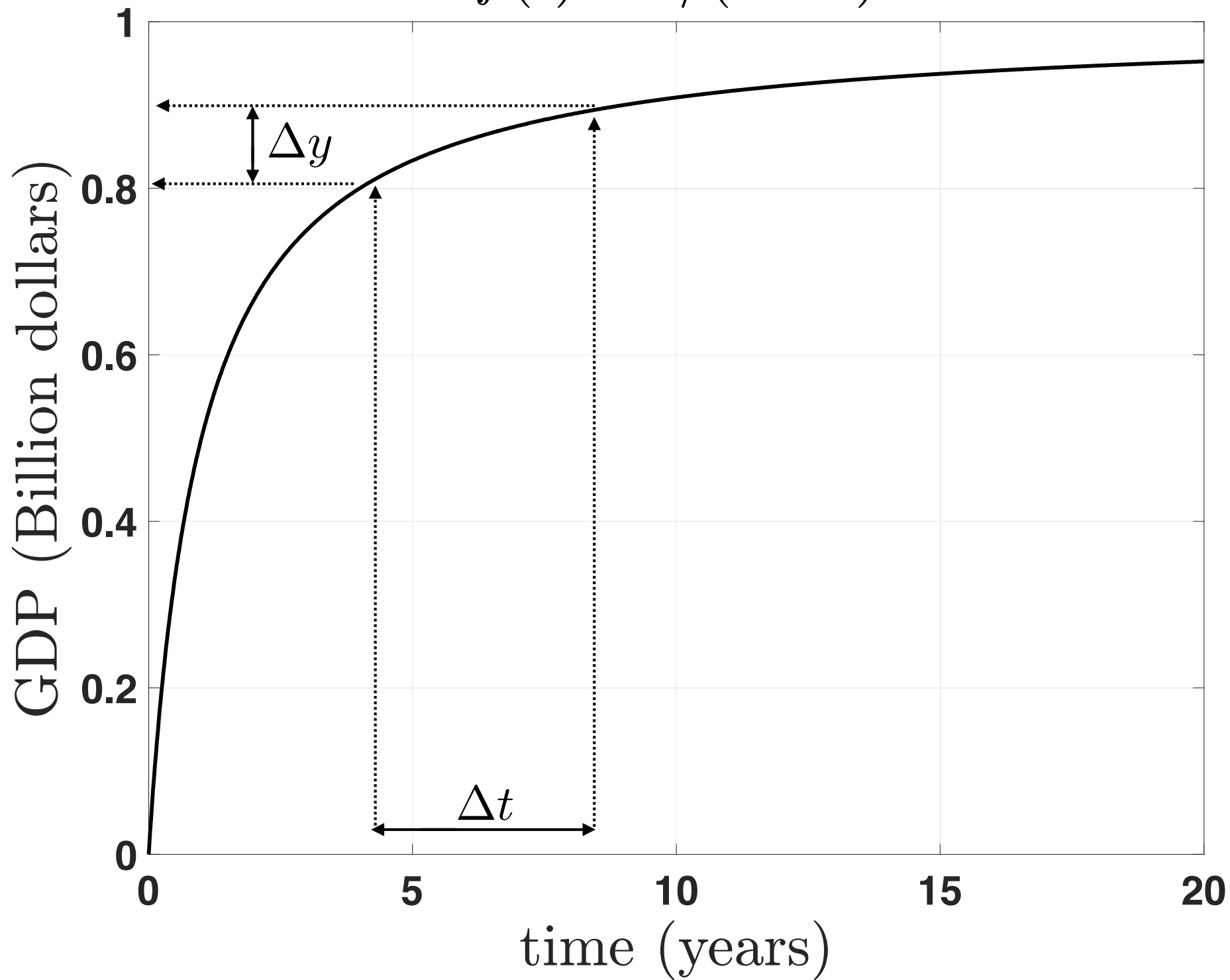
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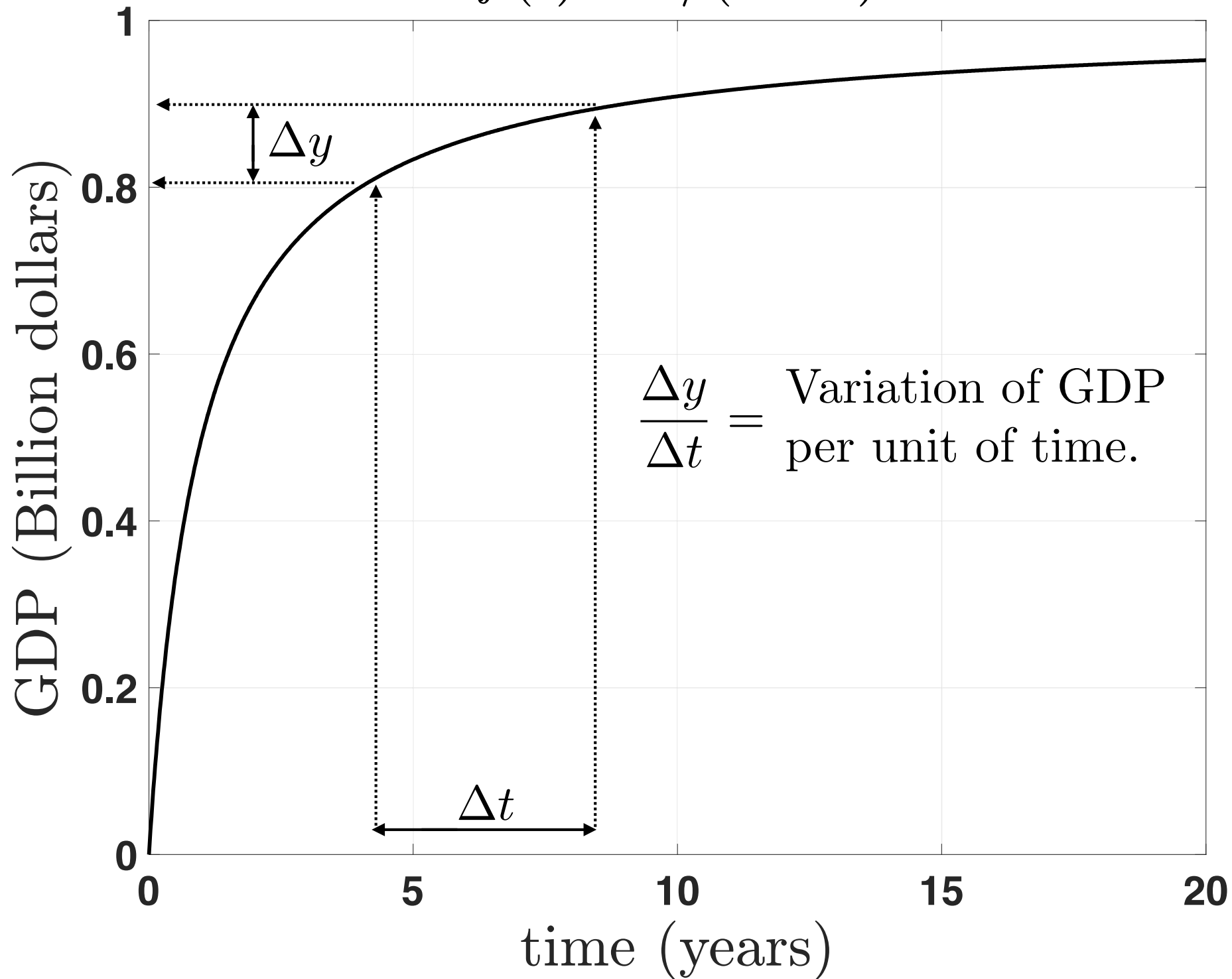
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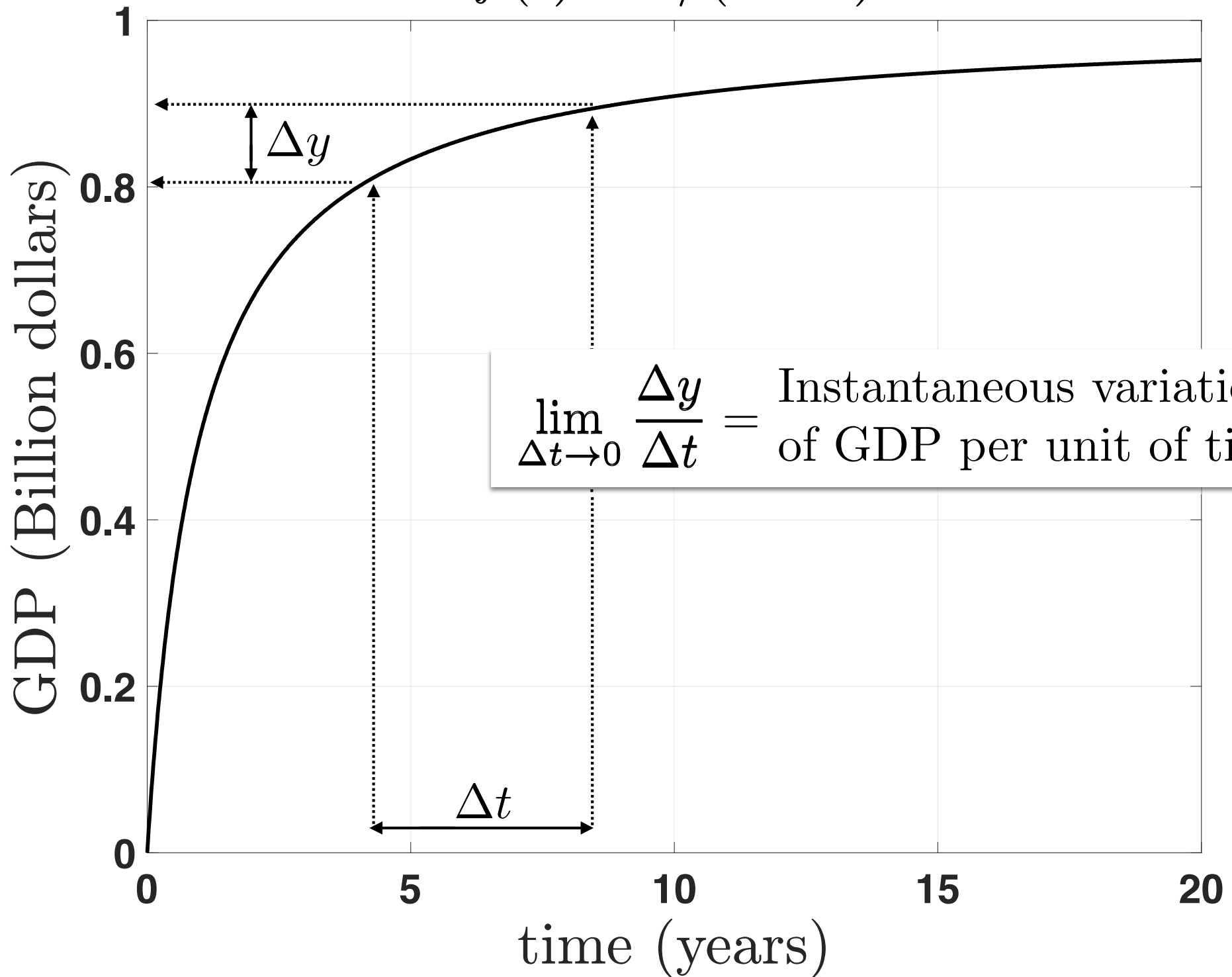
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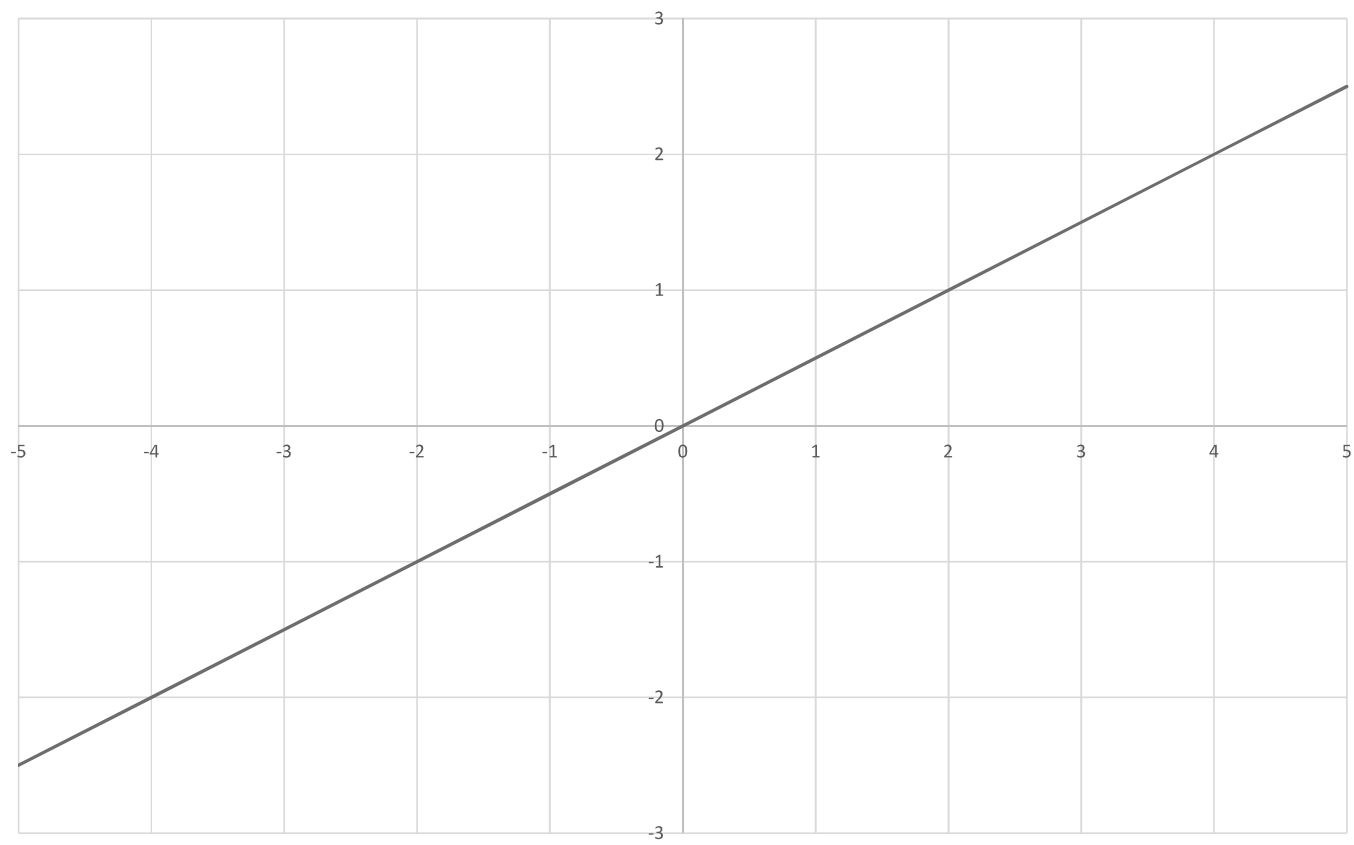


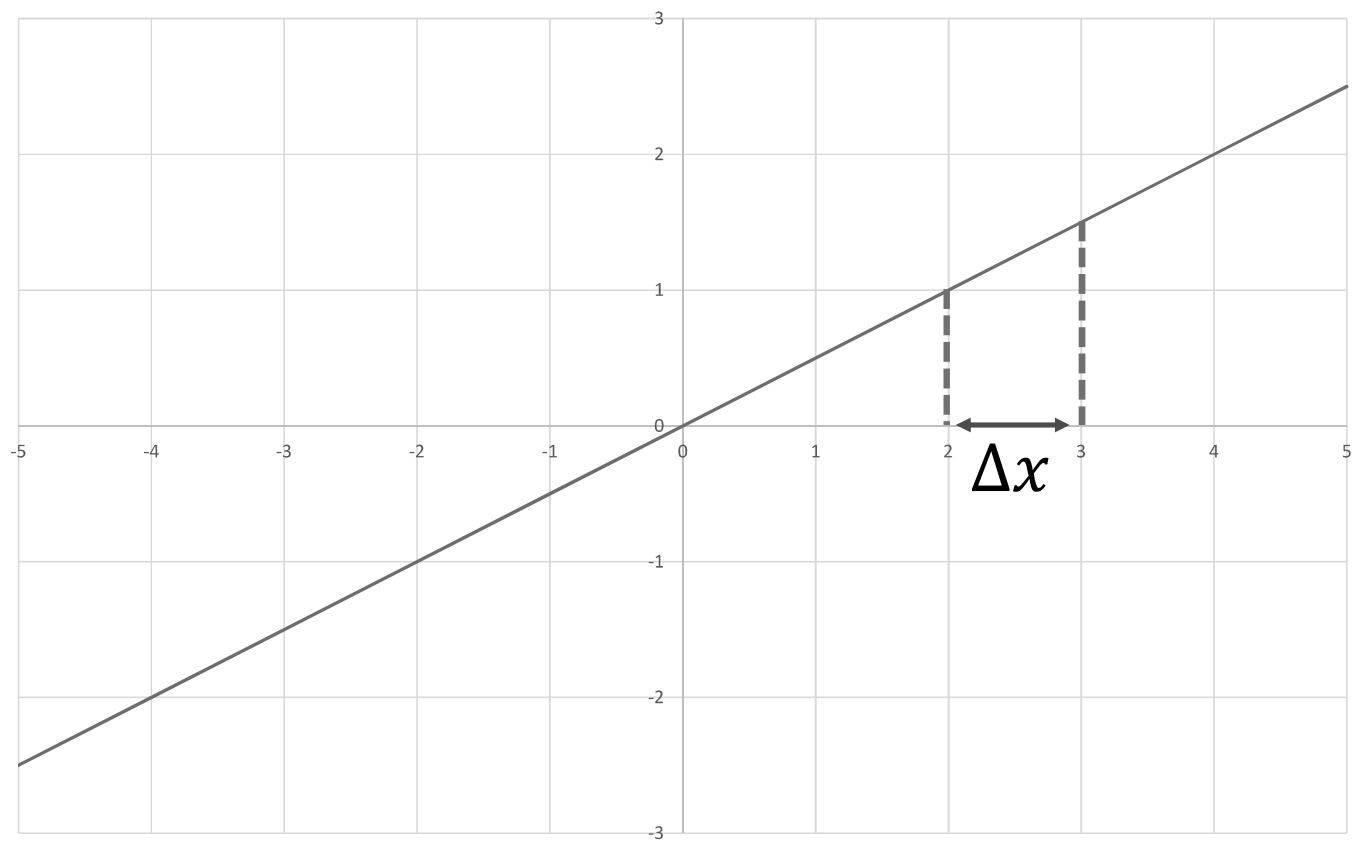
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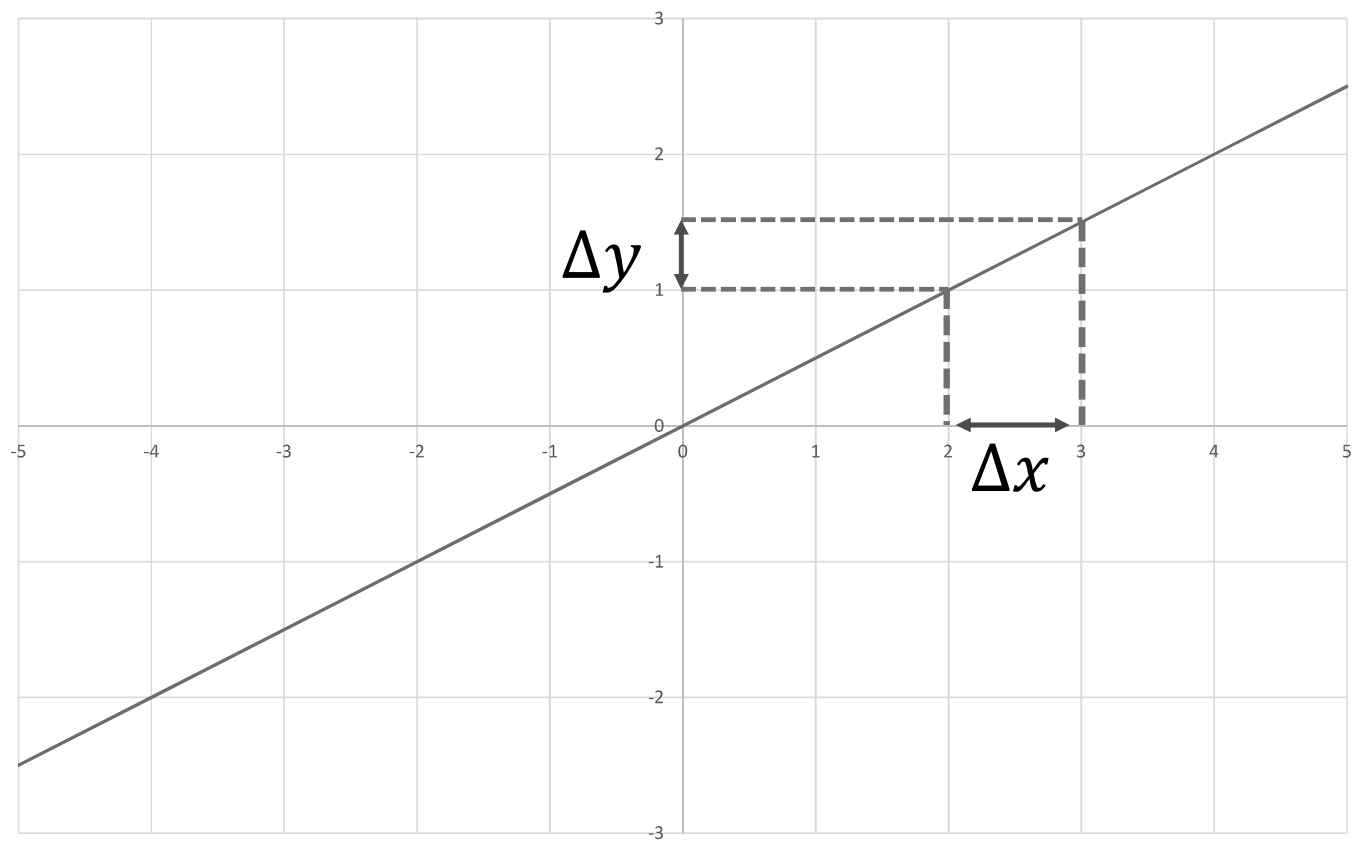


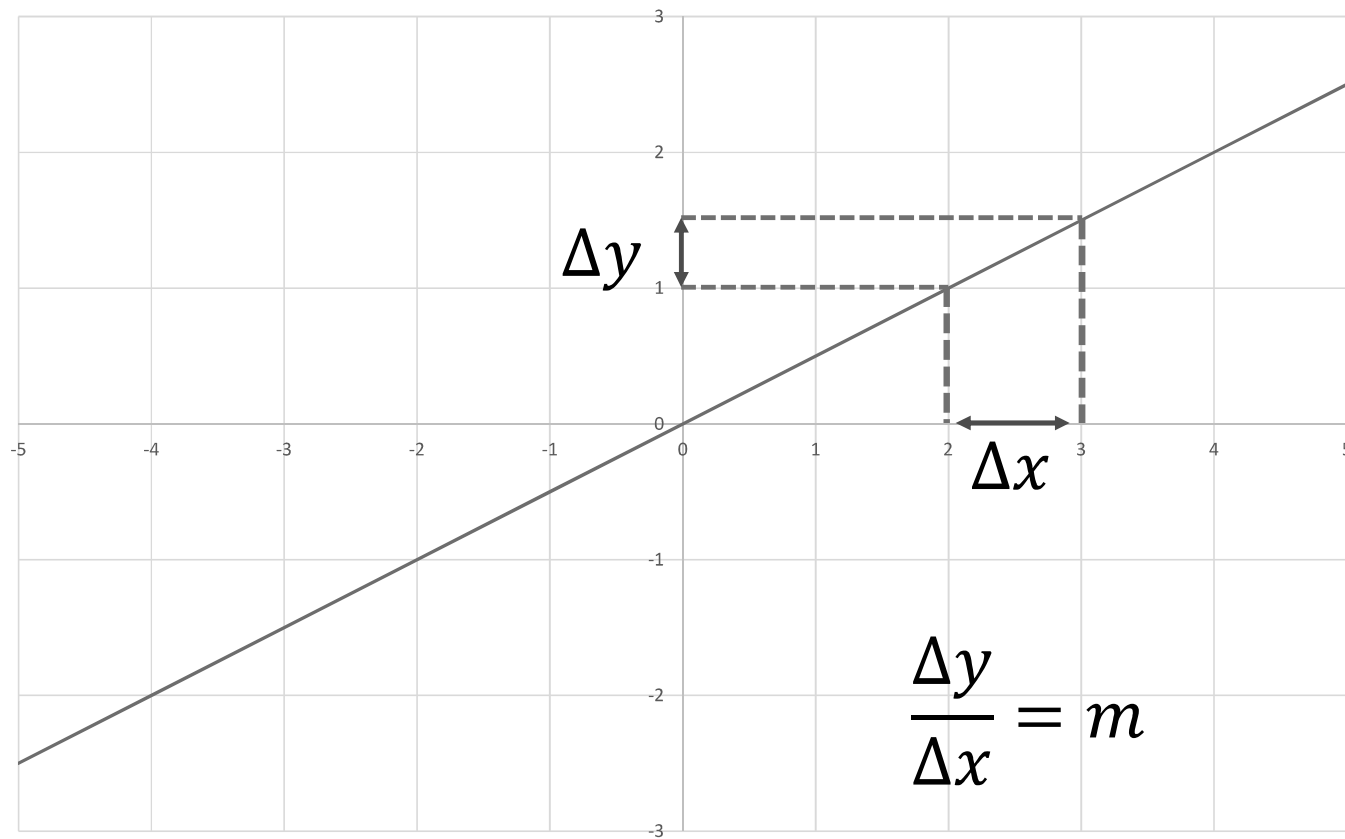
$$f(t) = t/(1+t)$$











Slope of the line indicates the rate of change

Incremental Ratio

Definition

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Let $x_0, x_1 \in (a, b)$. We call incremental ratio the ratio

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

This ratio represents the slope of the line through $A = (x_0; f(x_0))$ and $B = (x_1; f(x_1))$

Incremental Ratio

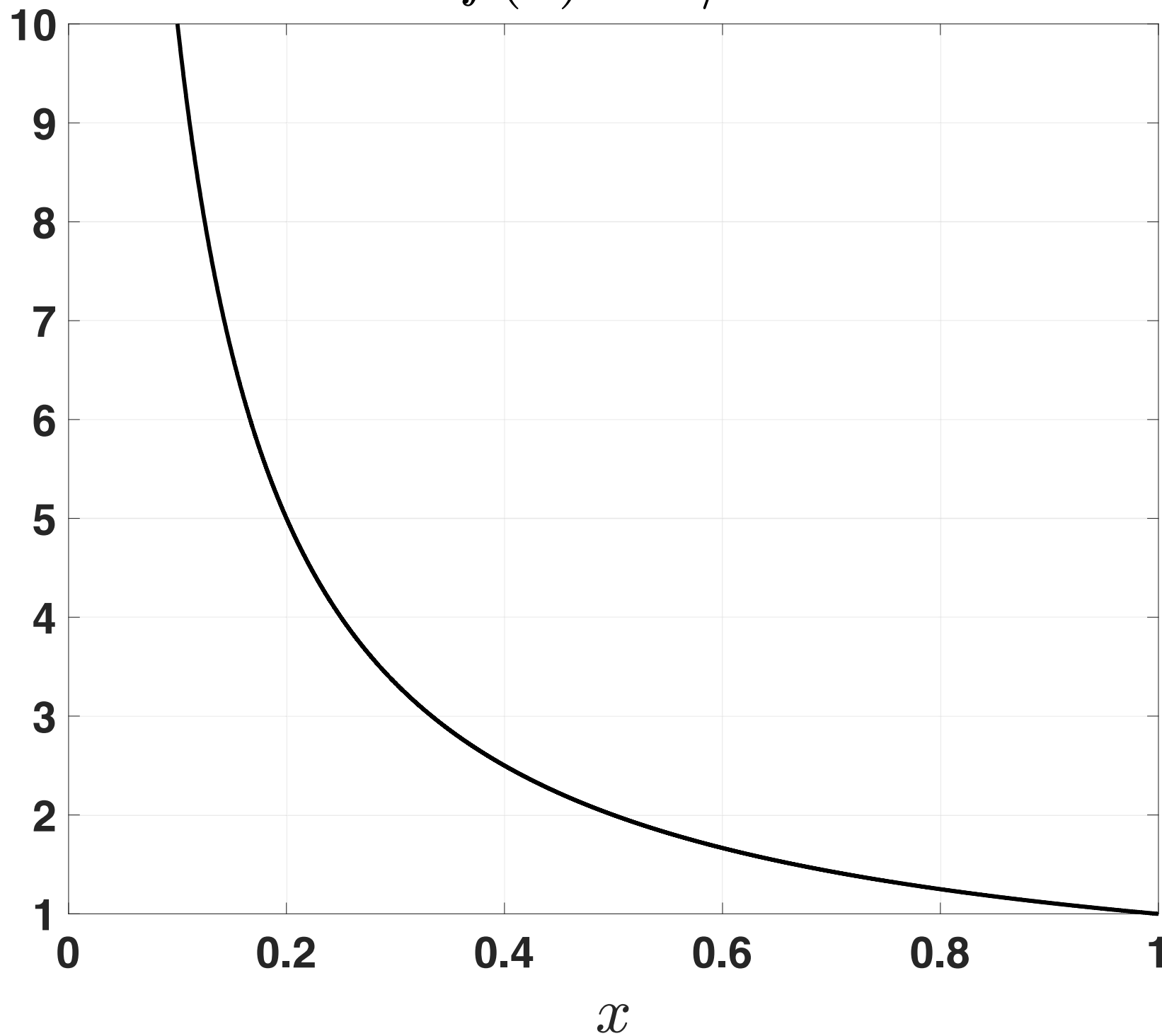
Definition

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Let $x_0 \in (a, b)$ and let $h \in \mathbb{R}$ such that $x_0 + h \in (a, b)$. We call incremental ratio the ratio

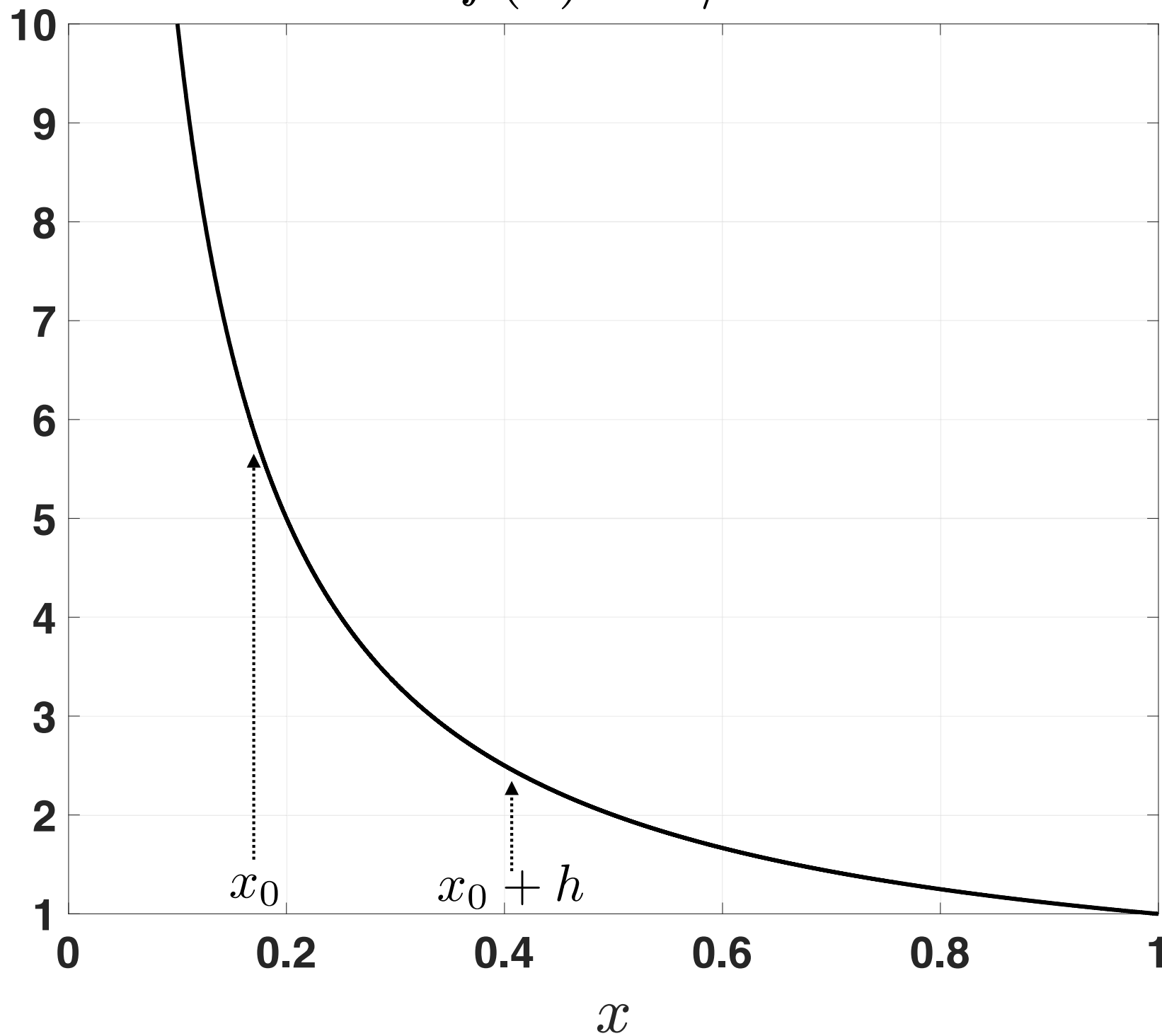
$$\frac{f(x_0 + h) - f(x_0)}{h}$$

This ratio represents the slope of the line through $A = (x_0; f(x_0))$ and $B = (x_0 + h; f(x_0 + h))$

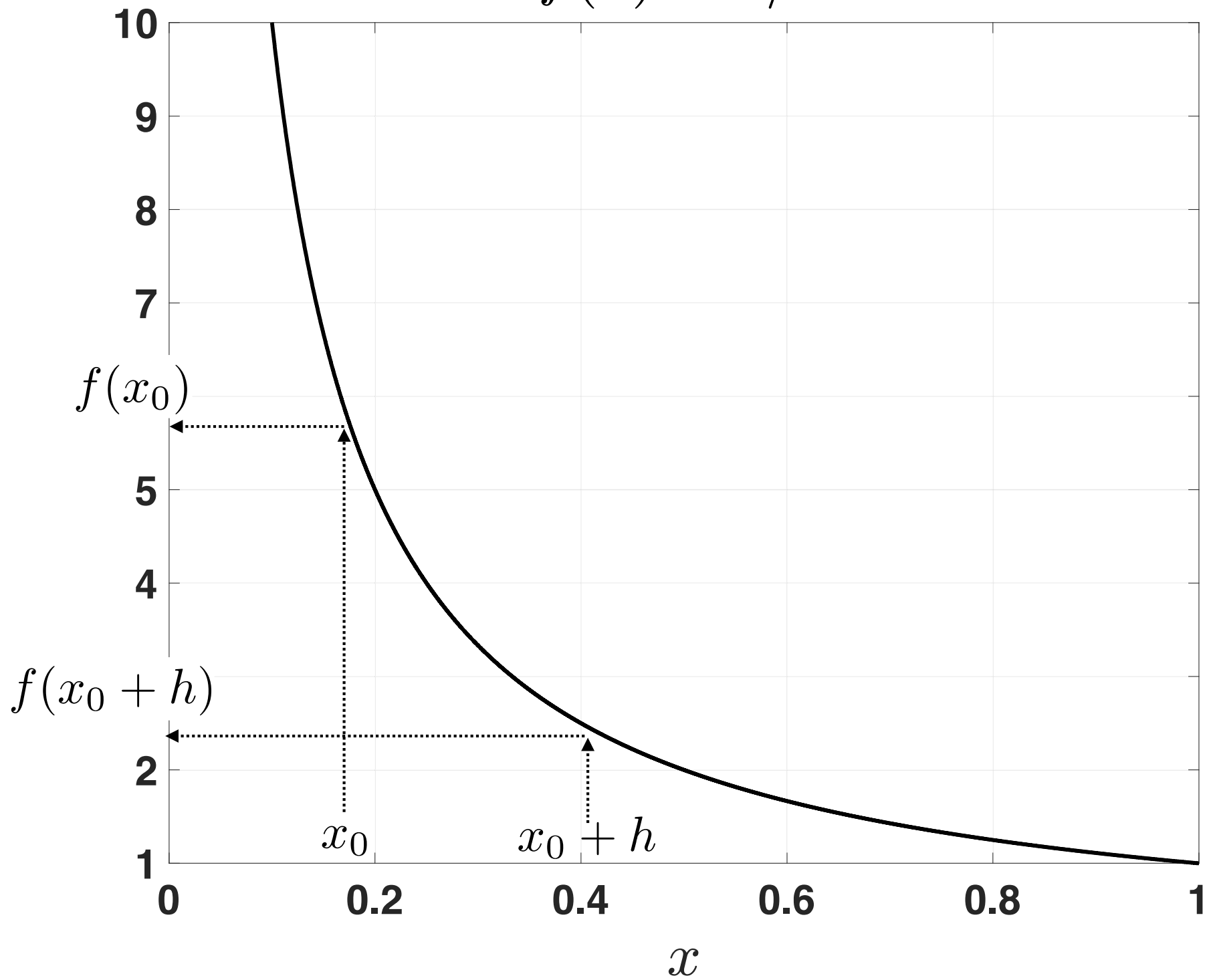
$$f(x) = 1/x$$



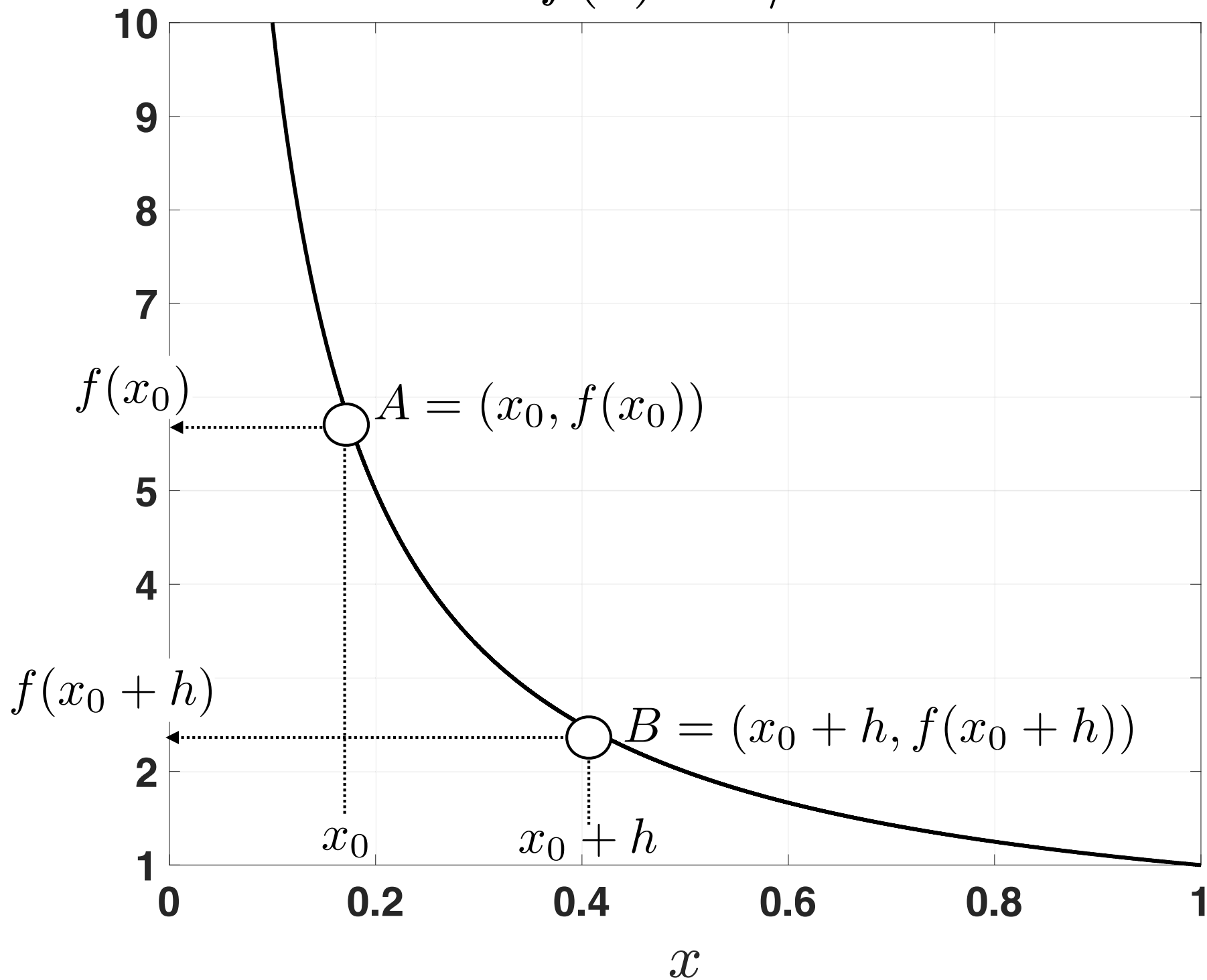
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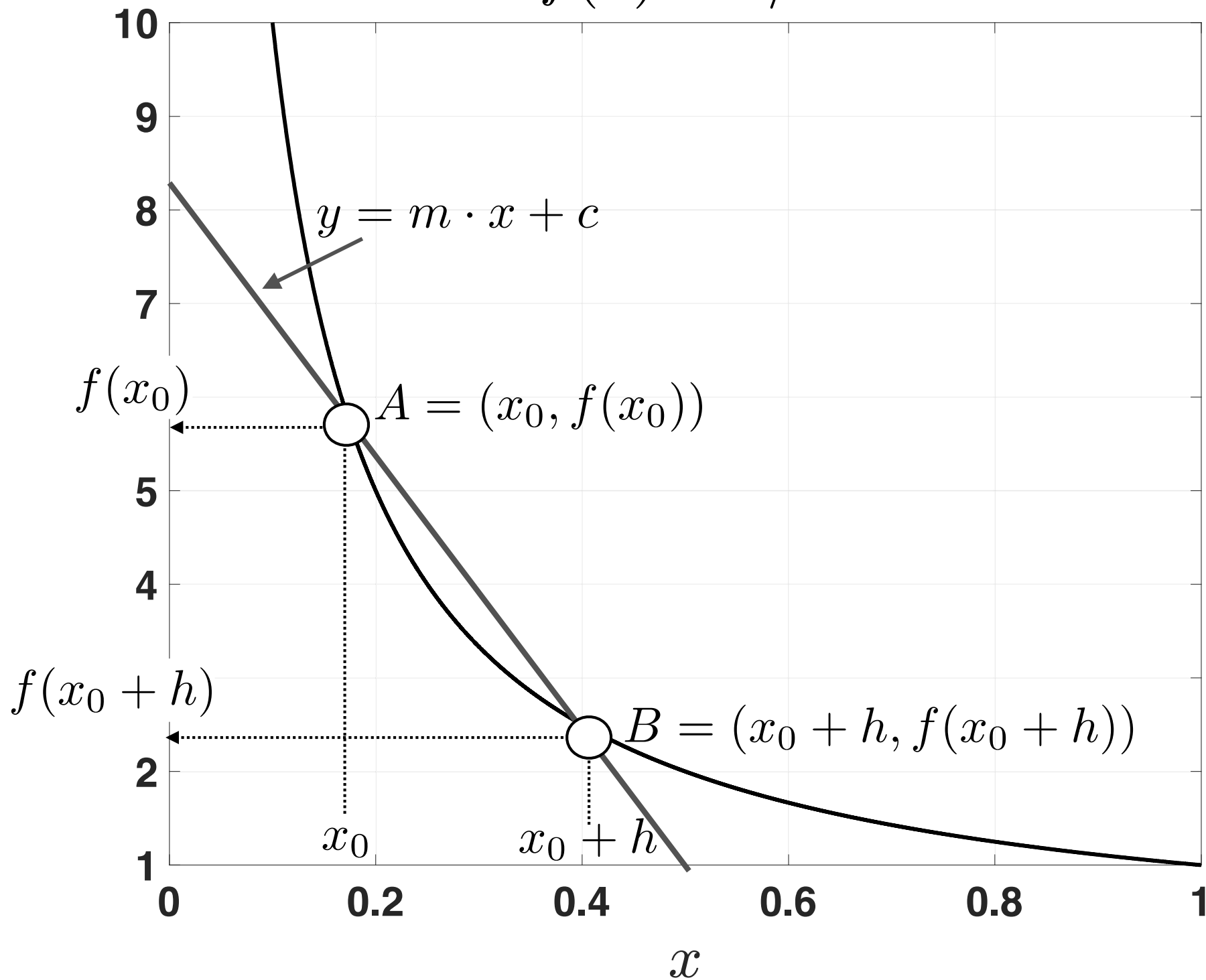
$$f(x) = 1/x$$



$$f(x) = 1/x$$



$$f(x) = 1/x$$



Derivatives

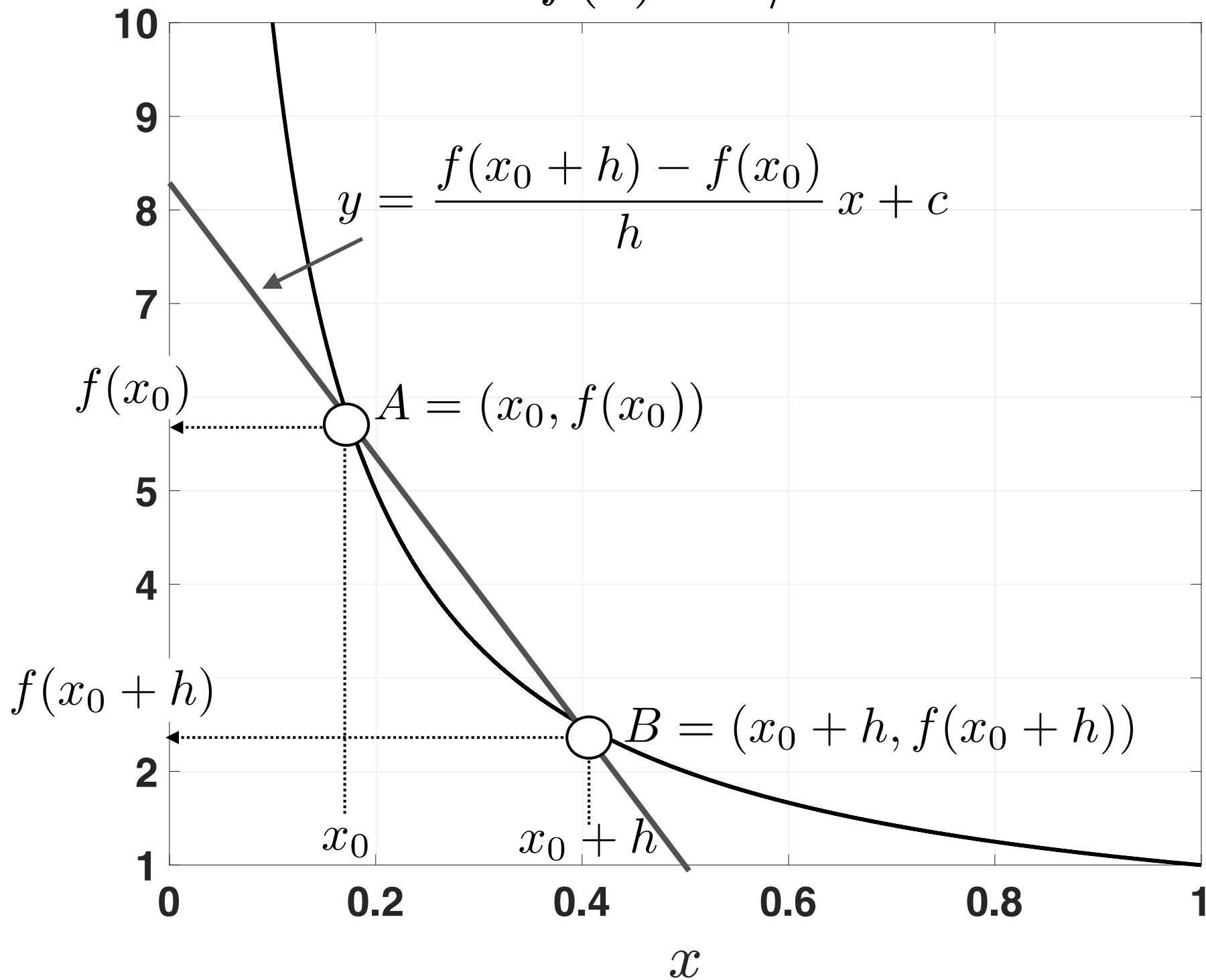
Definition

Let $f : D \rightarrow \mathbb{R}$ be a function and let $[a, b] \subset D$. We say that f is differentiable at x_0 if:

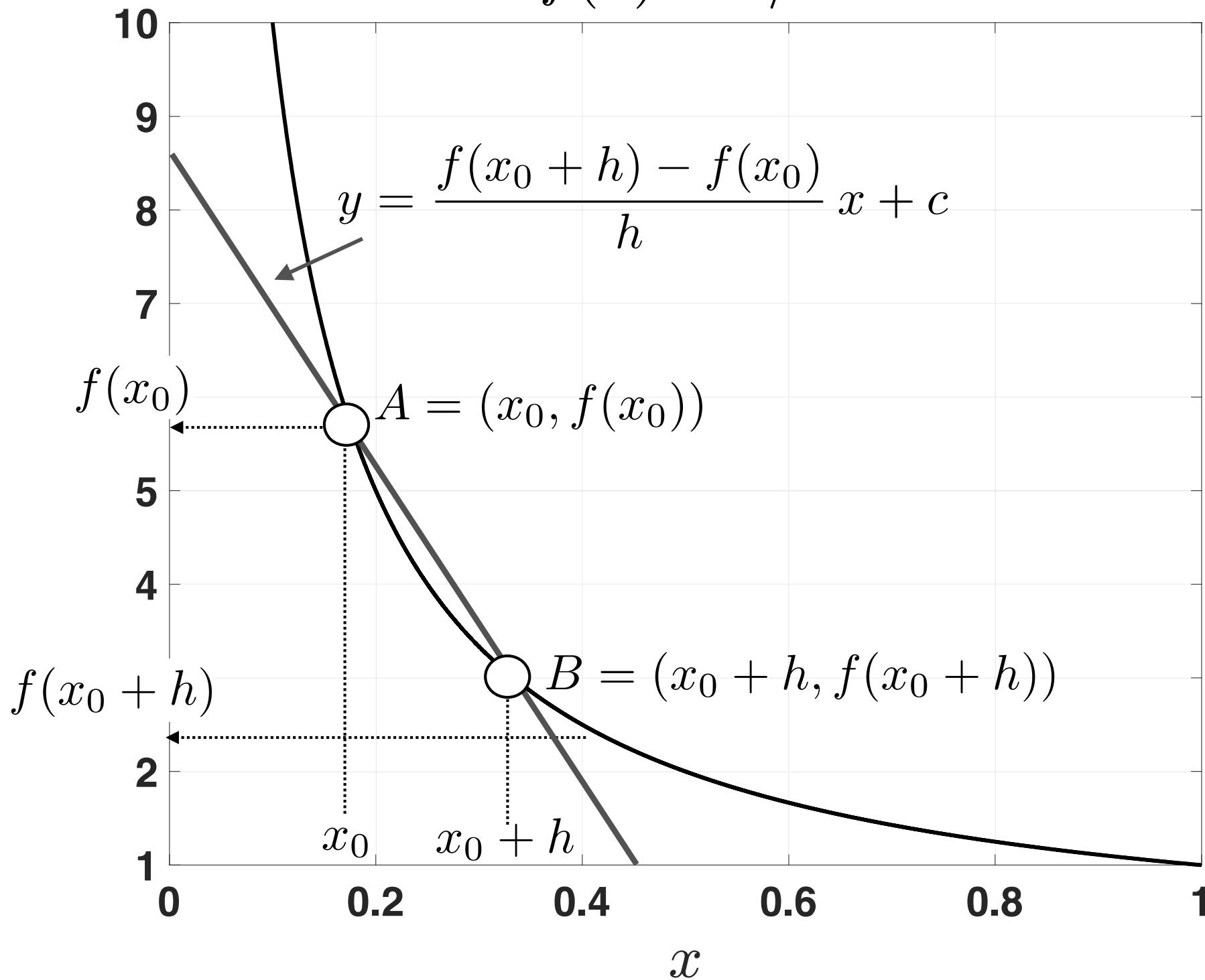
- $x_0 \in (a, b)$
- $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = d$ exists and it is finite.

If this is the case we call $d = f'(x_0)$ and we say that d is the derivative of f at the point x_0 .

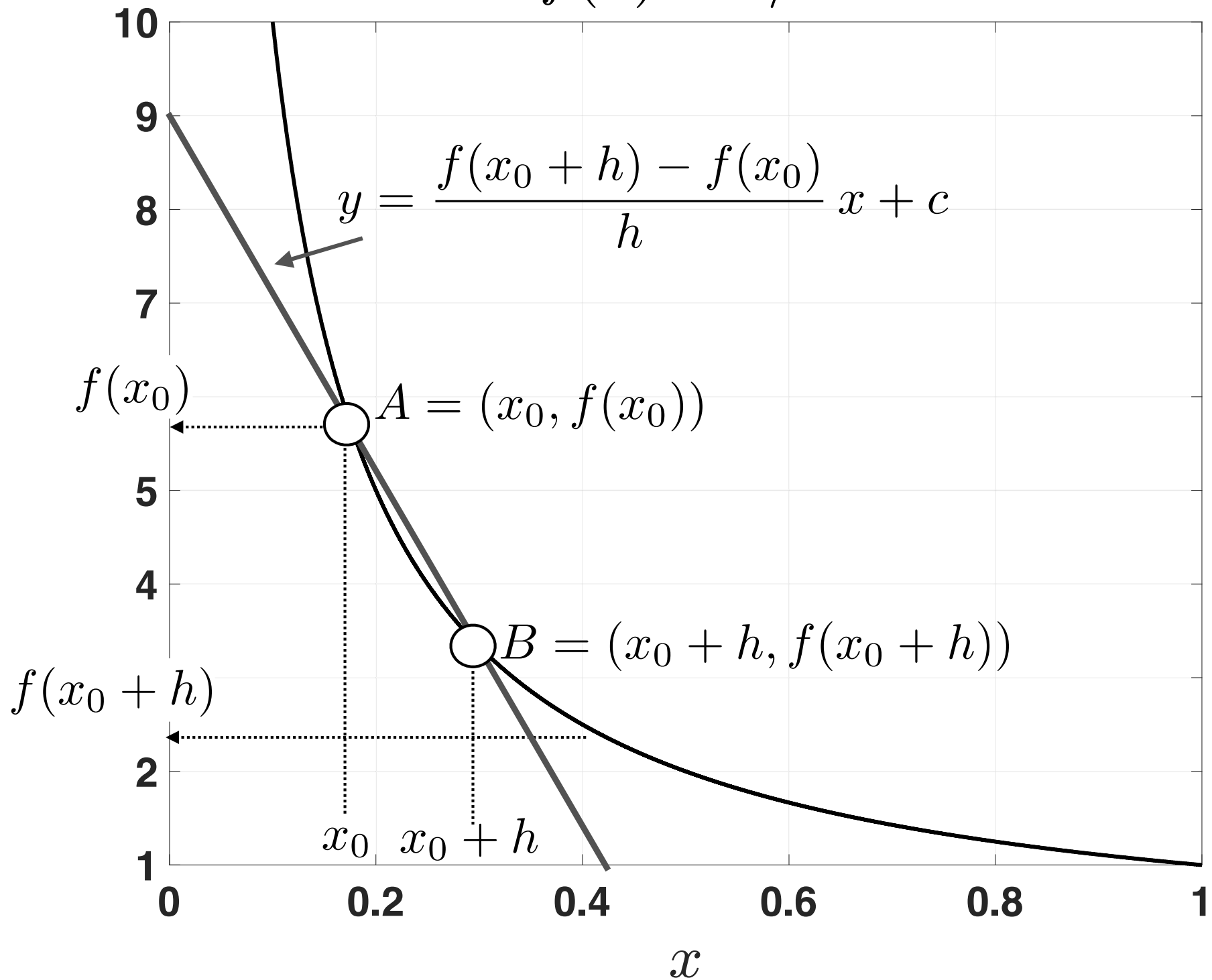
$$f(x) = 1/x$$



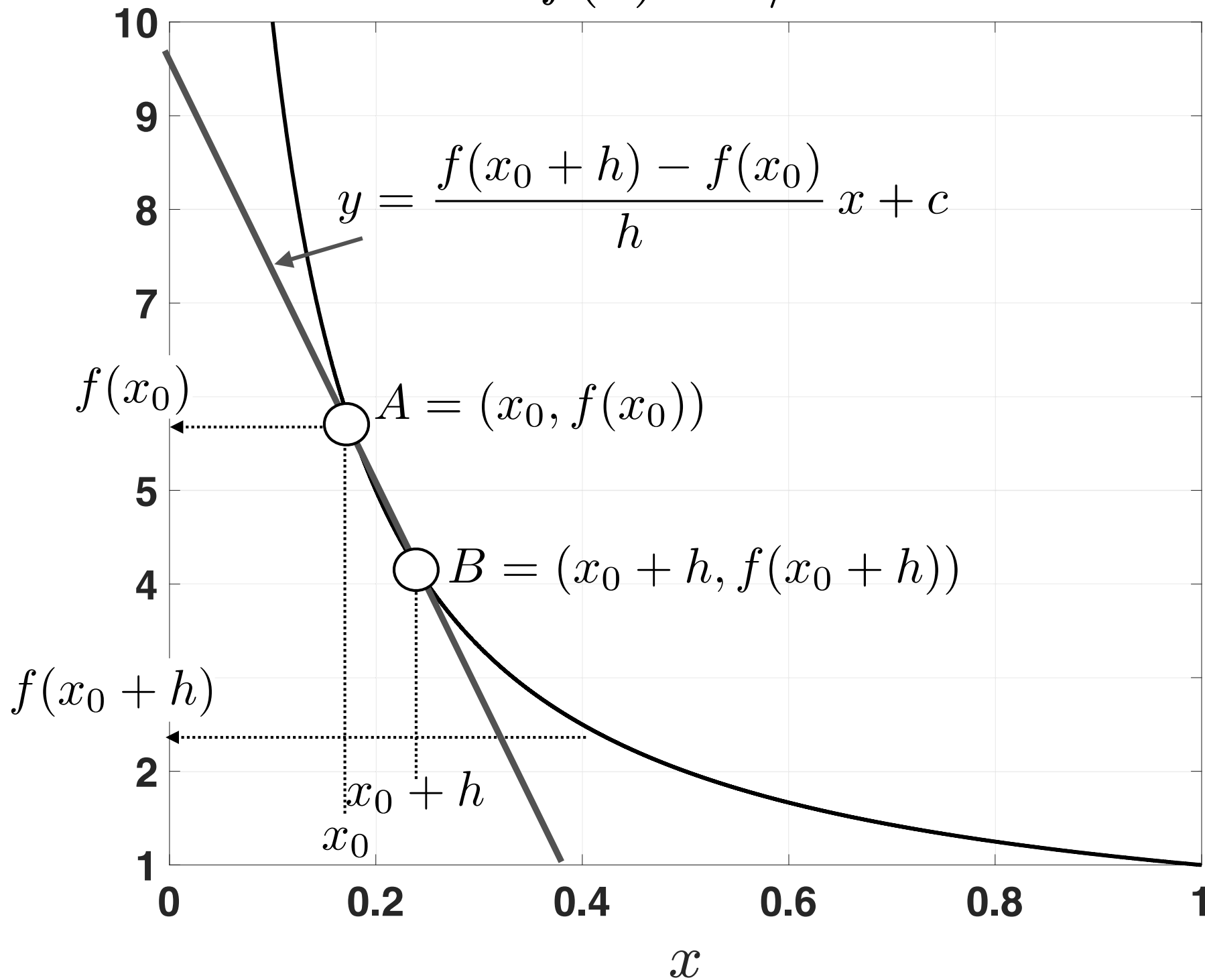
$$f(x) = 1/x$$



$$f(x) = 1/x$$



$$f(x) = 1/x$$



Derivatives of elementary functions

- Let $f(x) = k$, with $k \in \mathbb{R}$ (i.e. the constant function). $D = \mathbb{R}$.
Using the definition of derivative we get that, for any $x \in D$:

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{k - k}{h} = 0$$

The last equality is not an undetermined form since the numerator is equal to 0 no matter the value of h .

- Let $f(x) = x$. $D = \mathbb{R}$.
Using the definition of derivative we get that, for any $x \in D$:

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{x+h-x}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1$$

Derivatives of elementary functions

- Let $f(x) = x^2$. $D = \mathbb{R}$.

Using the definition of derivative we get that, for any $x \in D$:

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} = 2x\end{aligned}$$

- Let $f(x) = e^x$. $D = \mathbb{R}$.

Using the definition of derivative we get that, for any $x \in D$:

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} e^x \underbrace{\frac{e^h - 1}{h}}_1 = e^x$$

Derivatives of elementary functions

- Let $f(x) = \log(x)$. $D = (0, +\infty)$.

Using the definition of derivative we get that, for any $x \in D$:

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\log(x+h) - \log(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\log\left(1 + \frac{h}{x}\right)}{h} = \lim_{h \rightarrow 0} \underbrace{\frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}}}_1 \frac{1}{x} = \frac{1}{x}\end{aligned}$$

- Let $f(x) = \sin(x)$. $D = \mathbb{R}$.

Using the definition of derivative we get that, for any $x \in D$:

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \underbrace{\frac{\sin(x)(\cos(h) - 1)}{h}}_0 + \lim_{h \rightarrow 0} \cos(x) \underbrace{\frac{\sin(h)}{h}}_1 = \cos(x)\end{aligned}$$

Derivatives of elementary functions

- Let $f(x) = \cos(x)$. $D = \mathbb{R}$.
Using the definition of derivative we get that, for any $x \in D$:

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\&= \lim_{h \rightarrow 0} \underbrace{\frac{\cos(x)(\cos(h) - 1)}{h}}_0 - \lim_{h \rightarrow 0} \sin(x) \underbrace{\frac{\sin(h)}{h}}_1 = -\sin(x)\end{aligned}$$

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0
linear	x	1

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0
linear	x	1
power	$x^\alpha, \alpha \neq 0$	$\alpha x^{\alpha-1}$

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0
linear	x	1
power	$x^\alpha, \alpha \neq 0$	$\alpha x^{\alpha-1}$
sine	$\sin(x)$	$\cos(x)$

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
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linear	x	1
power	$x^\alpha, \alpha \neq 0$	$\alpha x^{\alpha-1}$
sine	$\sin(x)$	$\cos(x)$
cosine	$\cos(x)$	$-\sin(x)$

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0
linear	x	1
power	$x^\alpha, \alpha \neq 0$	$\alpha x^{\alpha-1}$
sine	$\sin(x)$	$\cos(x)$
cosine	$\cos(x)$	$-\sin(x)$
exponential	e^x	e^x

Derivatives of elementary functions

Function	$f(x)$	$f'(x)$
constant	k	0
linear	x	1
power	$x^\alpha, \alpha \neq 0$	$\alpha x^{\alpha-1}$
sine	$\sin(x)$	$\cos(x)$
cosine	$\cos(x)$	$-\sin(x)$
exponential	e^x	e^x
logarithm	$\log x$	$\frac{1}{x}$

Operations with derivatives

Theorem

Let $f : D_f \rightarrow \mathbb{R}$ and $g : D_g \rightarrow \mathbb{R}$ be two functions such that $f'(x)$ and $g'(x)$ exists. Then

$$D(f(x) + g(x)) = f'(x) + g'(x)$$

$$D(f(x) - g(x)) = f'(x) - g'(x)$$

$$D(kf(x)) = kf'(x)$$

$$D(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$D\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$$

$$D((f \circ g)(x)) = D(f(g(x))) = f'(g(x)) \cdot g'(x)$$

Exercises: Compute the derivatives

① $f(x) = 4x^3 - 3x^2 + x + 3$

② $f(x) = \sin x + \cos x$

③ $f(x) = \sqrt{x}$

④ $f(x) = \tan x$

⑤ $f(x) = x^{\sqrt{2}}$

⑥ $f(x) = \log x + \log_5 x$

⑦ $f(x) = 2x + 3e^x$

⑧ $f(x) = (2x + 3)^4 + e^{2x}$

⑨ $f(x) = \cos(x^2) + \cos^2(x)$

⑩ $f(x) = x^2 \sin(3x)$

⑪ $f(x) = \frac{x \log(x)}{\sin(x)}$

⑫ $f(x) = \frac{x^2 + \sqrt{x}}{(2x-1)^3}$

Checklist

Let $f : D \rightarrow \mathbb{F}$ and let $x_0 \in D$. Then f is differentiable at x_0 if the following three conditions hold:

- ① f is continuous in x_0
- ② left and right limits of the incremental ratio must coincide
- ③ the limit must be finite

Non-differentiability points

- if f is not defined at x_0 , that is $x_0 \notin D$, then clearly it cannot be differentiable (because in this case the function does not exist at the point x_0)
- if

$$\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = d_1, \quad d_1 \neq \pm\infty$$
$$\lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = d_2, \quad d_2 \neq \pm\infty$$

but $d_1 \neq d_2$, then the function is NOT differentiable at x_0 , and $(x_0, f(x_0))$ is called an **angle point**

Non-differentiability points

- if

$$\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = +\infty, \quad (-\infty)$$

$$\lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = +\infty, \quad (-\infty)$$

(i.e. left and right limits are infinite of the same sign!) then the function is NOT differentiable at x_0 , and $(x_0, f(x_0))$ is called an **inflection point with vertical tangent**

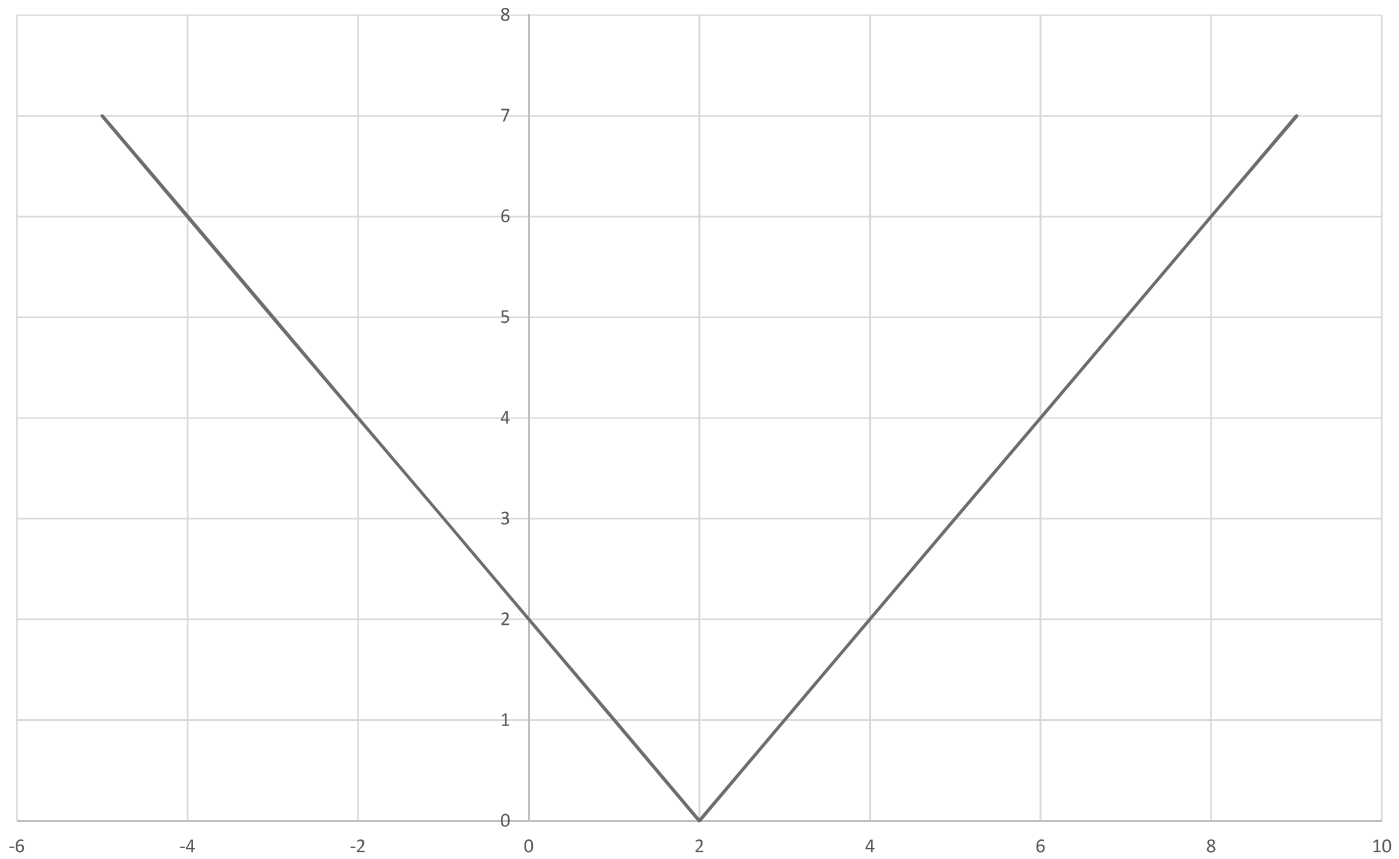
- if

$$\lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} = +\infty \quad (-\infty)$$

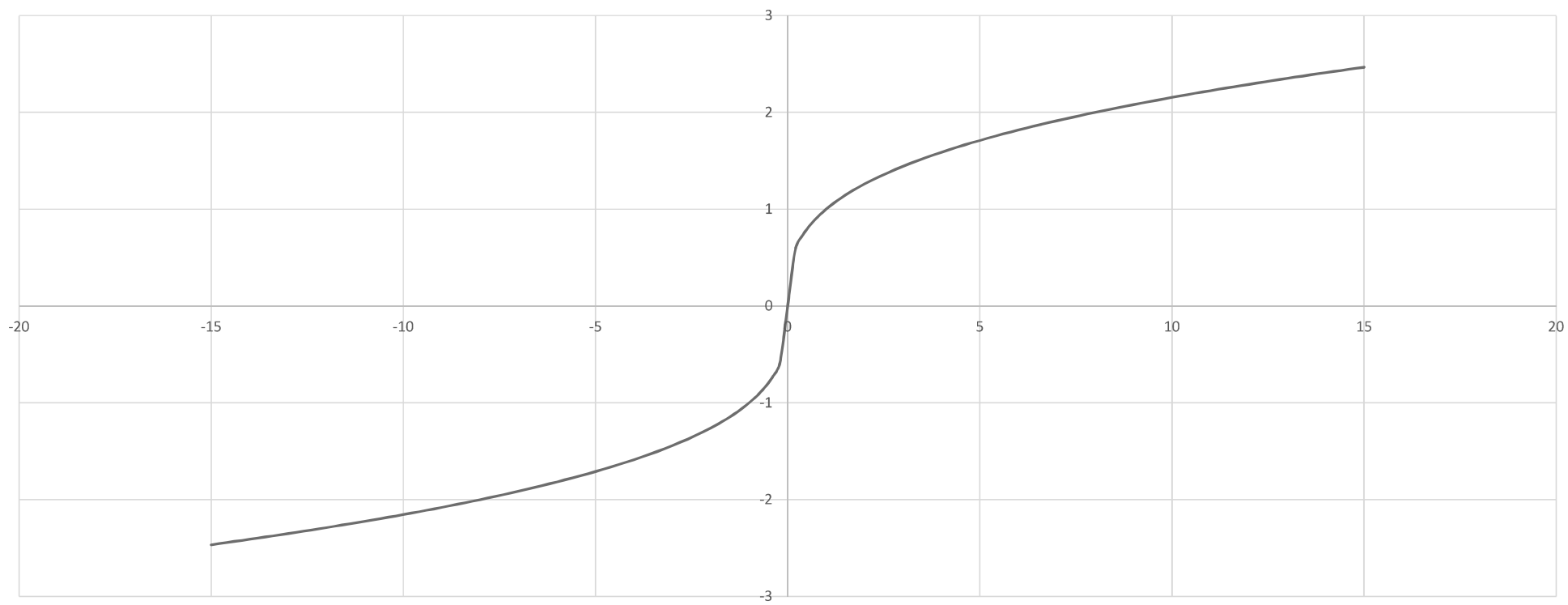
$$\lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} = -\infty, \quad (+\infty)$$

(i.e. left and right limits are infinite of the different sign!) then the function is NOT differentiable at x_0 , and $(x_0, f(x_0))$ is called a **cusp point**

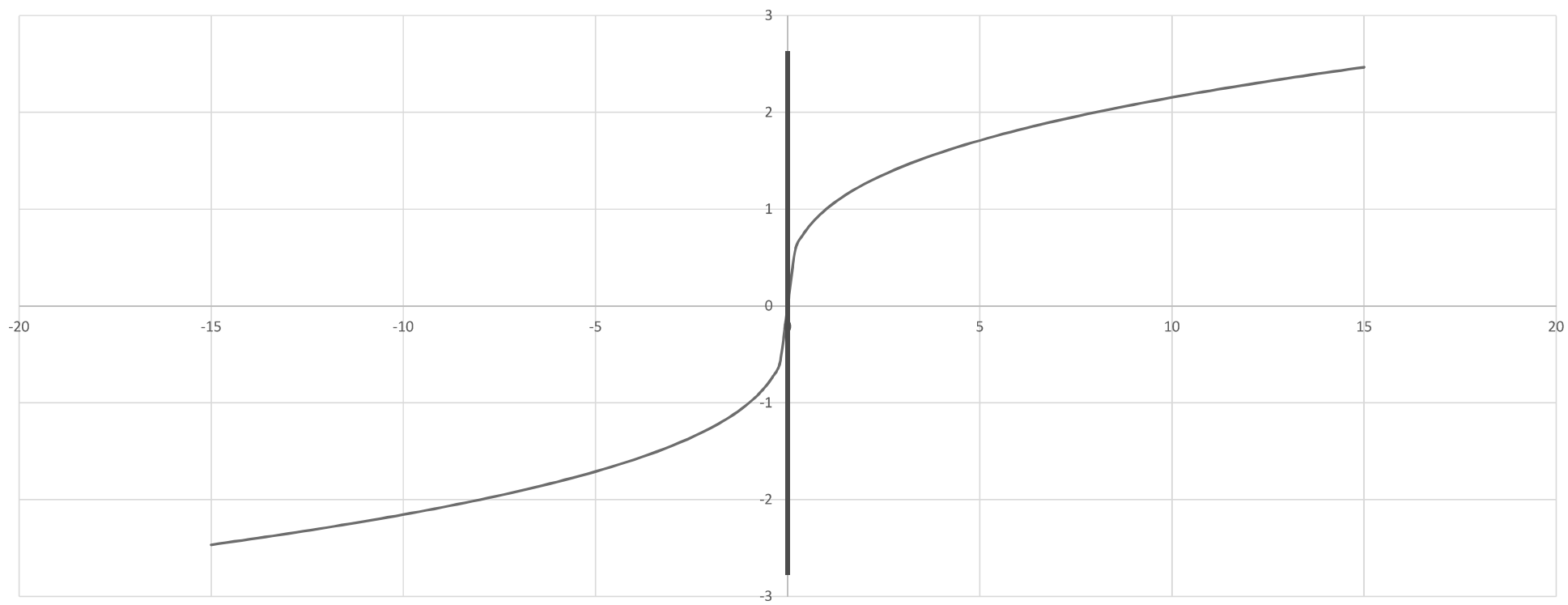
$$f(x) = |x - 2|$$



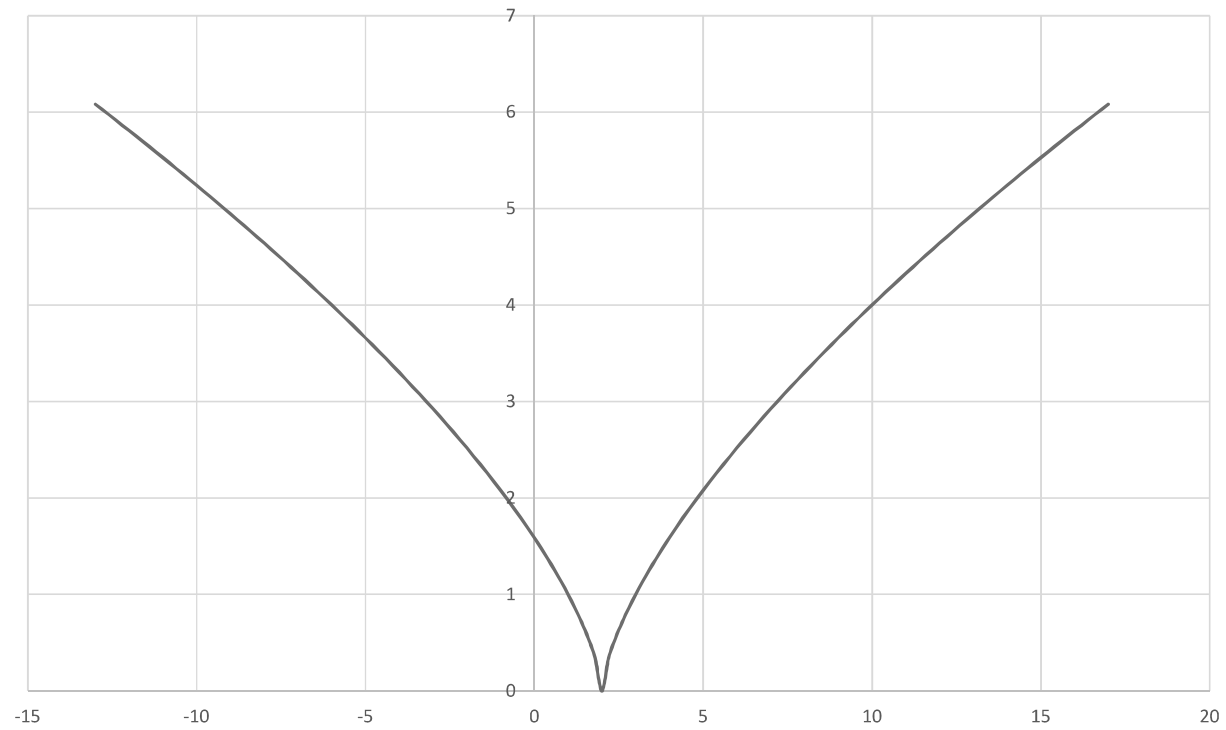
$$f(x) = \sqrt[3]{x}$$



$$f(x) = \sqrt[3]{x}$$



$$f(x) = \sqrt[3]{(x-2)^2}$$



First order Taylor approximation

Suppose that

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{(x - x_0)} = f'(x_0) \quad (1)$$

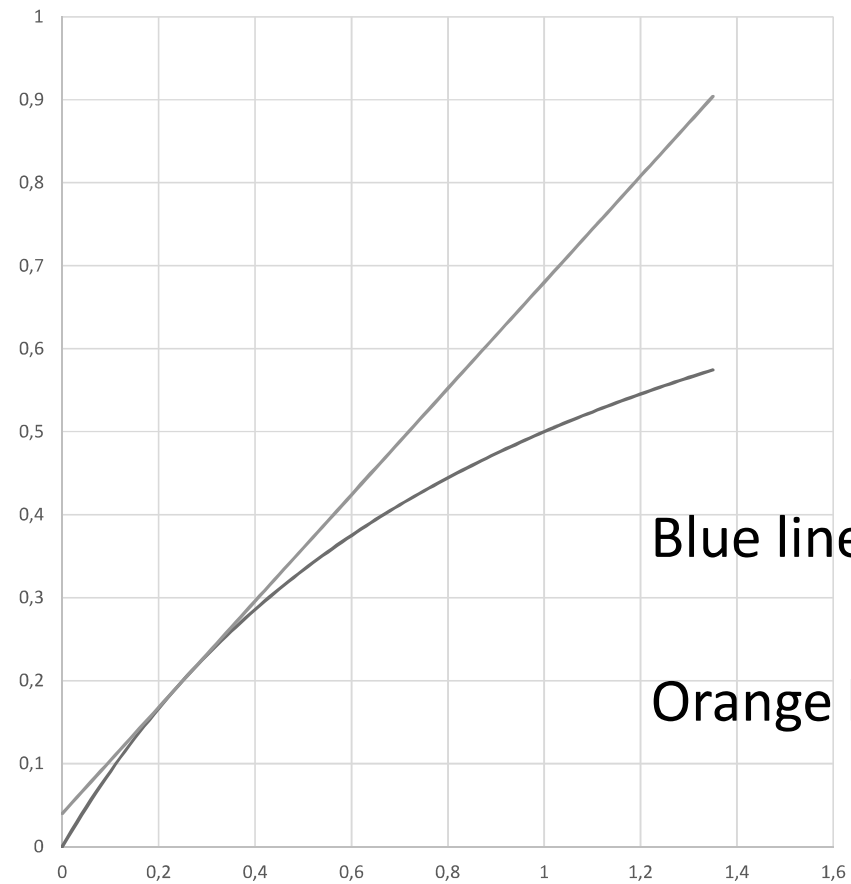
Then if x is close to x_0 , we can say that

$$f'(x_0) \sim \frac{f(x) - f(x_0)}{(x - x_0)}$$

Equivalently

$$f(x) \sim f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) \sim f(x_0) + f'(x_0)(x - x_0)$$



Blue line: $f(x)$

Orange line: tangent at $x_0 = \frac{1}{4}$

First order Taylor approximation

- The function

$$P(x) = f(x_0) + f'(x_0)(x - x_0)$$

is called the first order Taylor approximation of f or the linearization of f or the first order Taylor polynomial of f .

- The quantity

$$R(x_1) = f(x_1) - P(x_1)$$

is the reminder or the (absolute) error, for every $x_1 \neq x_0$.

- The relative error is $\epsilon(x_1) = \frac{R(x_1)}{f(x_1)}$

The approximation $P(x)$ is good if

- if x is close to x_0 , i.e. $|x - x_0| < \delta$
- f is almost flat

Differentiability and continuity

Theorem

Let $f : D \rightarrow \mathbb{R}$ and let $x_0 \in D$. If f is differentiable at x_0 then it is continuous in x_0 .

Proof: Notice that we already know that $x_0 \in D$. Therefore we need to show that $\lim_{x \rightarrow x_0} f(x)$ exists and it is equal to $f(x_0)$.

For all $x \neq x_0$ we can write

$$f(x) = f(x) - f(x_0) + f(x_0)$$

$$f(x) = \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) + f(x_0)$$

Now, we take the limit as $x \rightarrow x_0$ on both sides:

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{f'(x_0)} \underbrace{(x - x_0)}_0 + f(x_0) = f(x_0)$$

This concludes the proof.

Differentiability and continuity

Problem If a function is continuous, is it also differentiable ? **No!**

Differentiability and continuity

Problem If a function is continuous, is it also differentiable ? **No!**

Example: The function absolute value

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases},$$

This function is continuous in for all $x \in \mathbb{R}$.

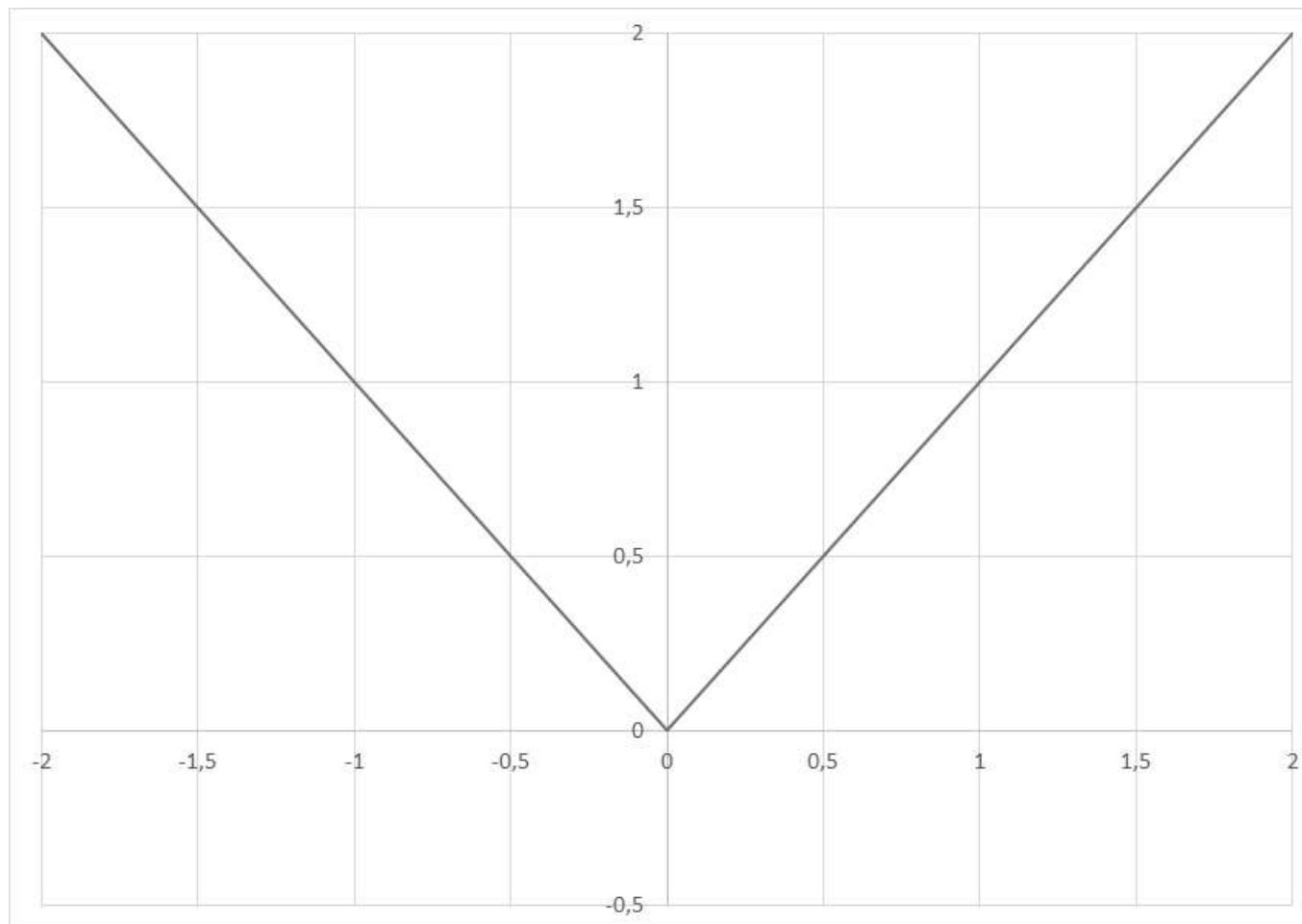
However, if we compute the left and right limit of incremental ratio at $x_0 = 0$ we get

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h} &= \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \\ \lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{h} &= \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1 \end{aligned}$$

Hence f is not differentiable at $x_0 = 0$ and $x_0 = 0$ is an angle point.

Example

$$f(x) = |x|$$



Derivatives: Rolle's Theorem

Theorem

If a function $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b)

Derivatives: Rolle's Theorem

Theorem

If a function $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$ then $\exists x_0 \in (a, b)$ such that $f'(x_0) = 0$.

Derivatives: Rolle's Theorem

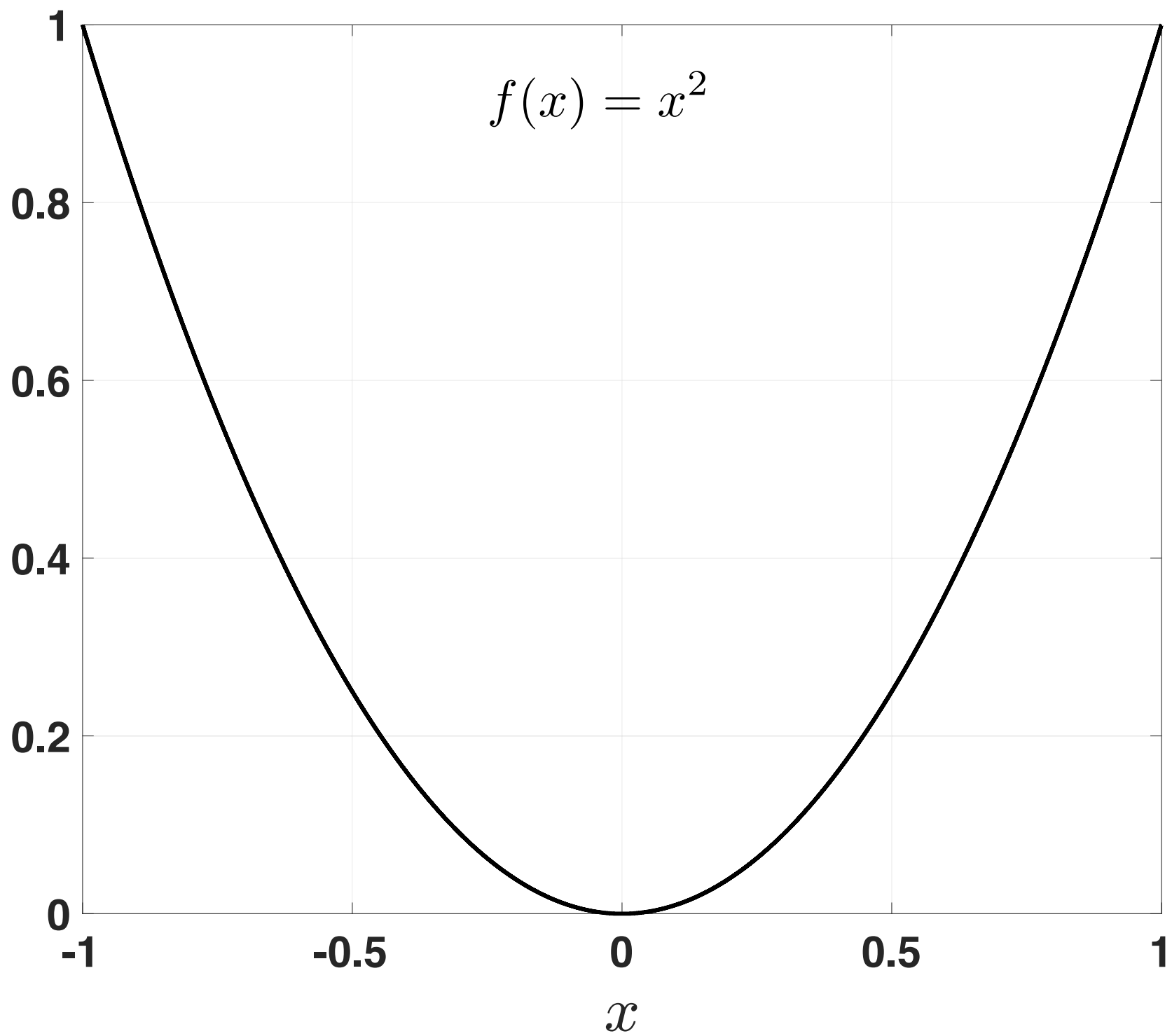
Theorem

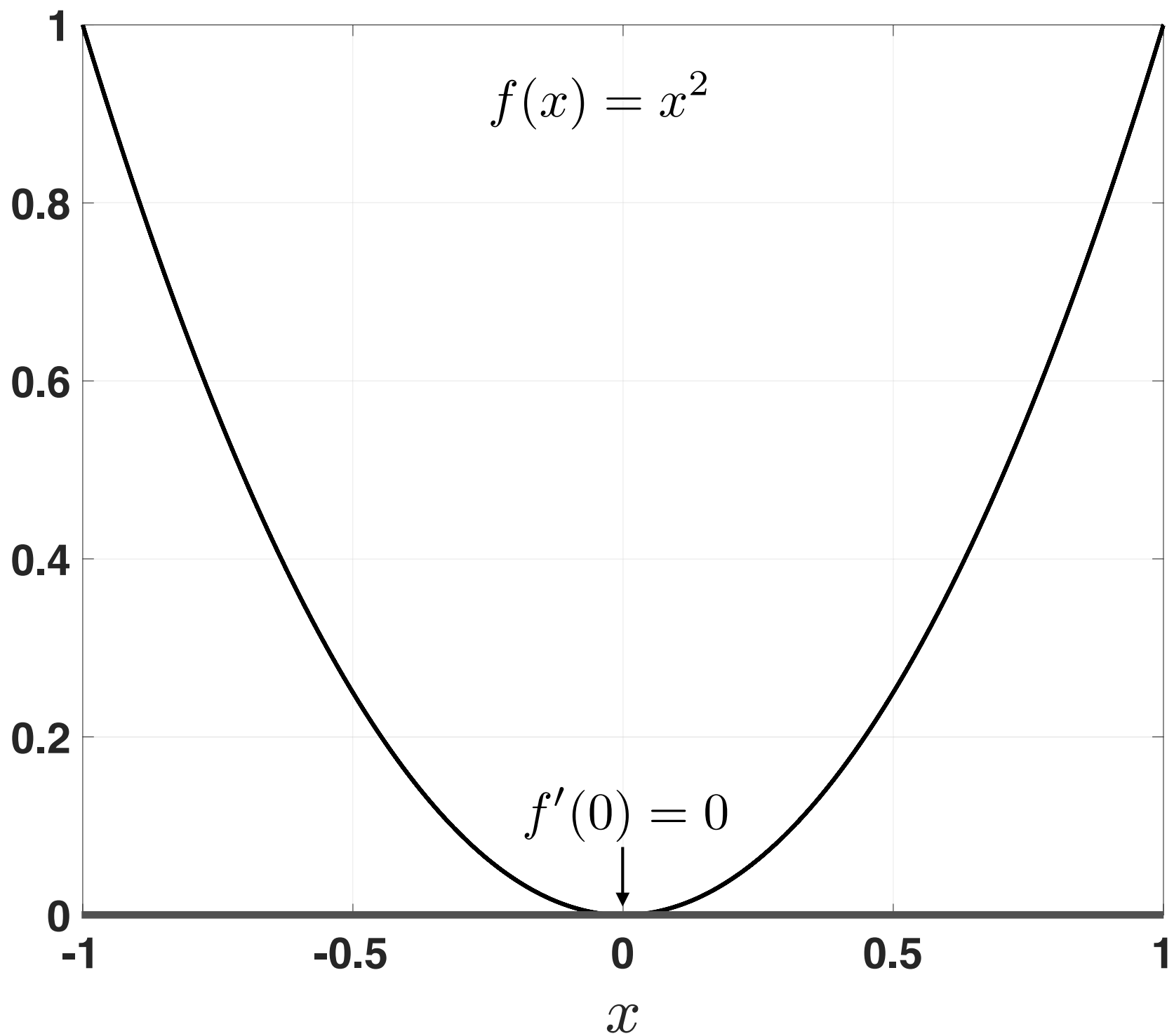
If a function $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$ then $\exists x_0 \in (a, b)$ such that $f'(x_0) = 0$.

Rolle Theorem says that there exists at least ONE point in $[a, b]$ with horizontal tangent line!

Checklist:

- $[a, b]$ is closed and bounded
- The function is continuous in $[a, b]$
- The domain of the derivative MUST include the set (a, b)
- $f(a) = f(b)$





Derivatives: Lagrange's Mean Value Theorem.

Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) .

Derivatives: Lagrange's Mean Value Theorem.

Theorem

*Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) .
Then there exists at least one point $x_0 \in (a, b)$ such that*

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

That means there is at least one point x_0 where the tangent to the graph of f is parallel to the line from $(a, f(a))$ to $(b, f(b))$

Derivatives: Lagrange's Mean Value Theorem.

Theorem

*Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) .
Then there exists at least one point $x_0 \in (a, b)$ such that*

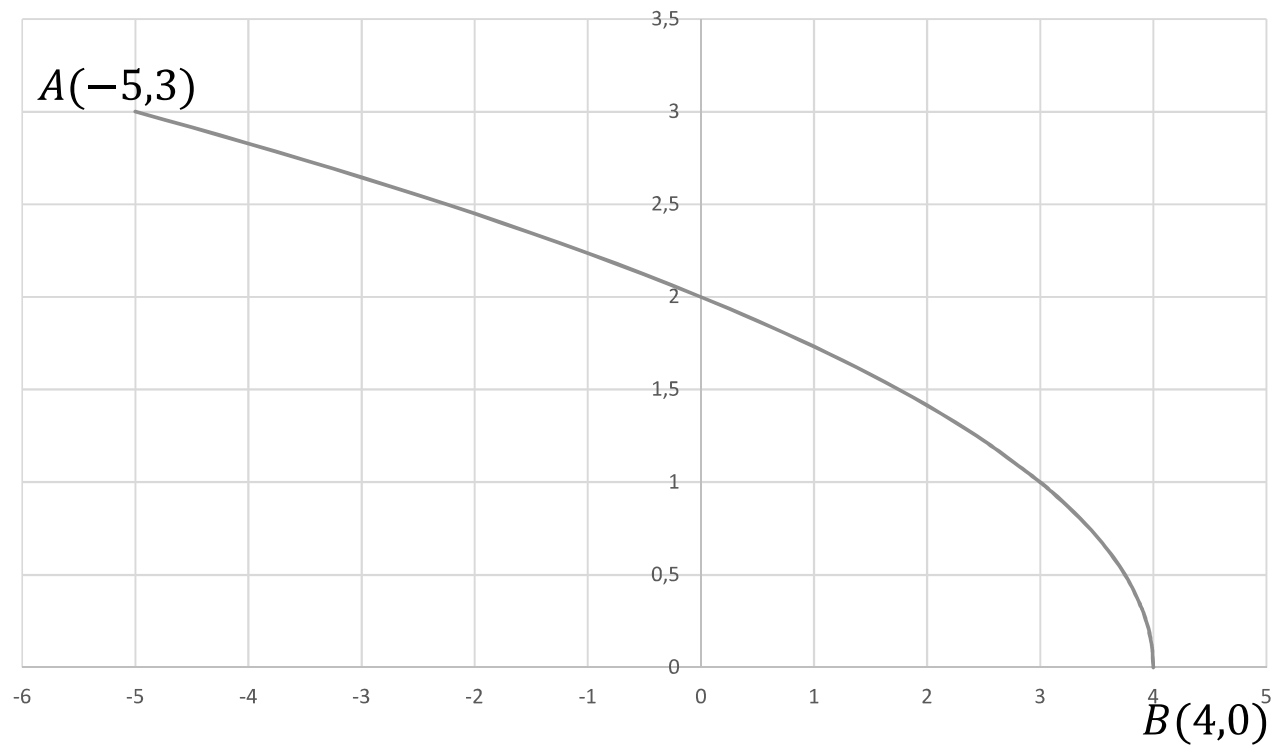
$$f'(x_0) = \frac{f(b) - f(a)}{b - a}.$$

That means there is at least one point x_0 where the tangent to the graph of f is parallel to the line from $(a, f(a))$ to $(b, f(b))$

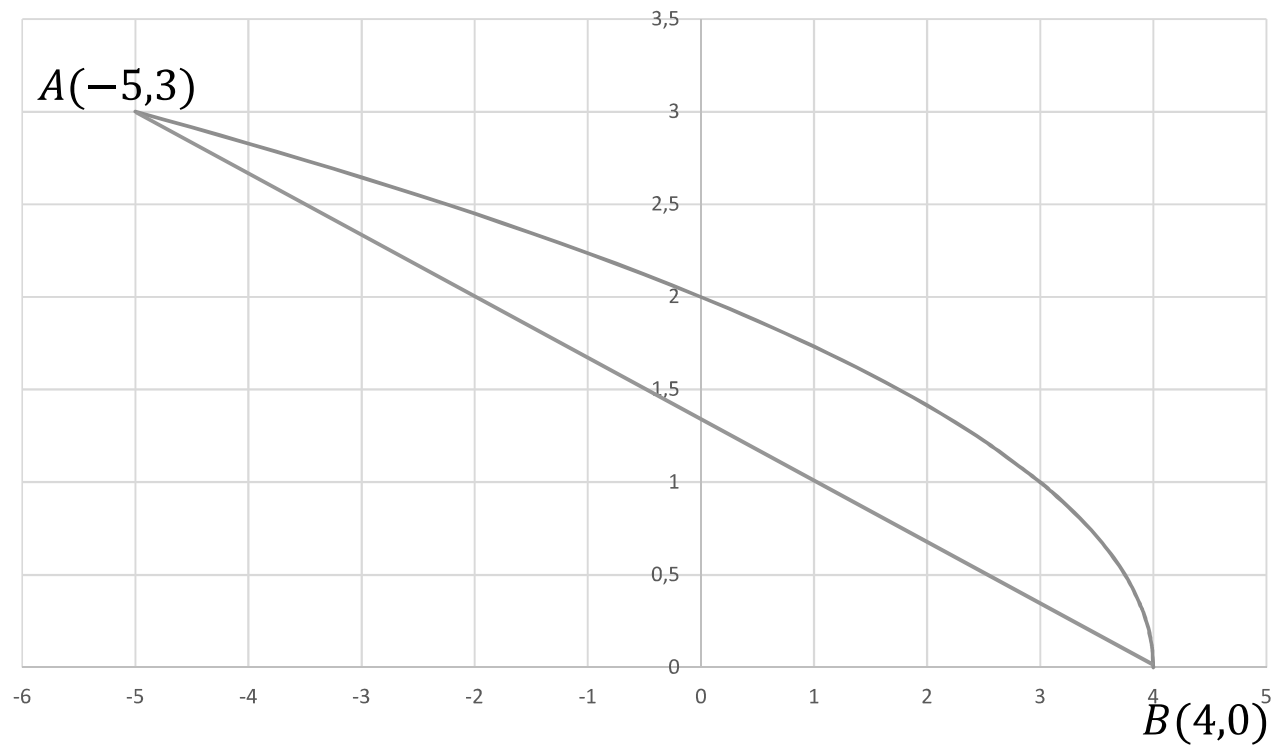
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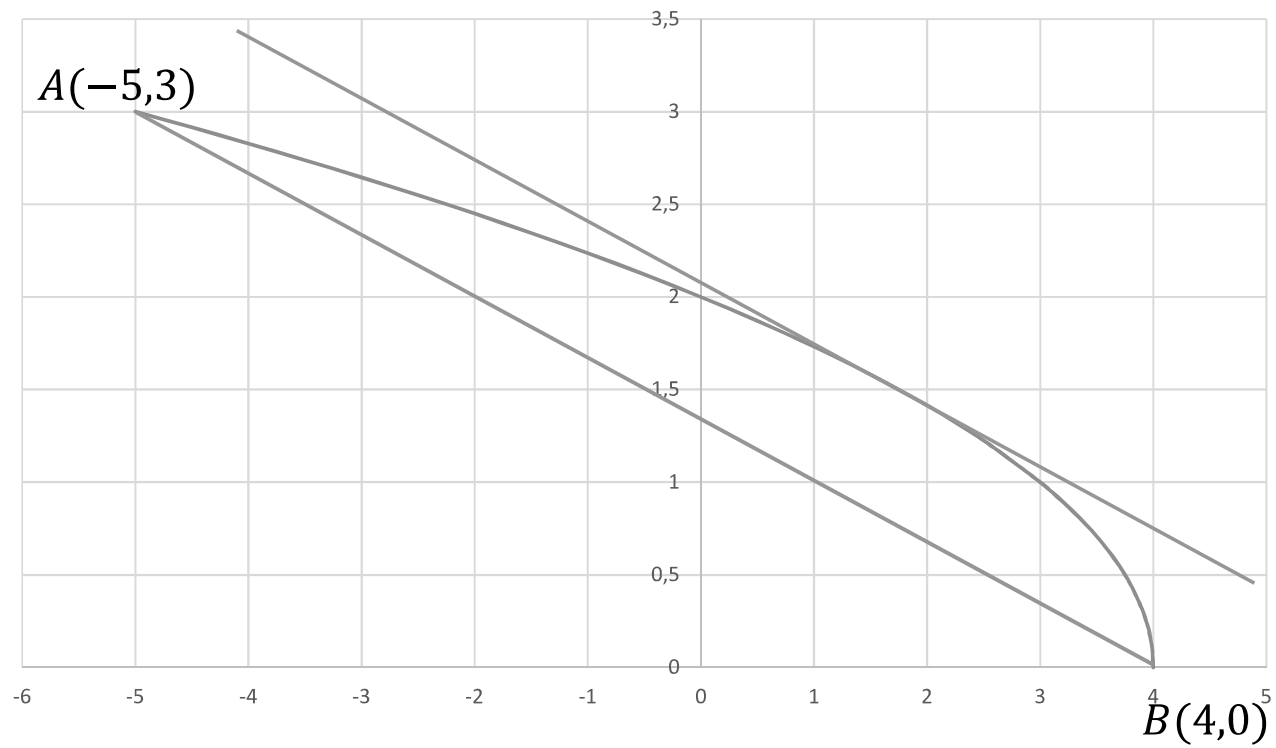
$$f(x) = \sqrt{4 - x} \text{ on } [-5, 4]$$



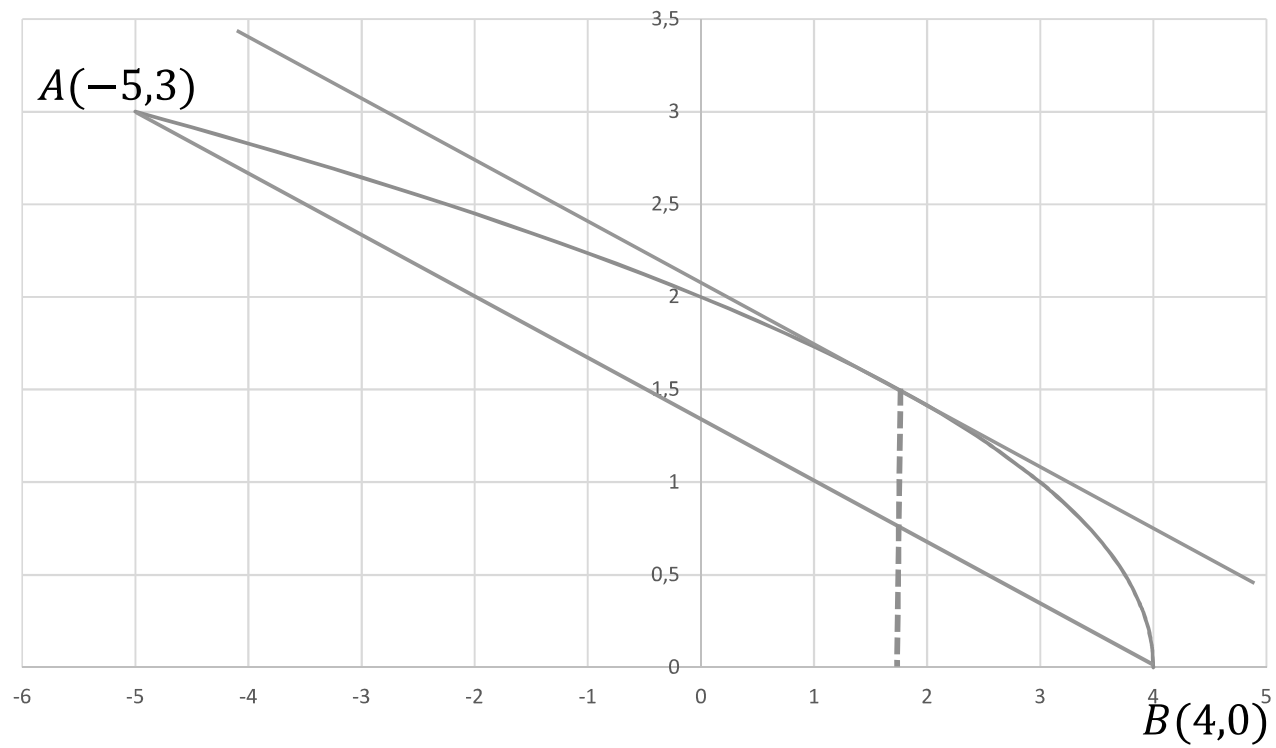
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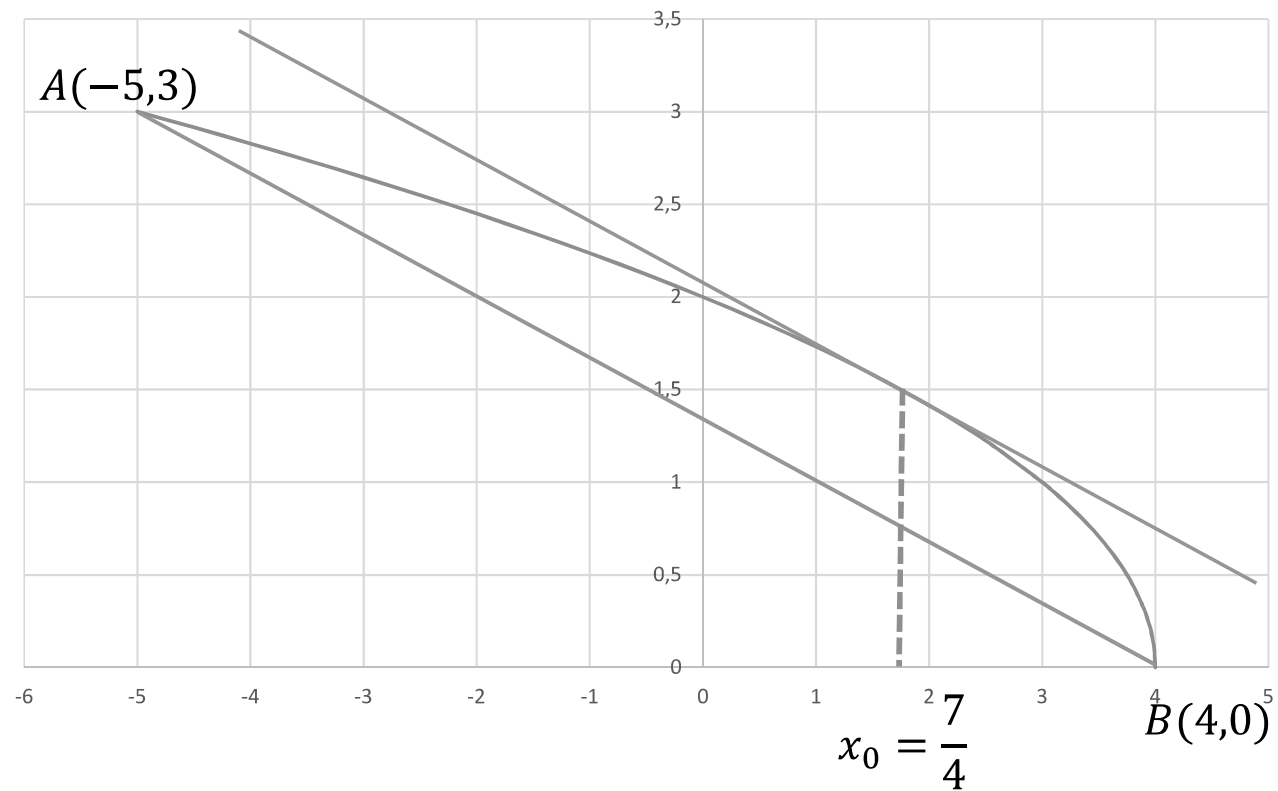
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Increasing and Decreasing functions

Theorem

Necessary and sufficient conditions for monotonicity of differentiable functions
Let $f : D \rightarrow \mathbb{R}$ be a function and assume that f is differentiable in any open interval $I \subseteq D$. Then:

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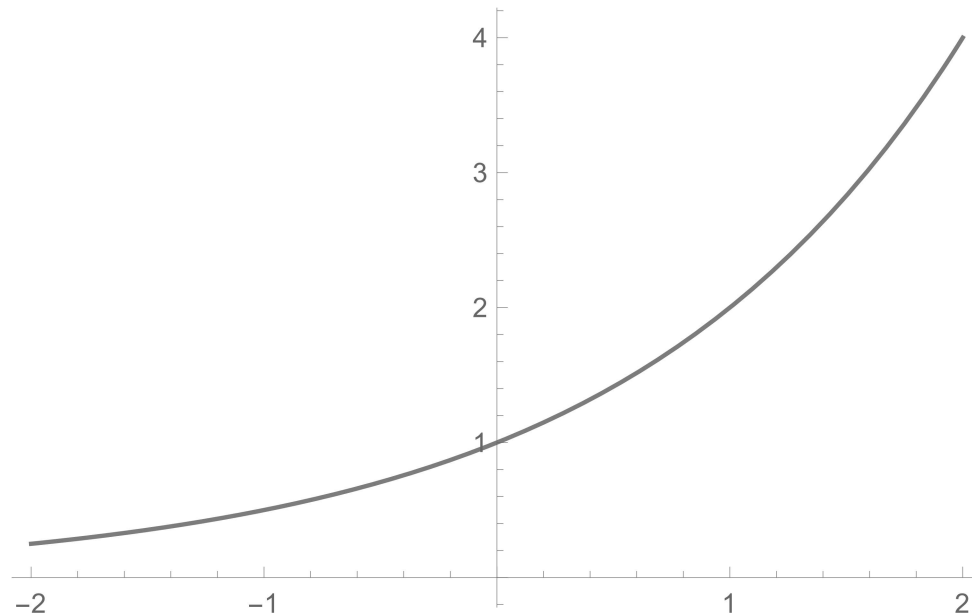
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Pay attention: The interval I can be unbounded.

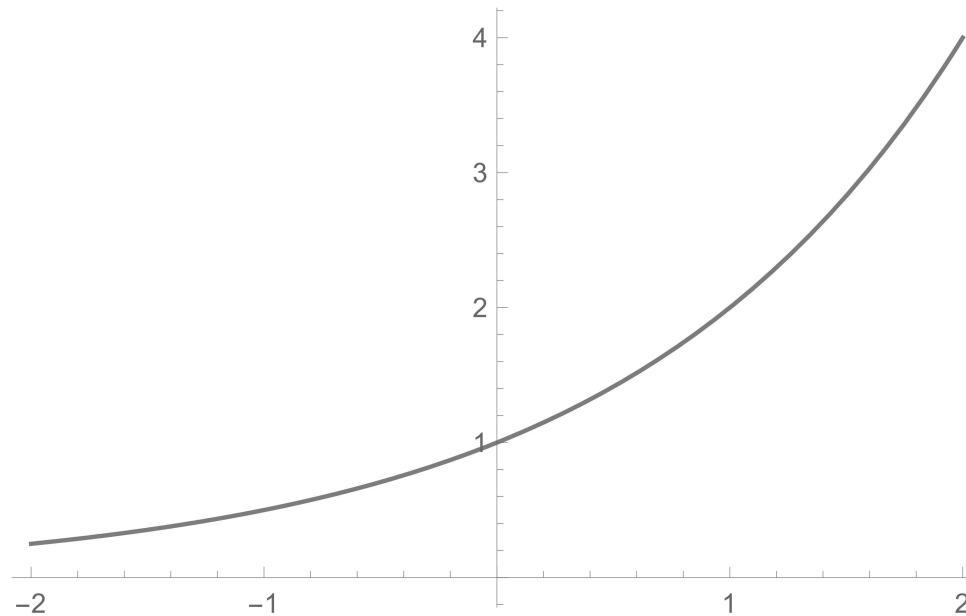
Increasing and decreasing functions: examples

$$f(x) = 2^x$$



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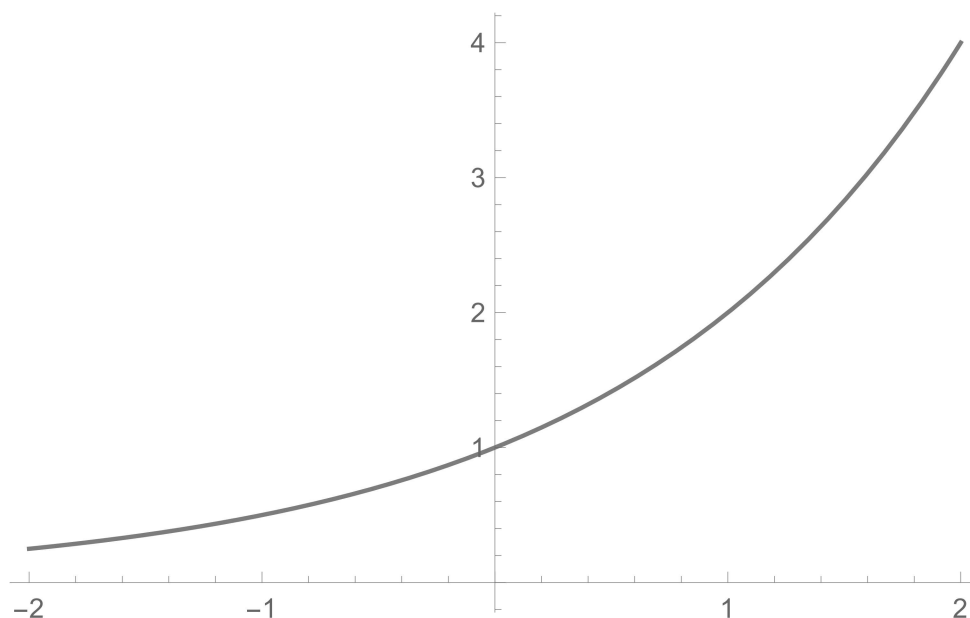
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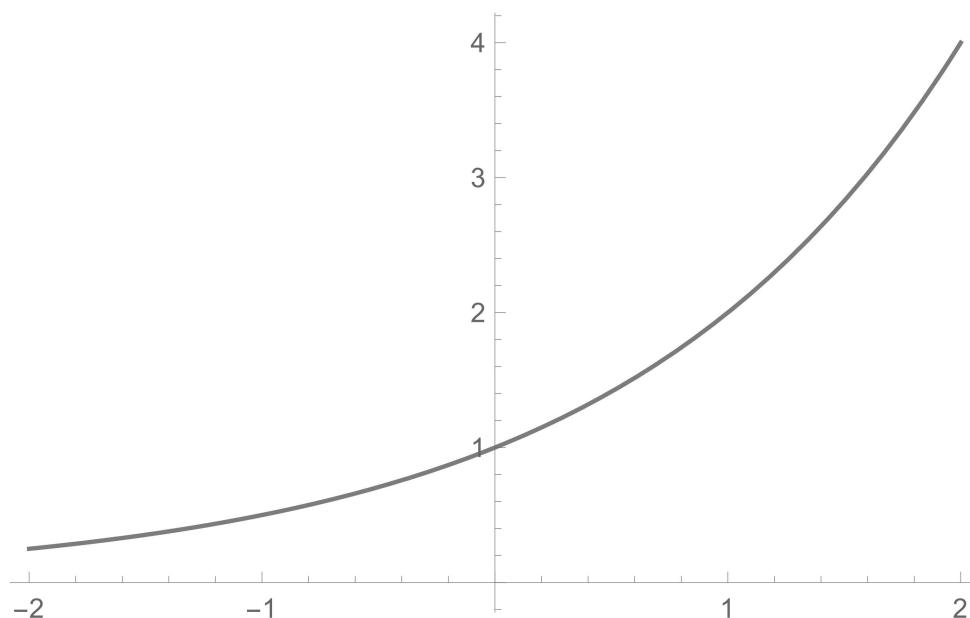
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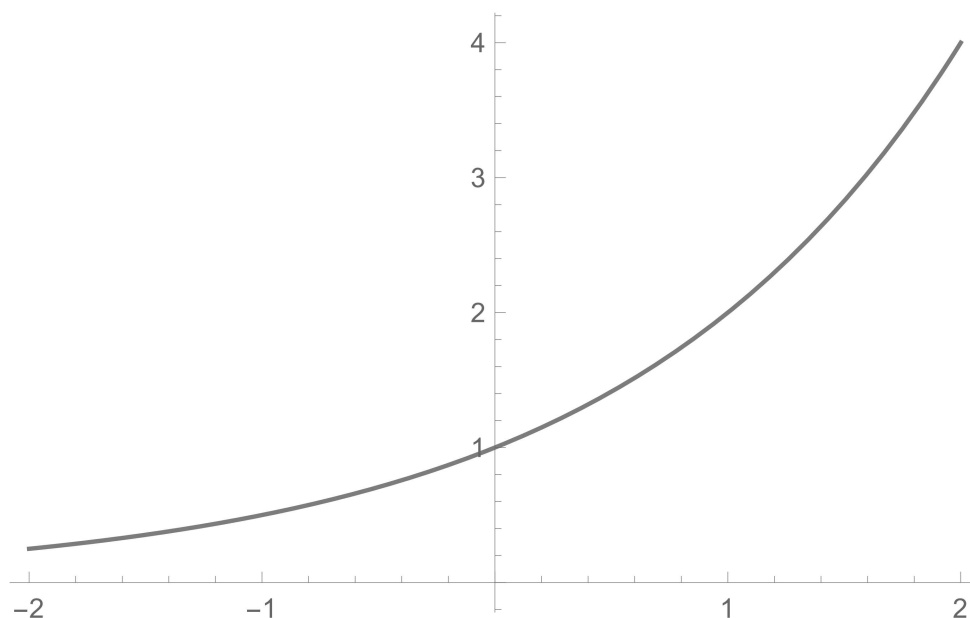
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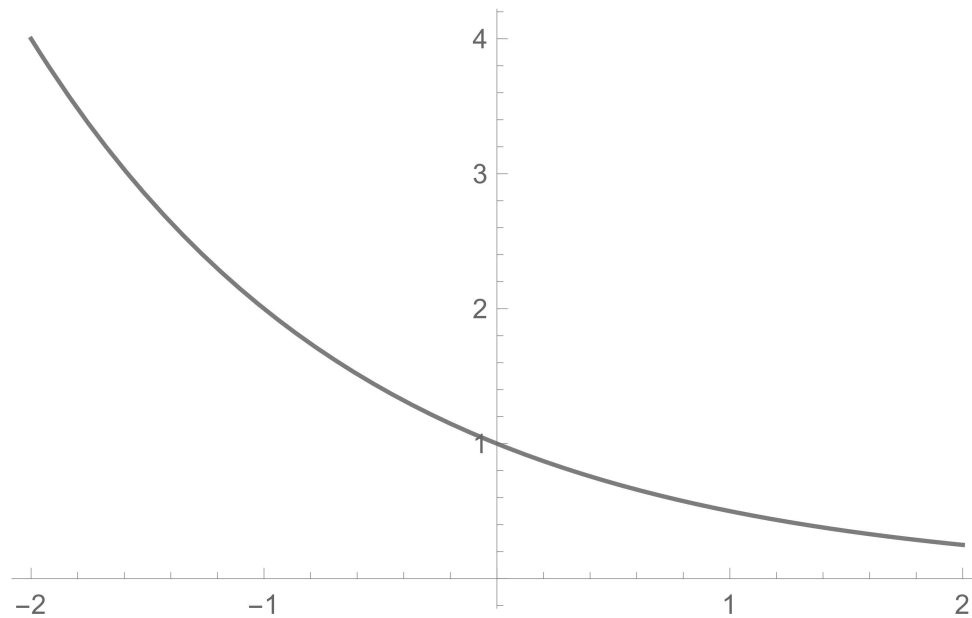
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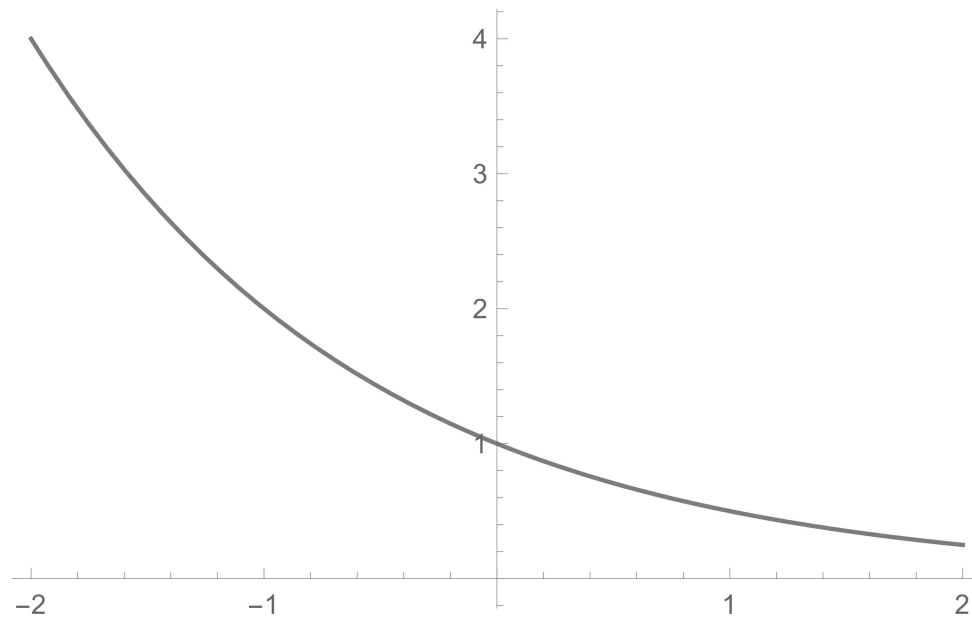
Increasing and decreasing functions: examples, cont'd

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Increasing and decreasing functions: examples, cont'd

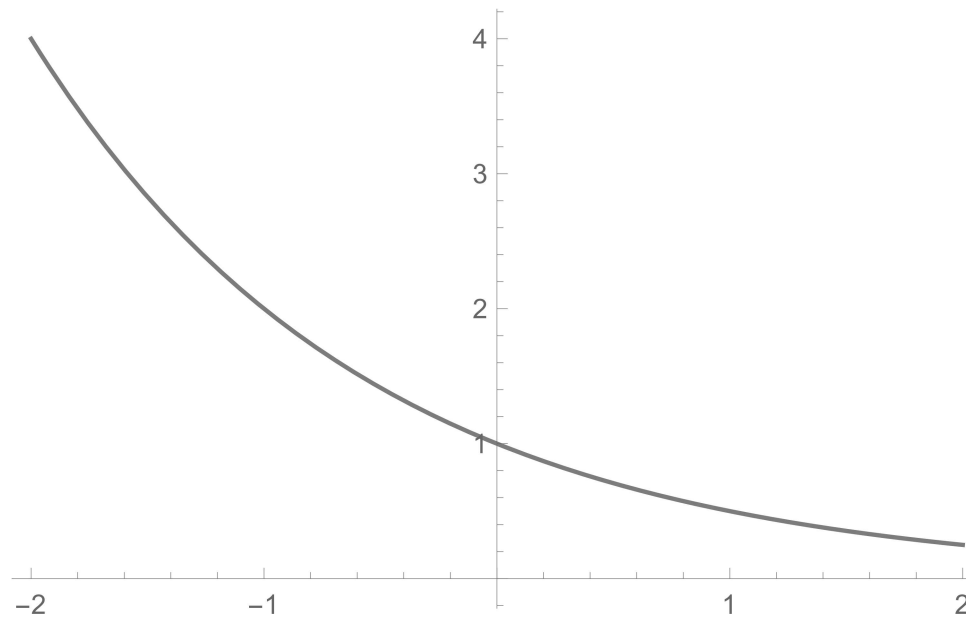
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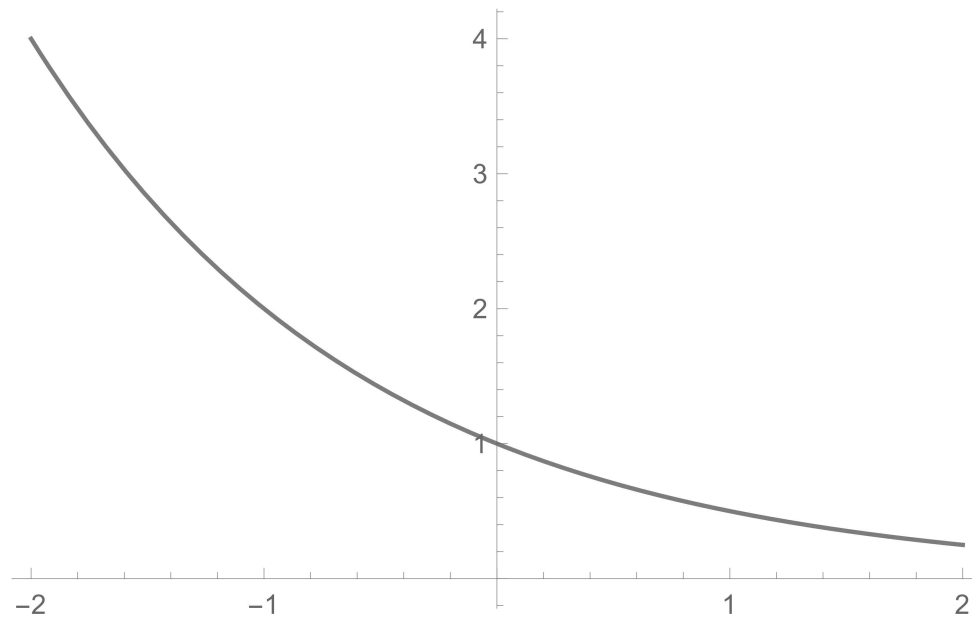
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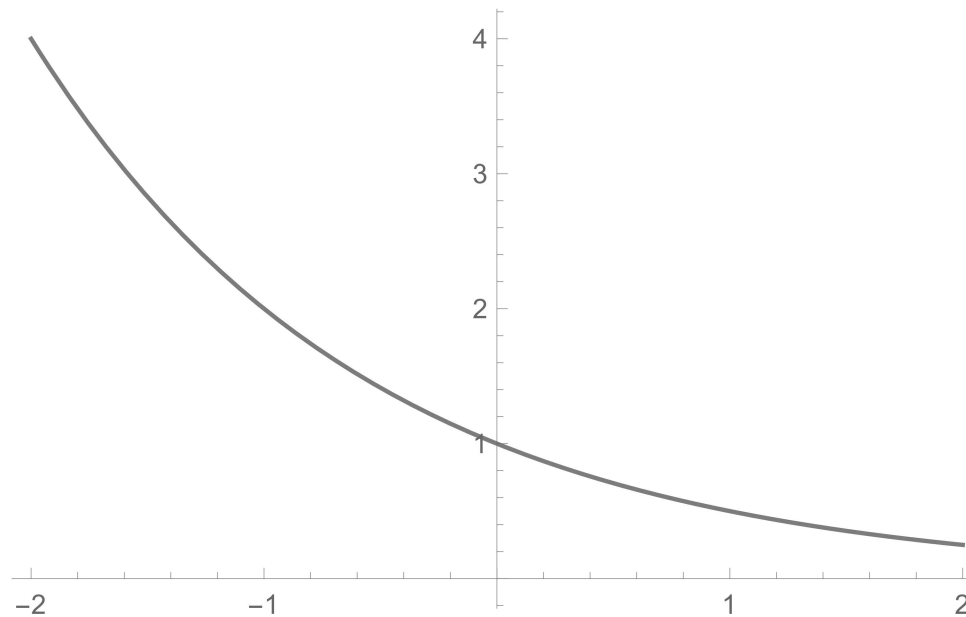
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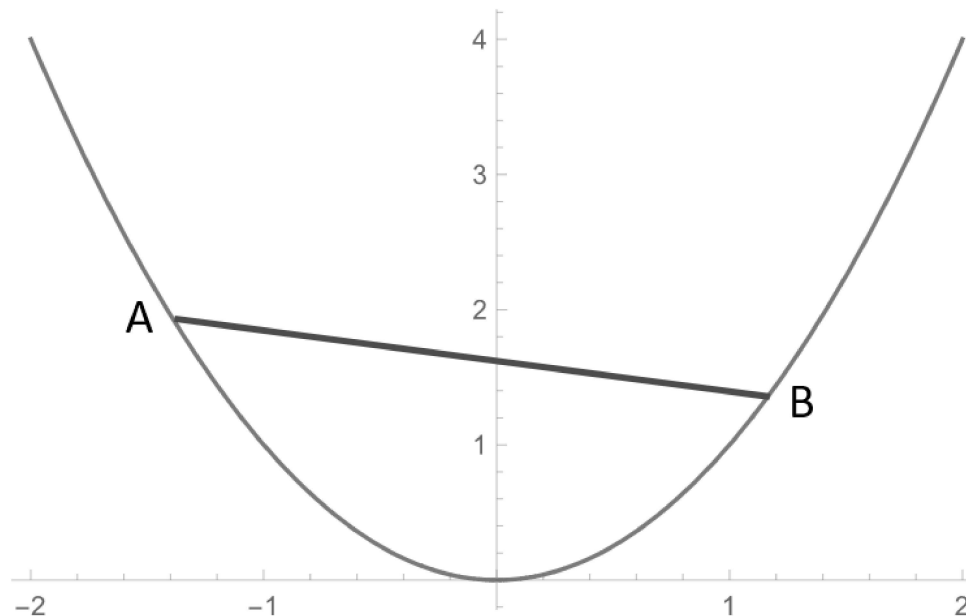
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Increasing and decreasing functions: examples

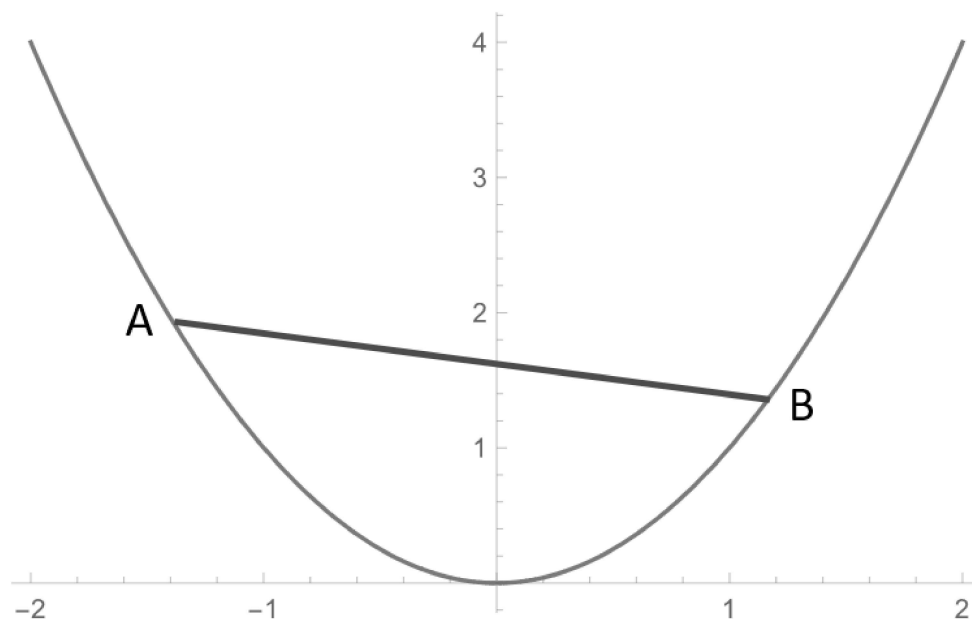
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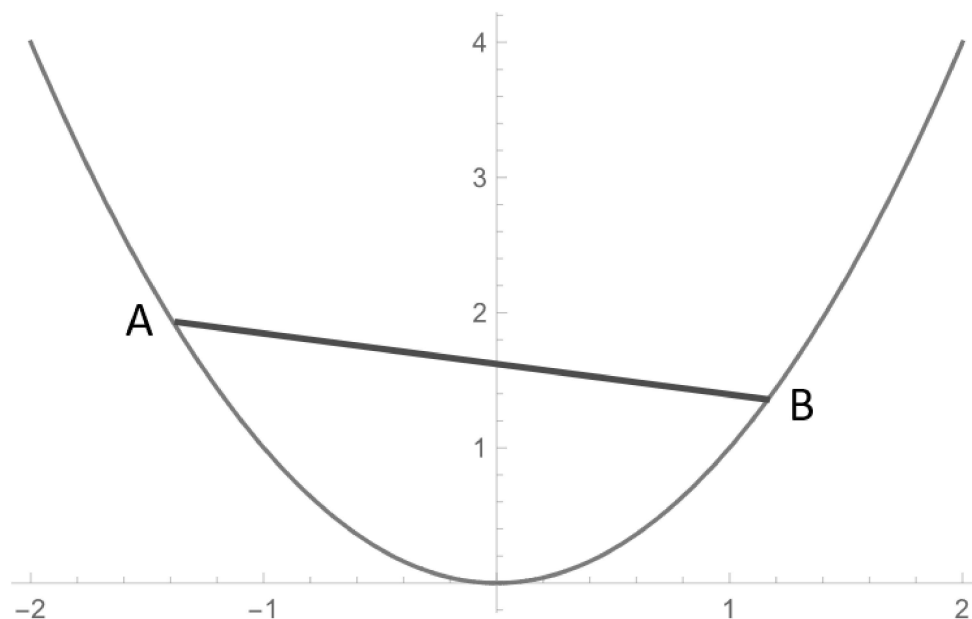
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Increasing and decreasing functions: exercises

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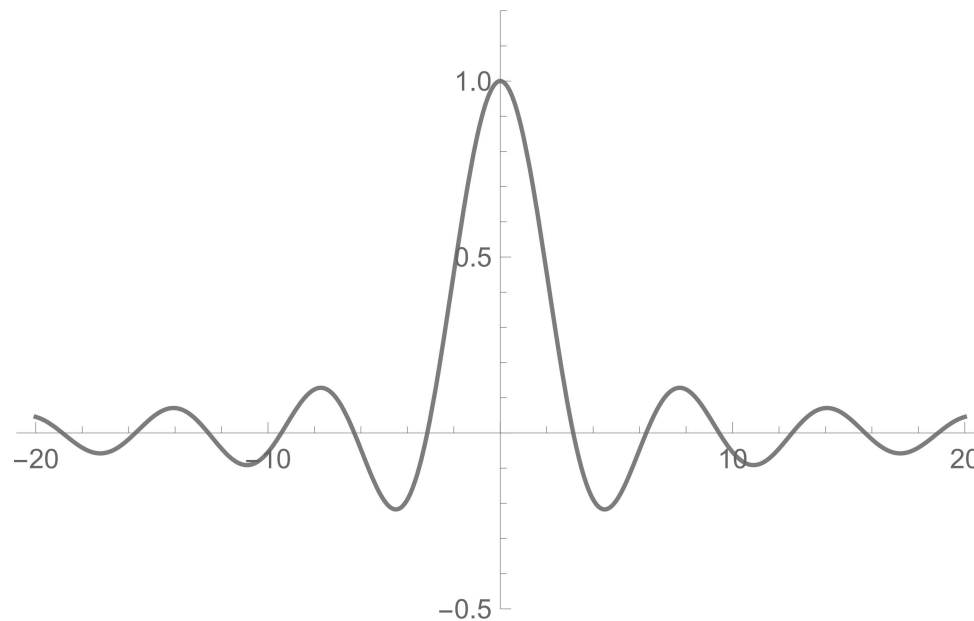
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In an interval $[a, b]$ we can have multiple local maxima/minima but ONLY one maximum/minimum.

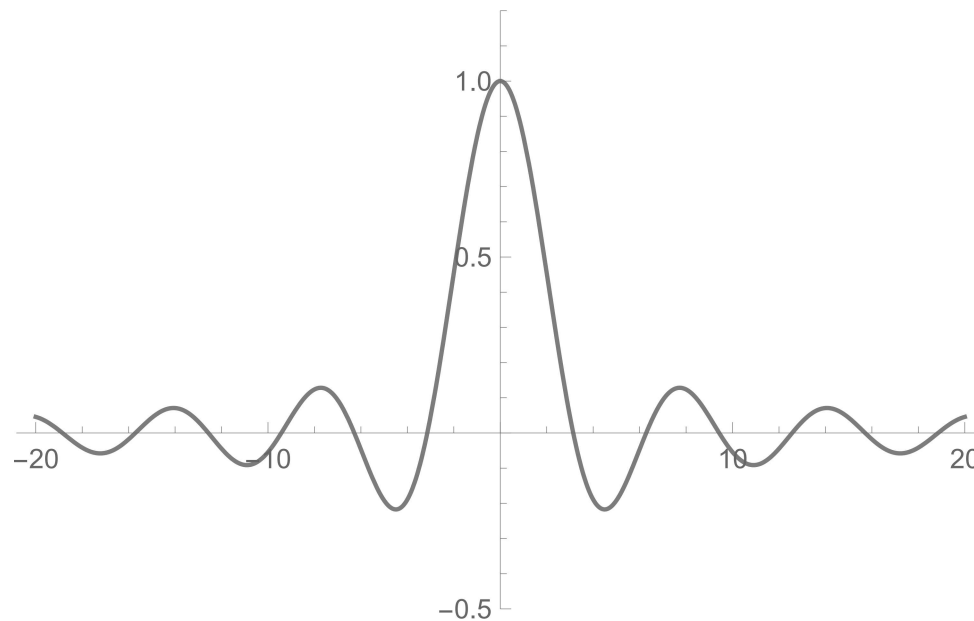
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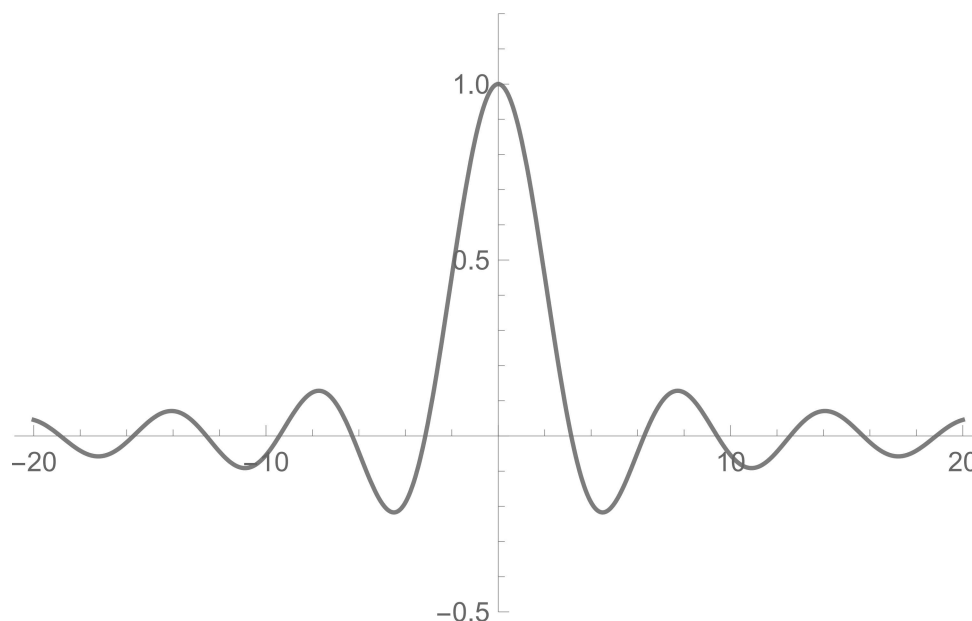
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This function is continuous in $\mathbb{R}$.

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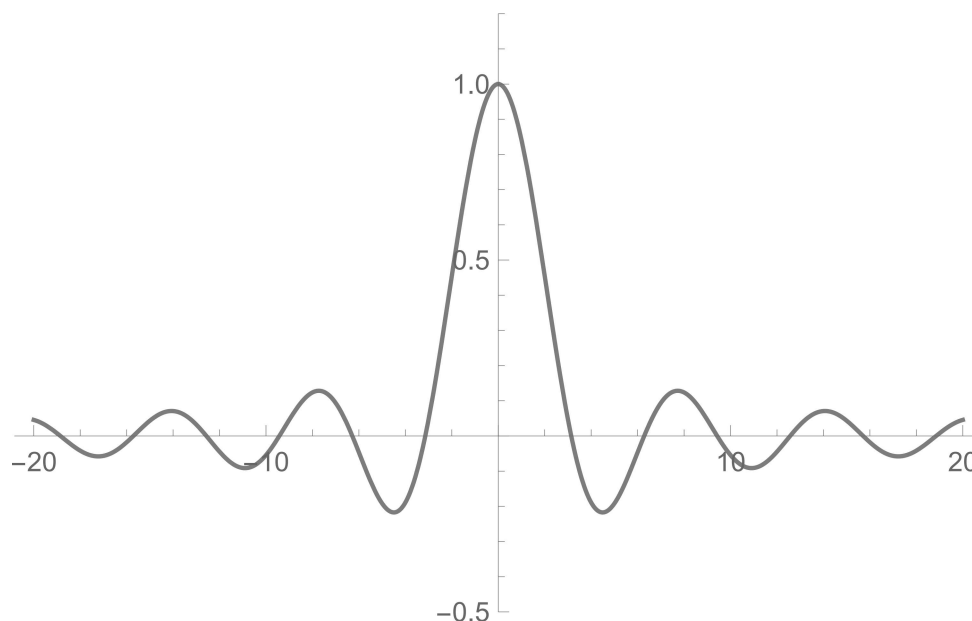
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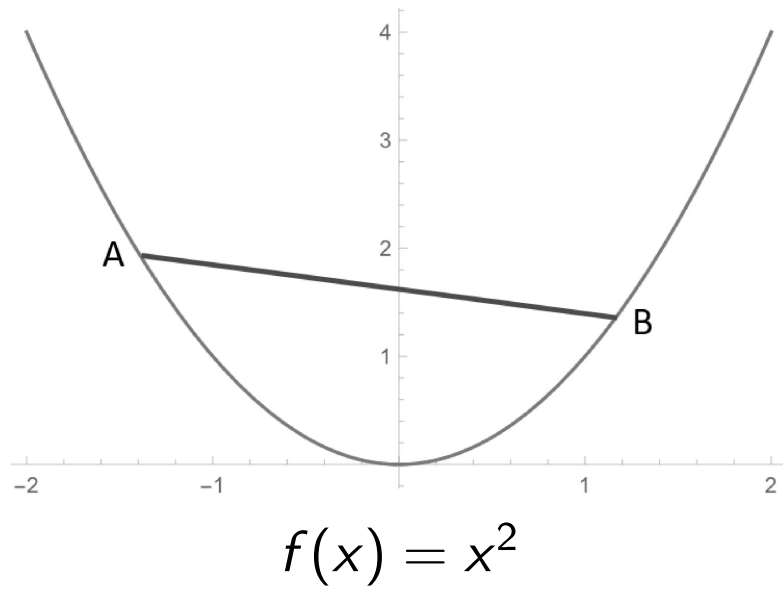
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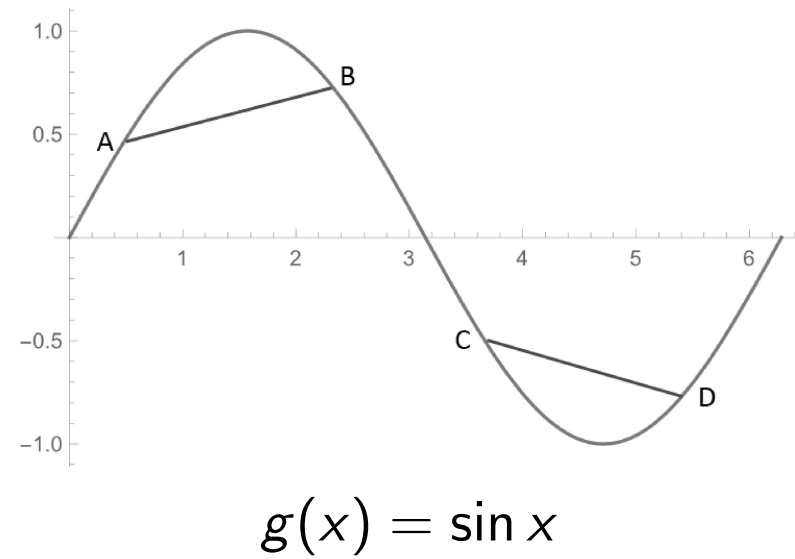
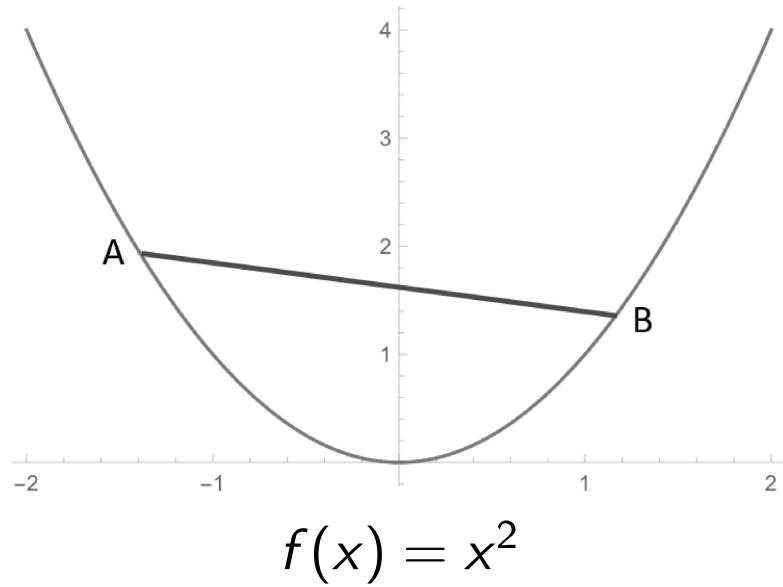


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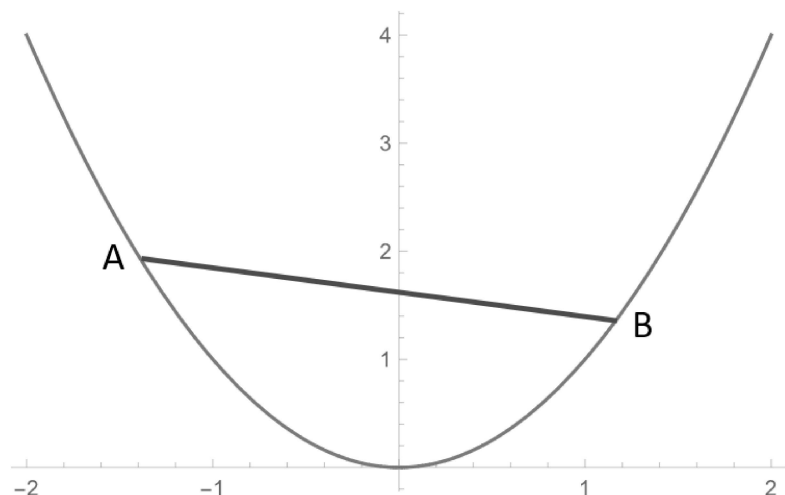
Fermat's theorem: the intuition



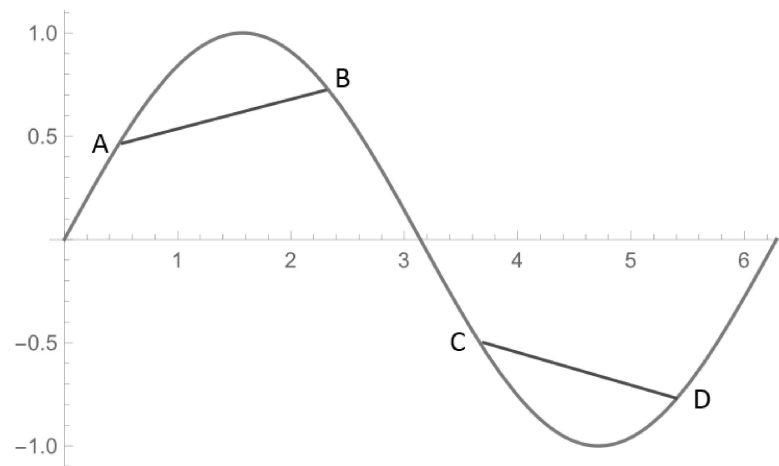
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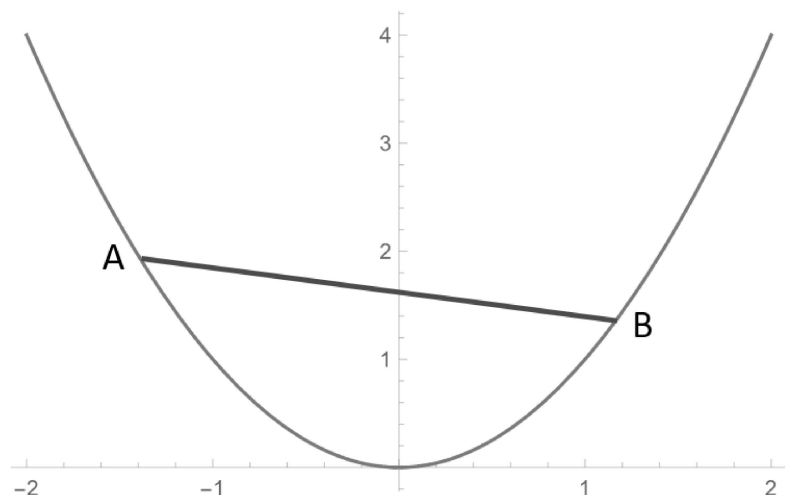
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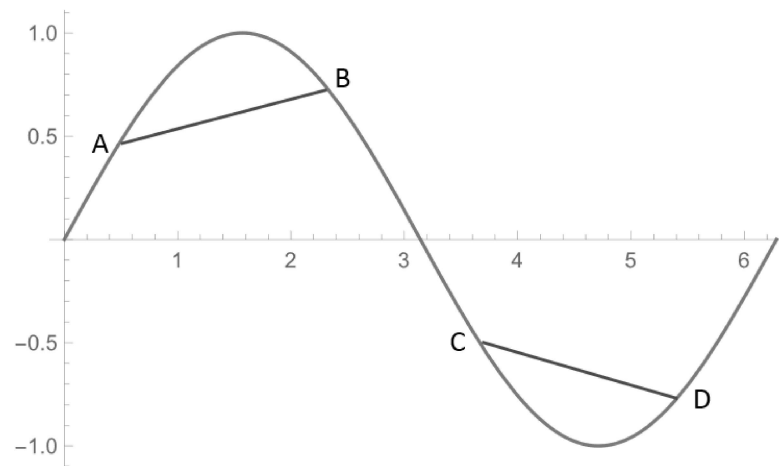
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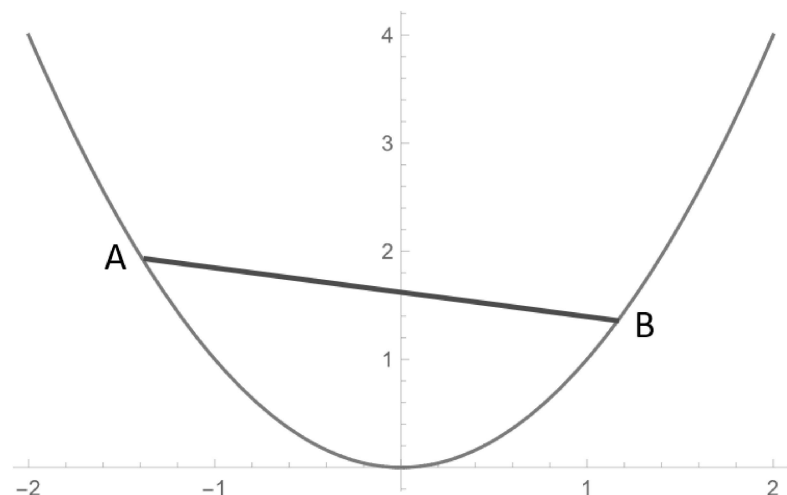
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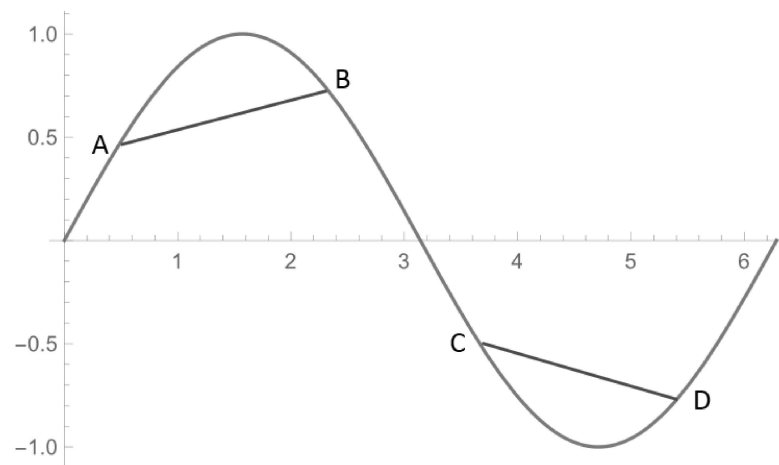
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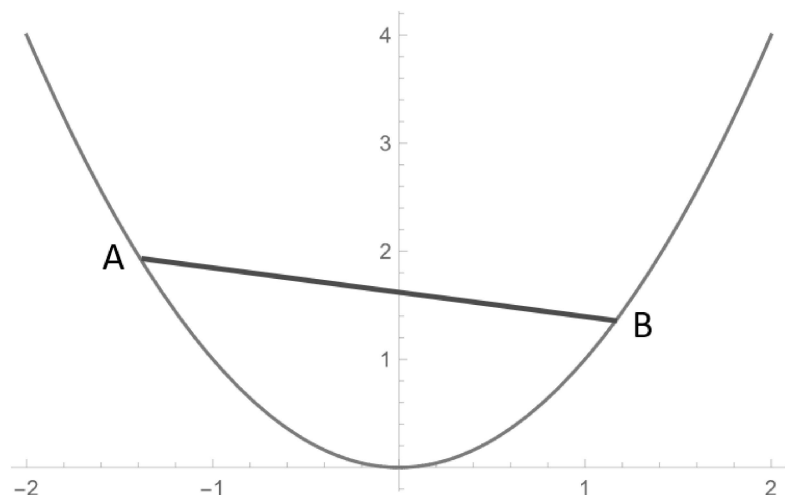
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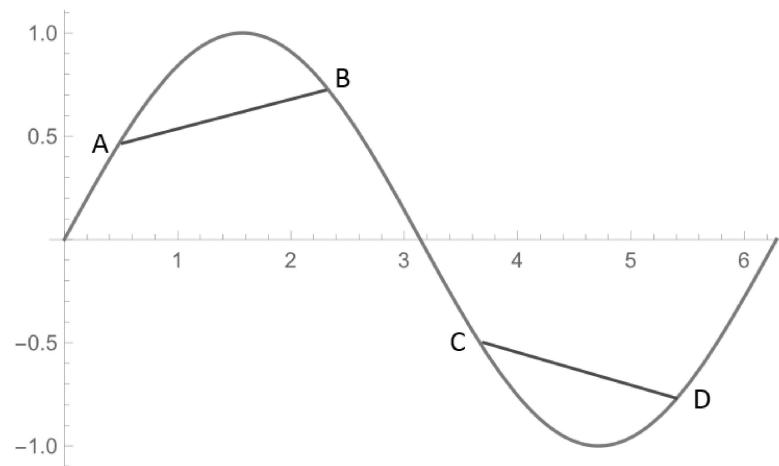
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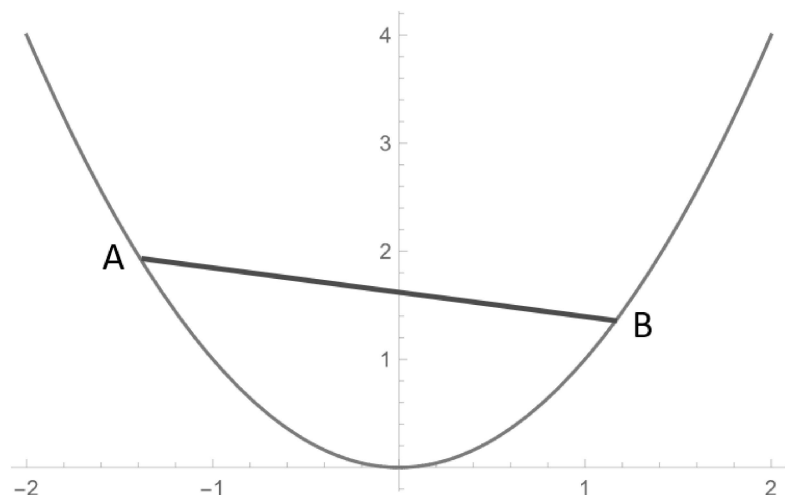


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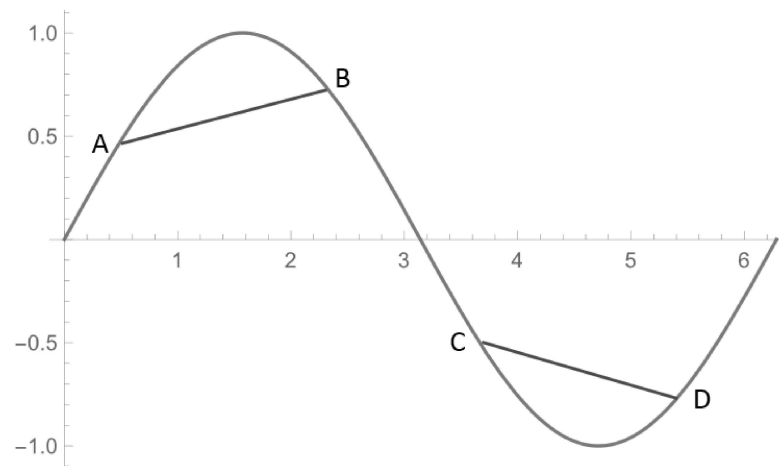
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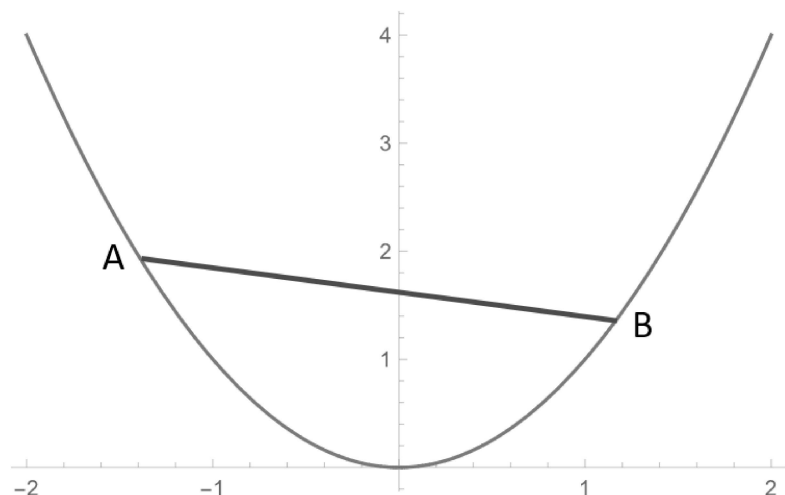


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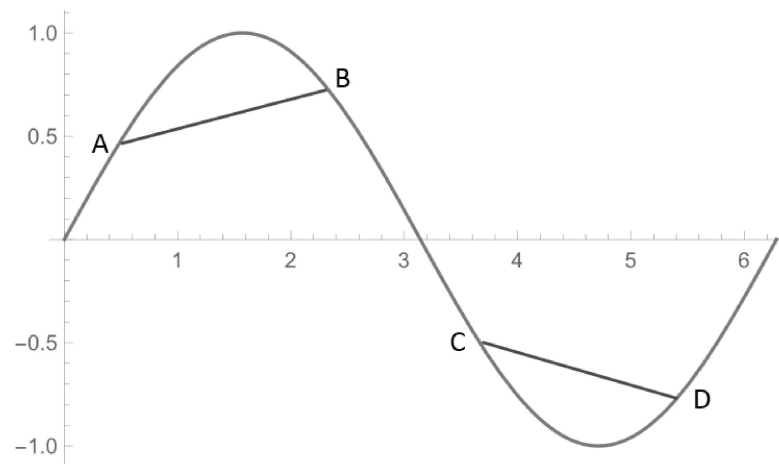
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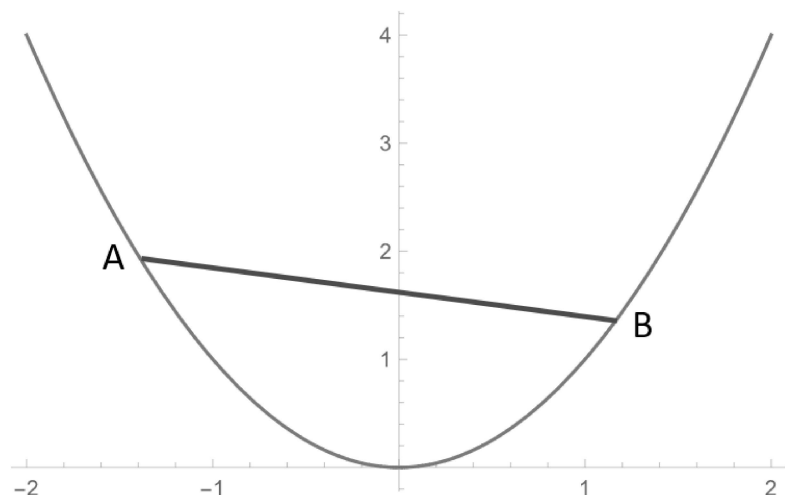
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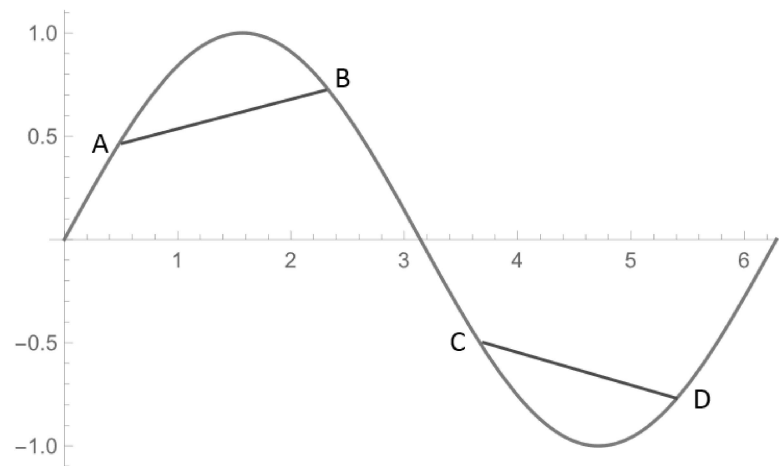
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Thus, if a function is differentiable, the derivative computed in its local maxima and minima is equal to zero

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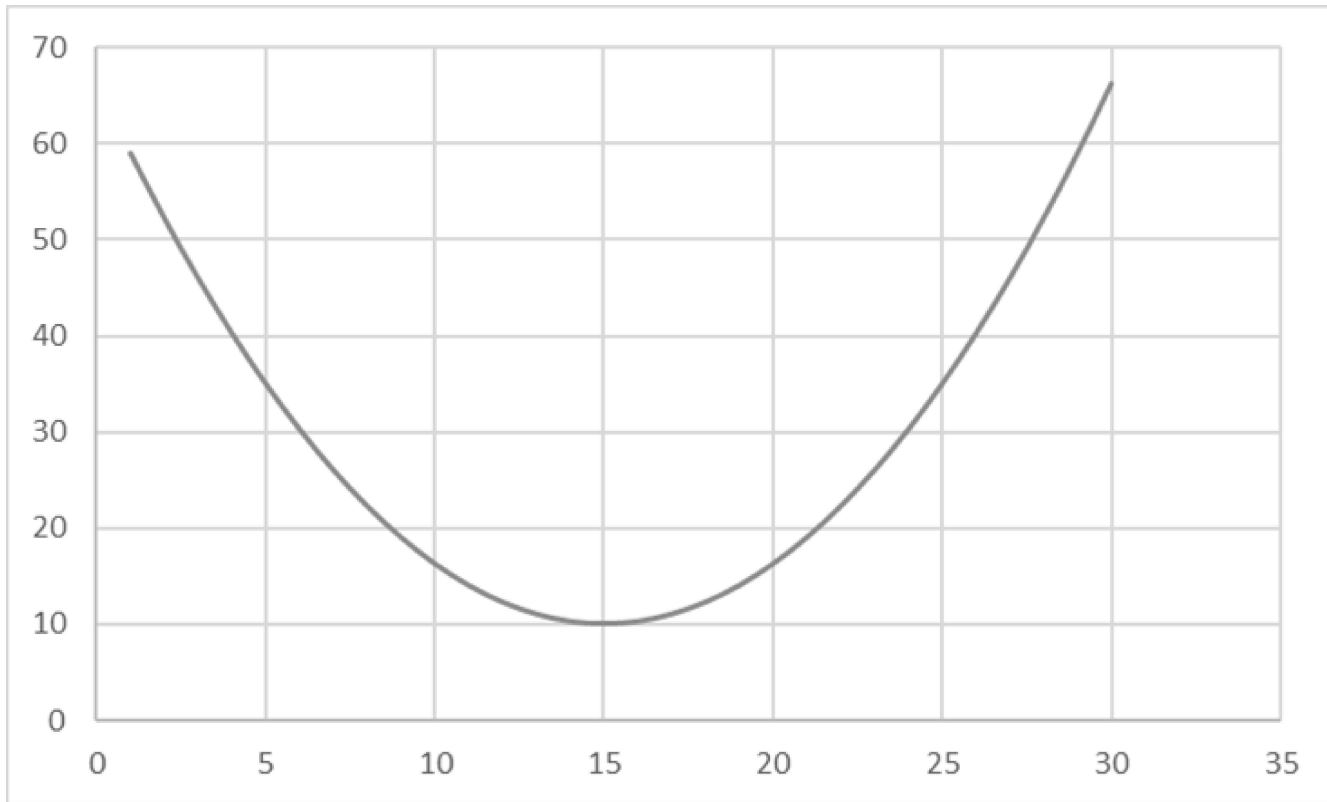
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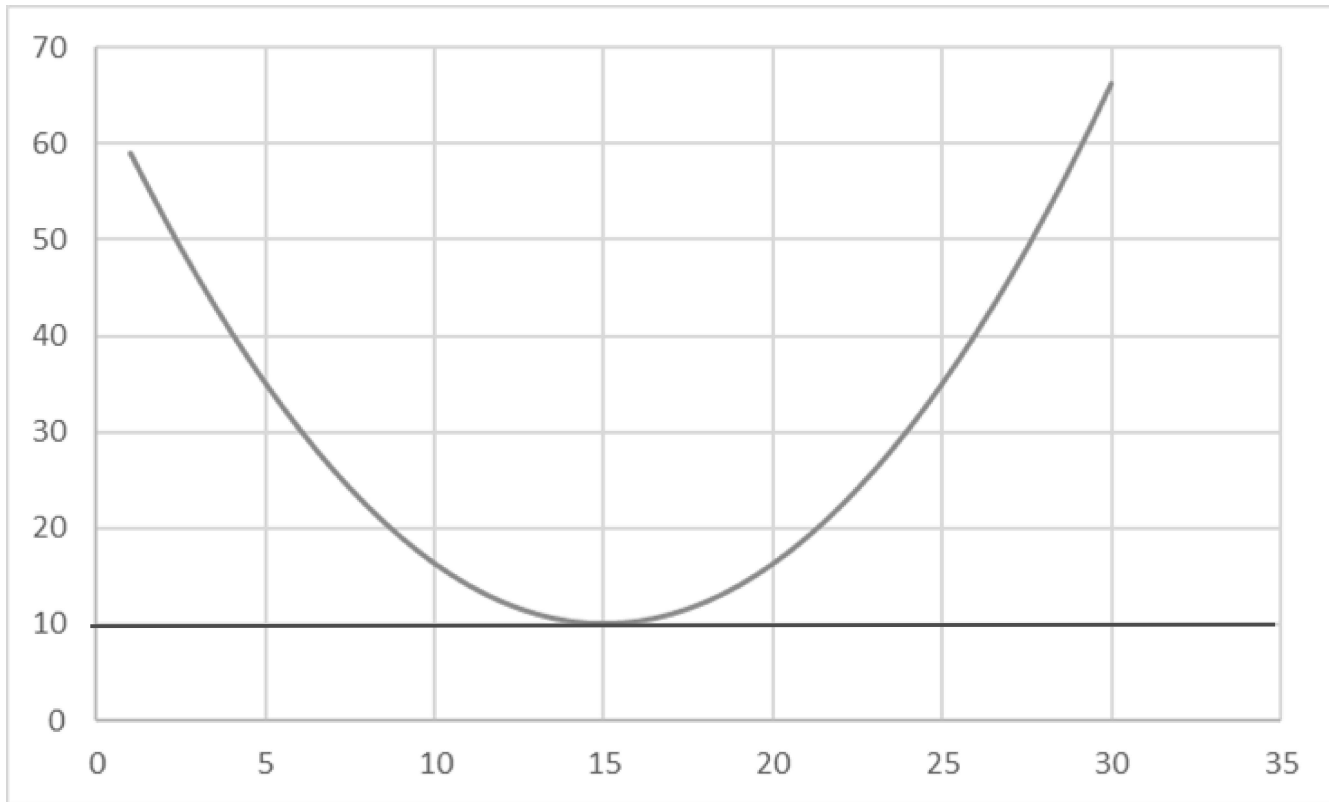
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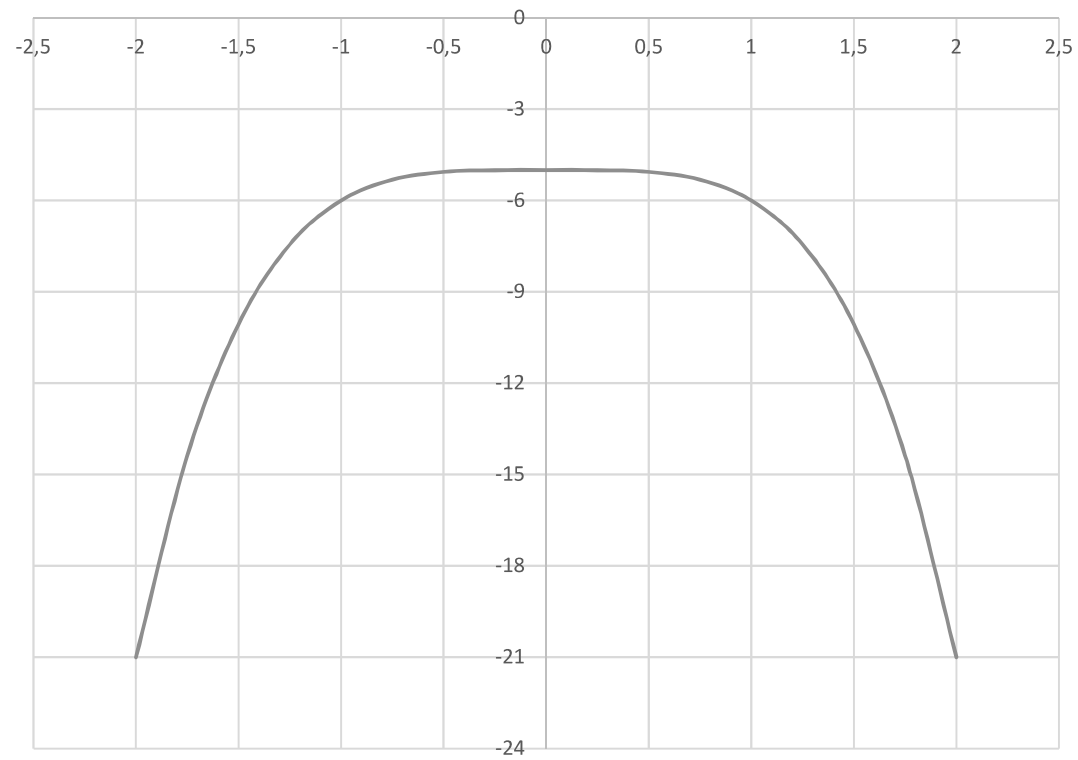
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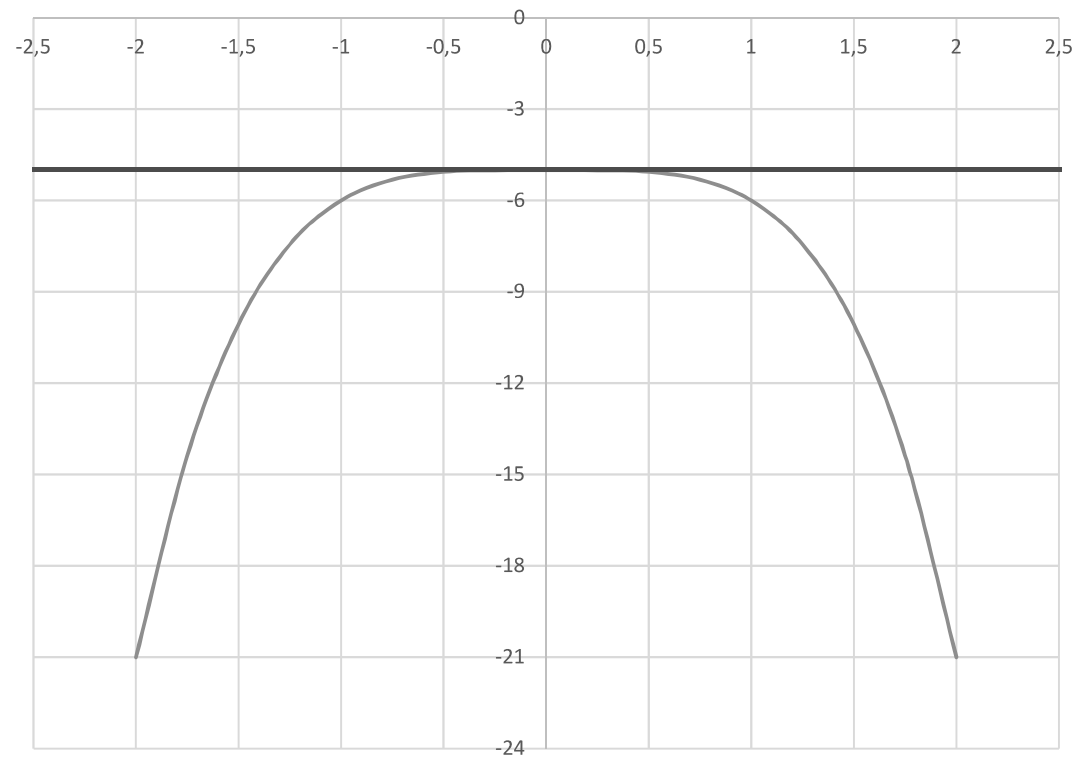
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Does the converse hold?

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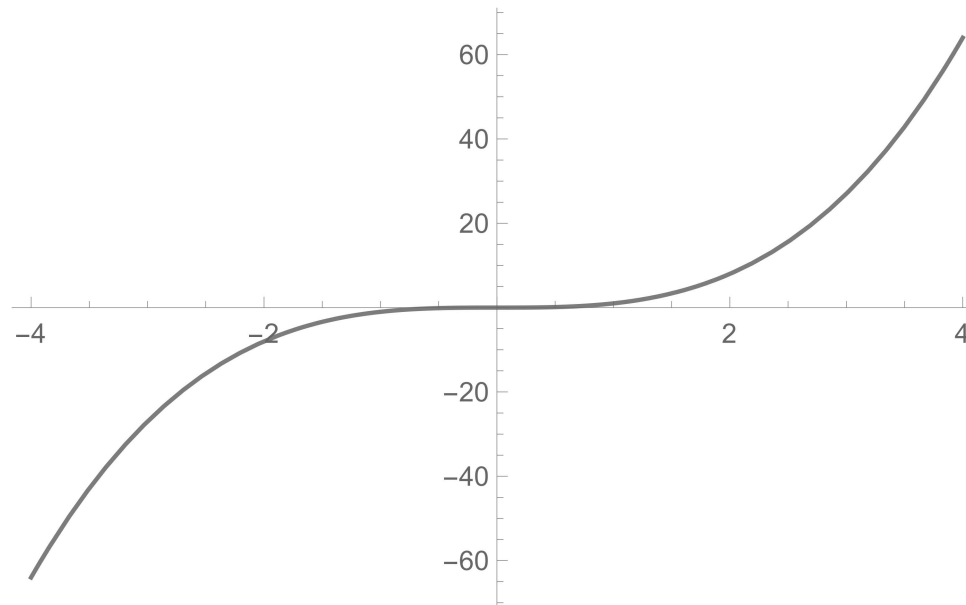
Does the converse hold? Namely, if a function $f : D \rightarrow \mathbb{R}$ is differentiable in $x_0 \in D$ and if $f'(x_0) = 0$, can we conclude that x_0 is a local maximum or a local minimum?

Fermat's theorem, cont'd

NO!

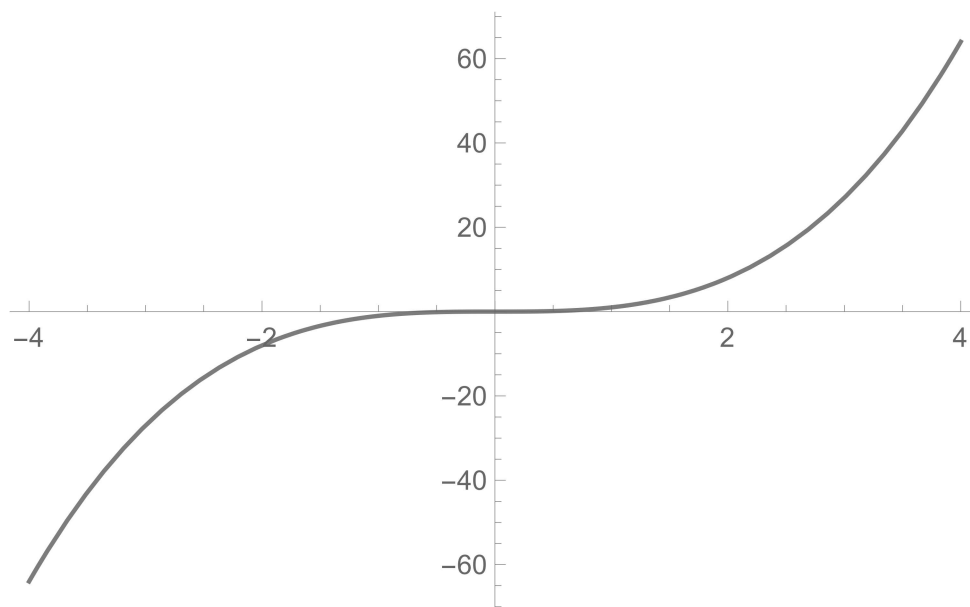
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Fermat's theorem, cont'd

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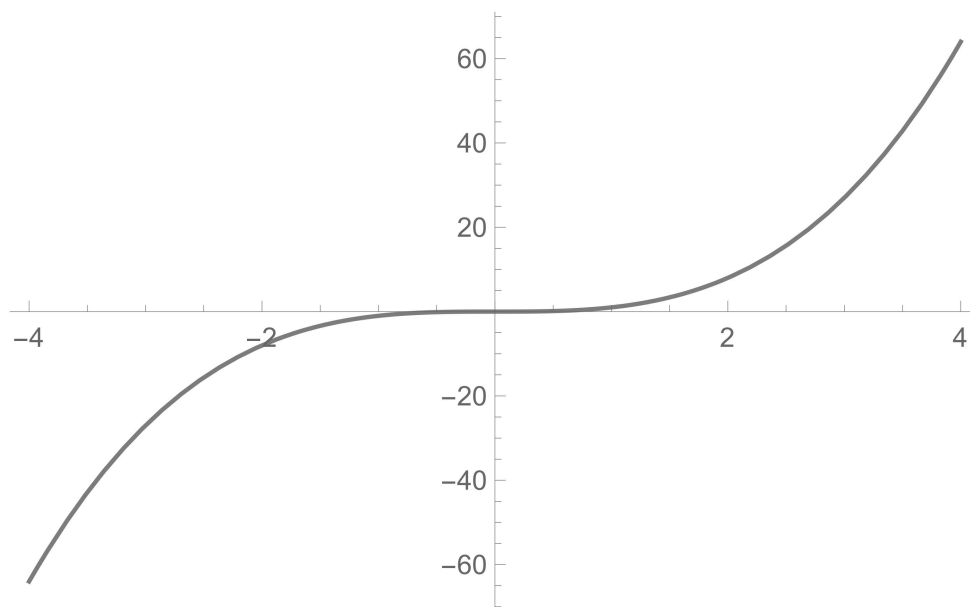


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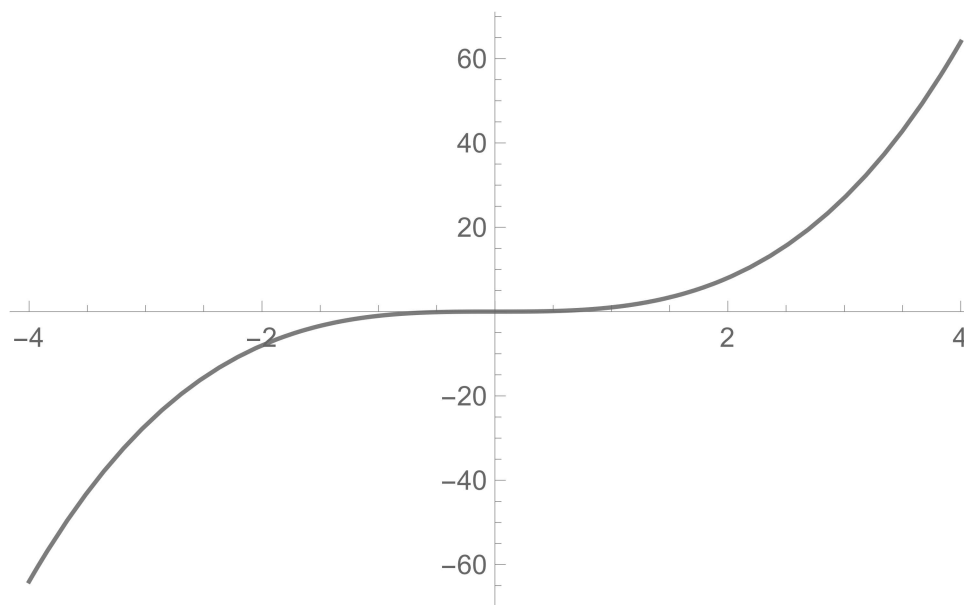


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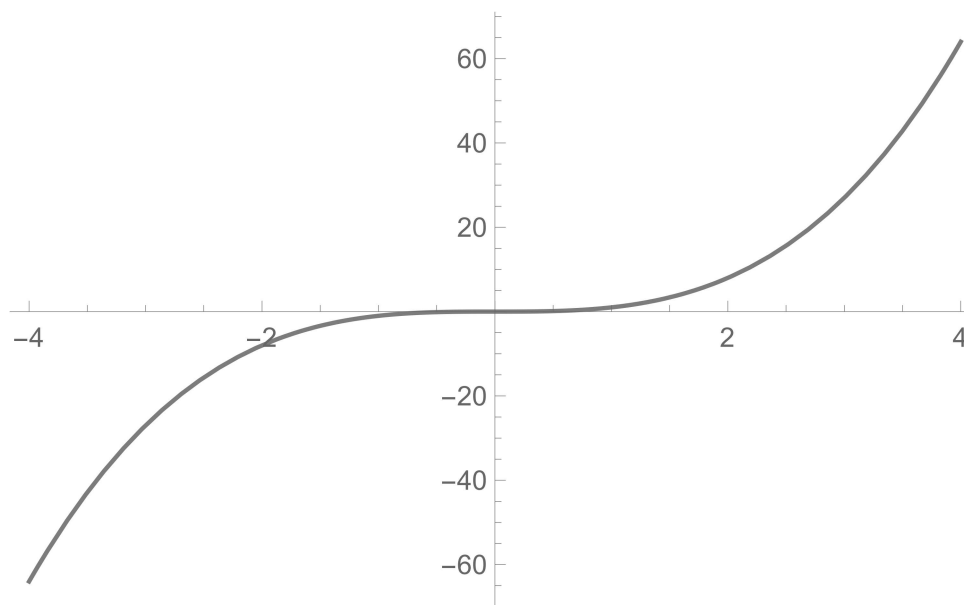
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However, $x = 0$ is NOT a local maximum/minimum.

Fermat's theorem, cont'd

NO! Indeed, consider the function $f(x) = x^3$:



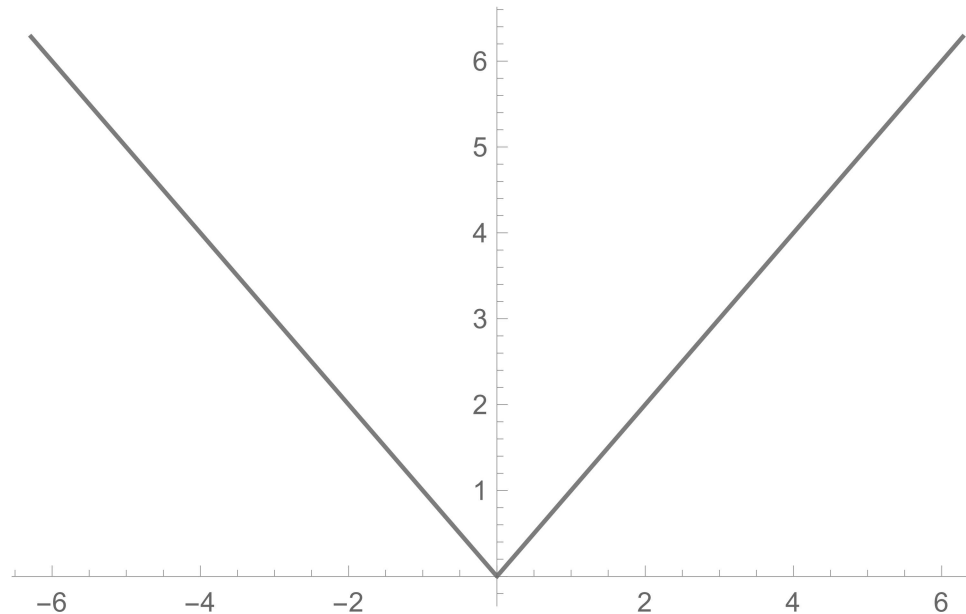
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However, $x = 0$ is NOT a local maximum/minimum. It is an **inflection point with an horizontal tangent**.

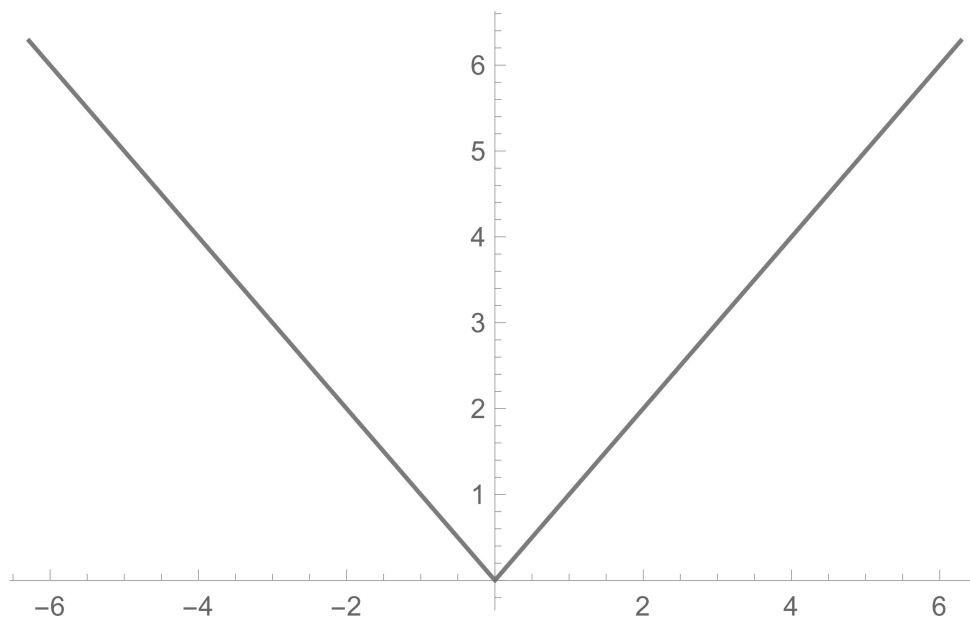
Fermat's theorem, cont'd

What if the function is not differentiable?



Fermat's theorem, cont'd

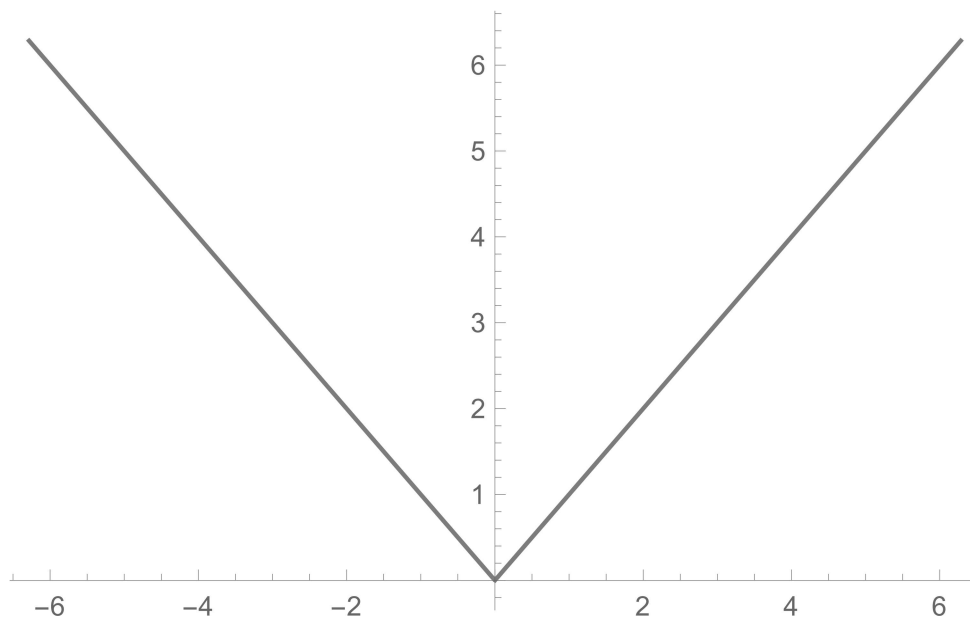
What if the function is not differentiable?



The function is not differentiable in $x = 0$, i.e. $f'(0)$ does not exist.

Fermat's theorem, cont'd

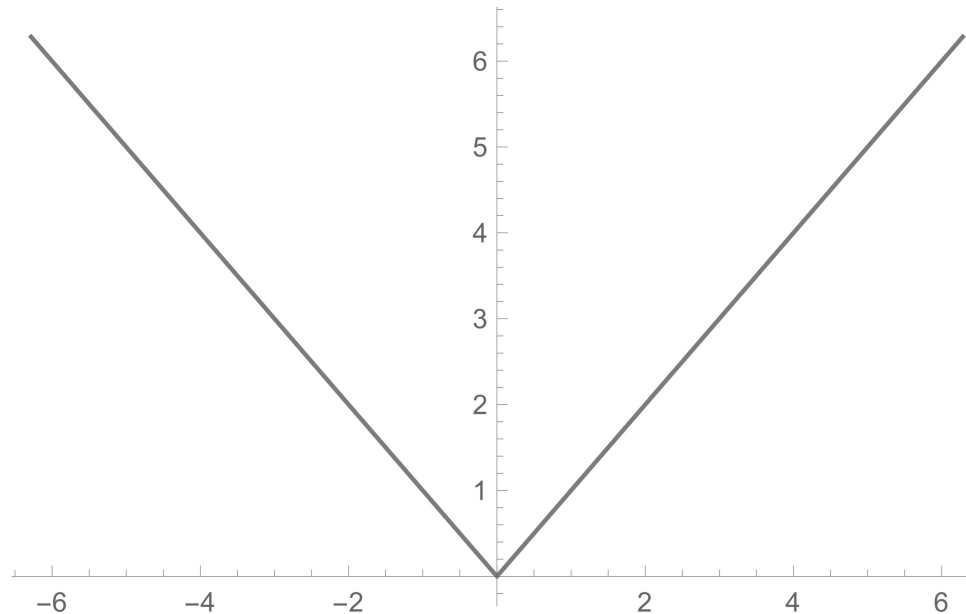
What if the function is not differentiable?



The function is not differentiable in $x = 0$, i.e. $f'(0)$ does not exist. However note that $x = 0$ is a local minimum.

Fermat's theorem, cont'd

What if the function is not differentiable?



The function is not differentiable in $x = 0$, i.e. $f'(0)$ does not exist. However note that $x = 0$ is a local minimum. **We CANNOT use the methods described in this course, which are based on derivatives, to find the local maxima/minima of a non-differentiable function.**

Critical points

Definition

Let $f : D \rightarrow \mathbb{R}$ be differentiable in $x_0 \in D$.

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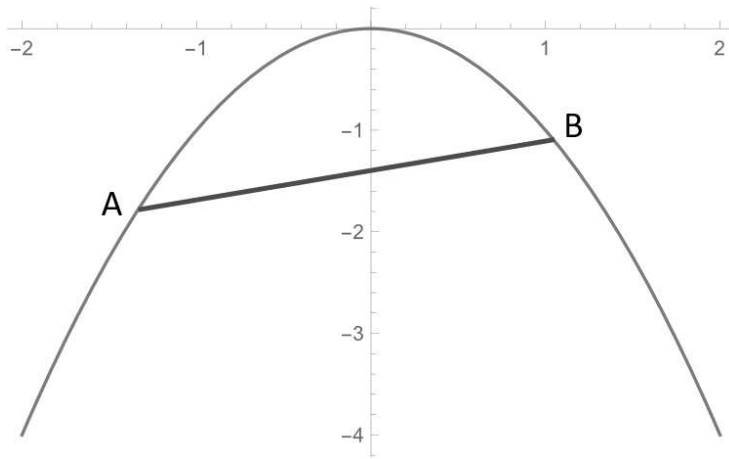
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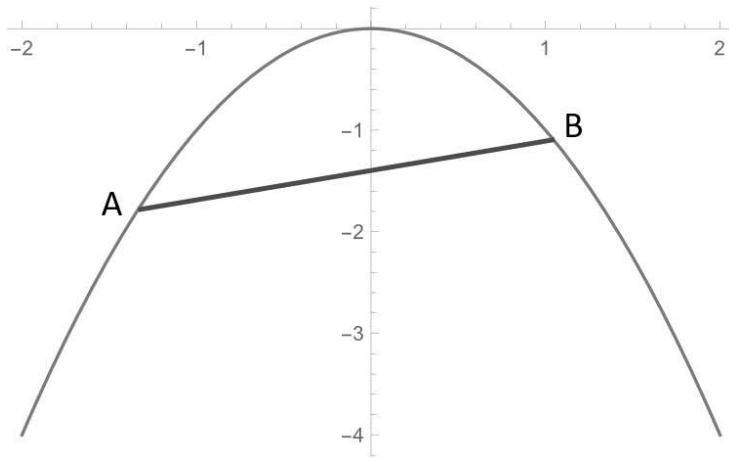
Thus, the condition $f'(x_0) = 0$ is only **necessary but not sufficient** for x_0 to be a local maximum or minimum for f .

In order to understand whether x_0 is a local minimum, a local maximum or an inflection point, we need additional conditions that involve the **second derivative**.

Concavity and Convexity: the intuition

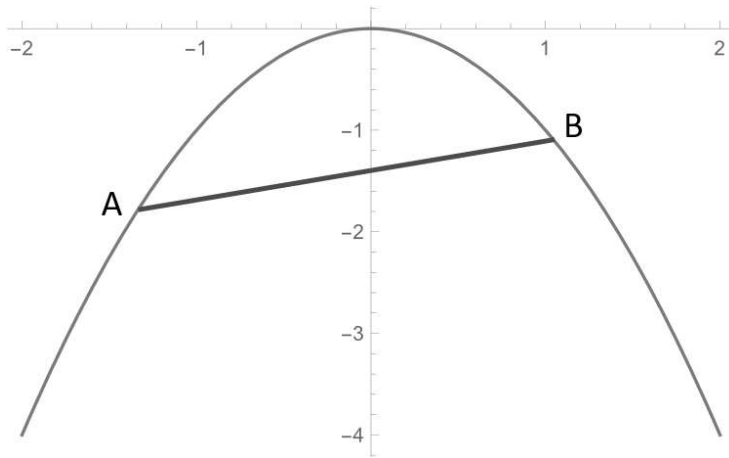


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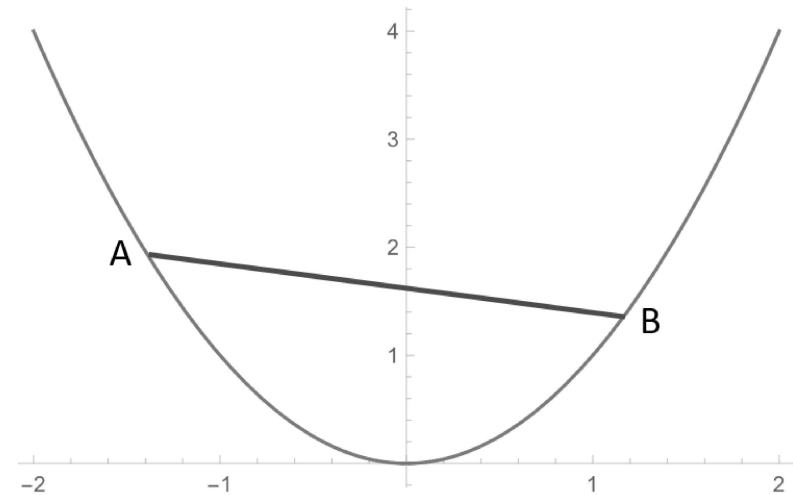


The function $f(x) = -x^2$ is concave because any segment joining two points A and B of the graph is below the function

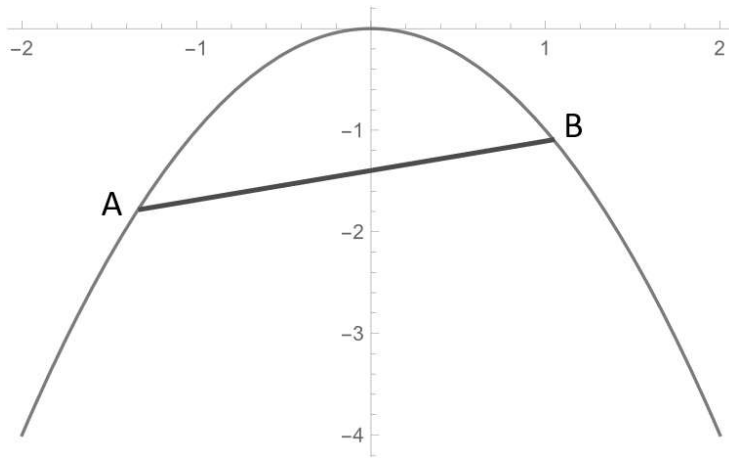
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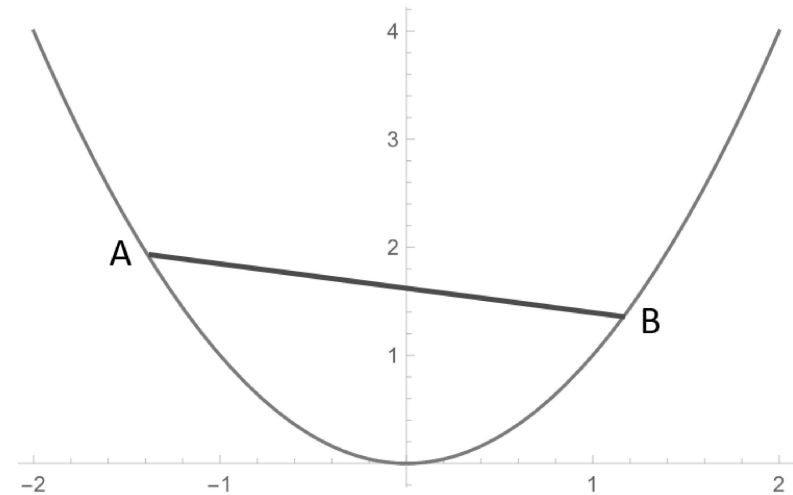
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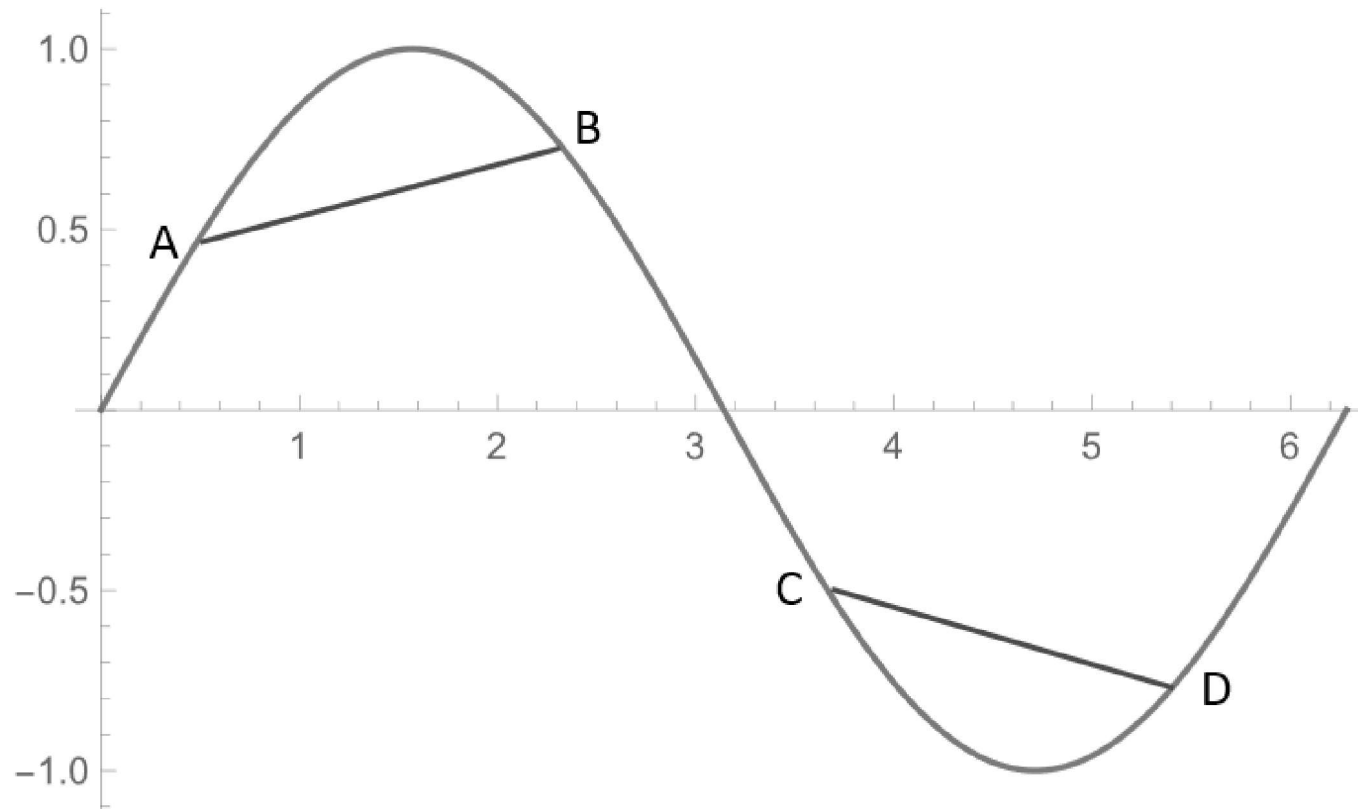


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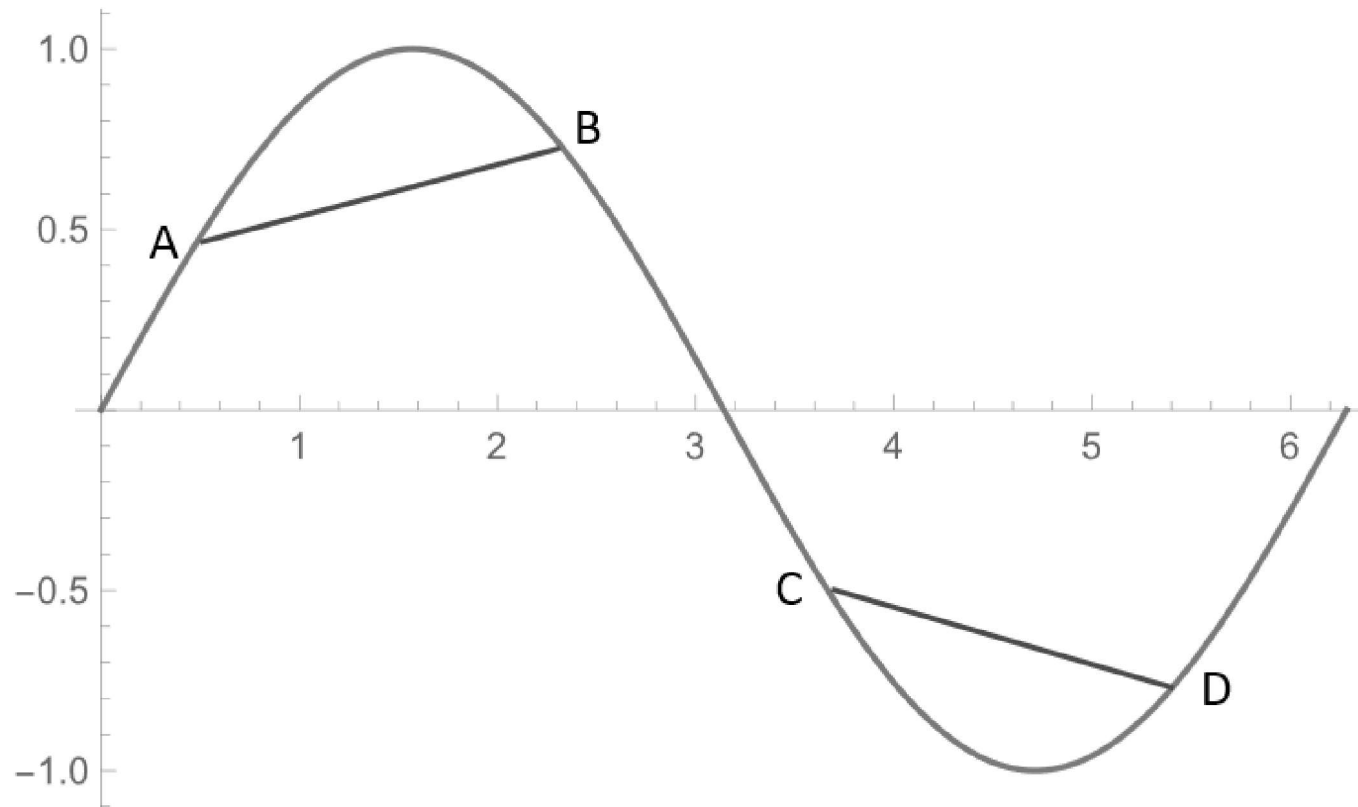


The function $f(x) = x^2$ is convex because any segment joining two points A and B of the graph is above the function

Concavity and Convexity: the intuition, cont'd



Concavity and Convexity: the intuition, cont'd



The function $f(x) = \sin(x)$ is concave in $(0, \pi)$ and convex in $(\pi, 2\pi)$

Concavity and Convexity: the definition

Definition

A function $f : D \rightarrow \mathbb{R}$ is said to be concave in $(a, b) \subseteq D$ if for all $x_1, x_2 \in (a, b)$ the segment from the point $(x_1, f(x_1))$ to the point $(x_2, f(x_2))$ lies below the graph of $f(x)$ in the interval (x_1, x_2) .

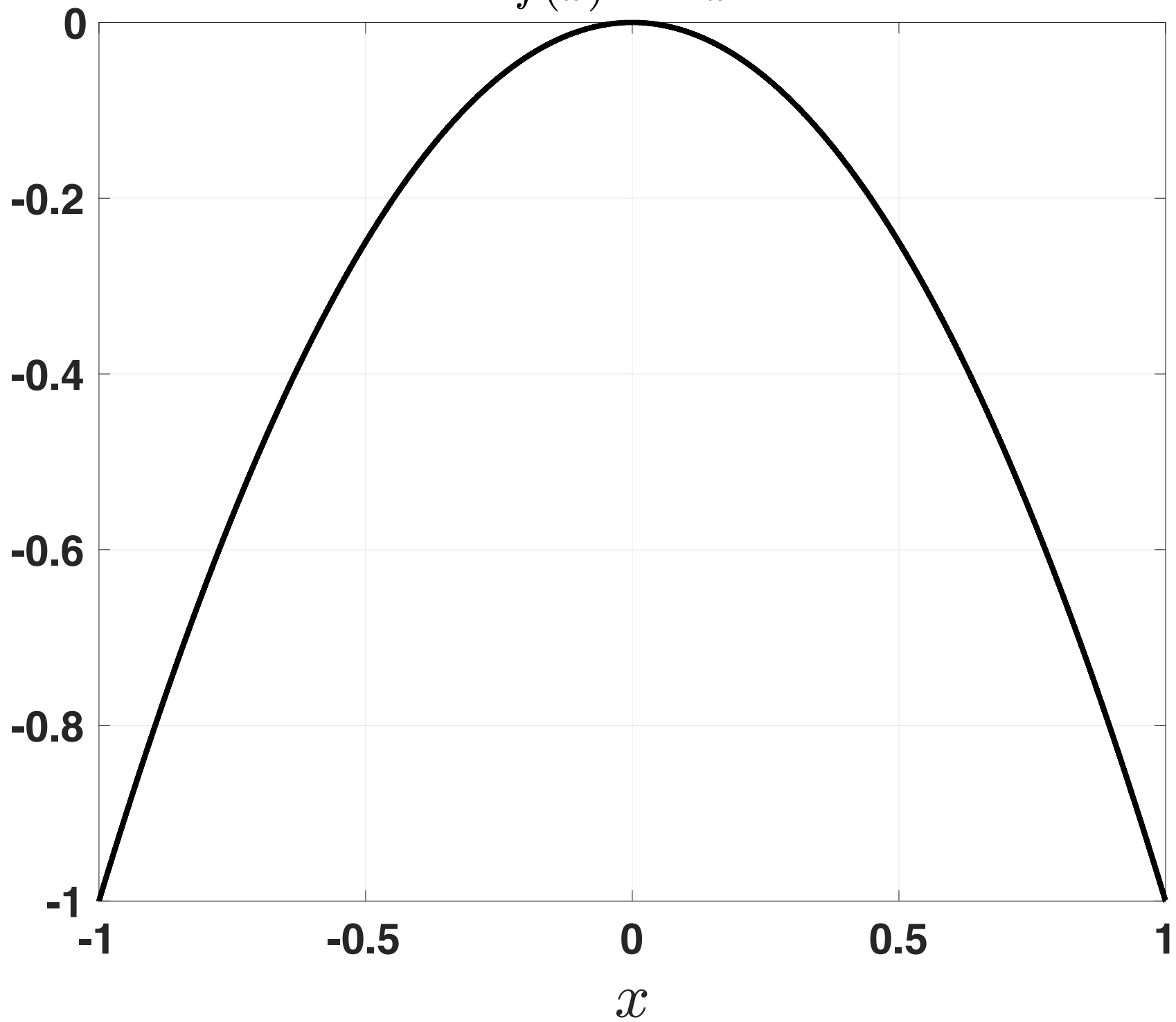
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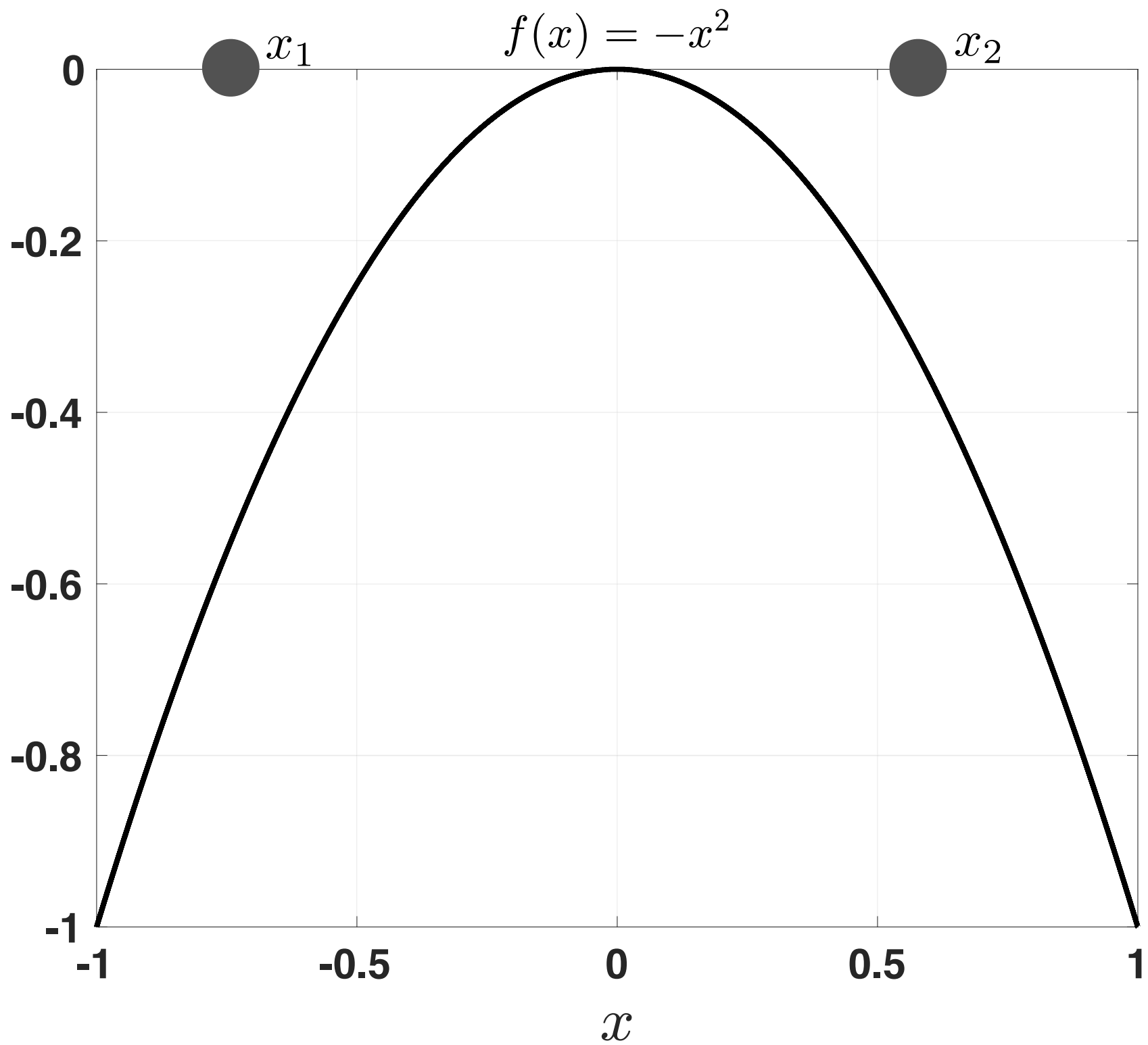
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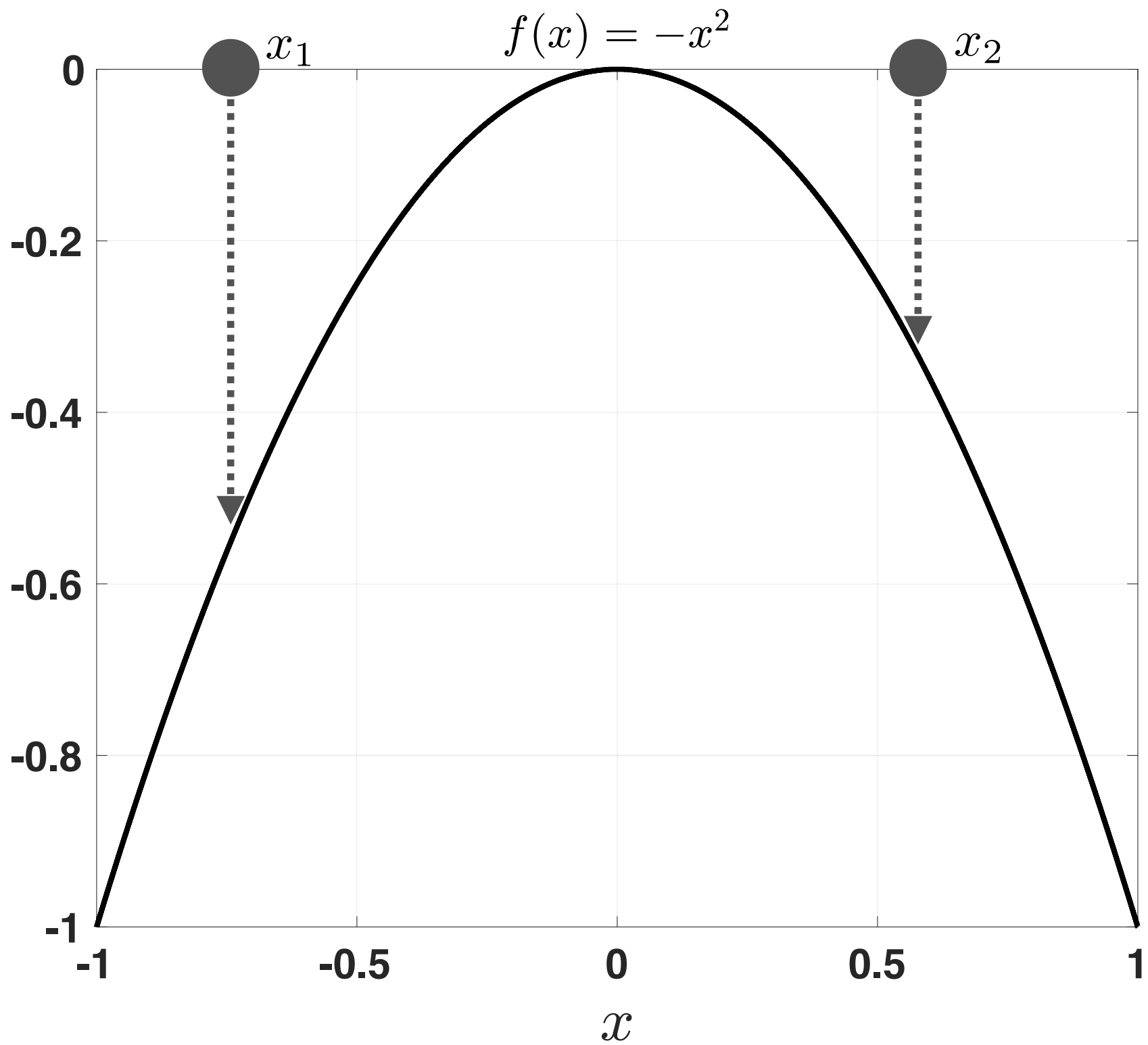
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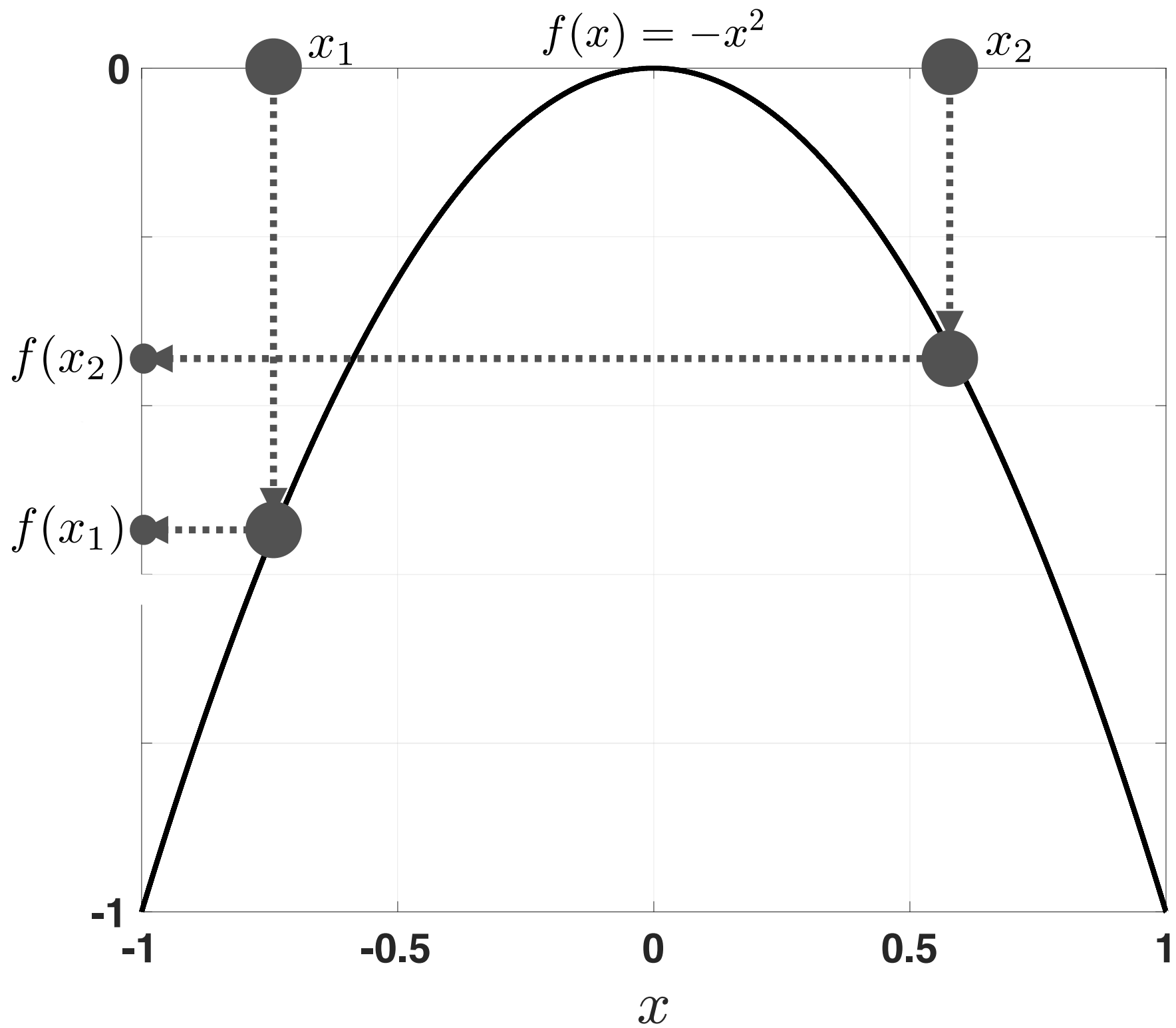
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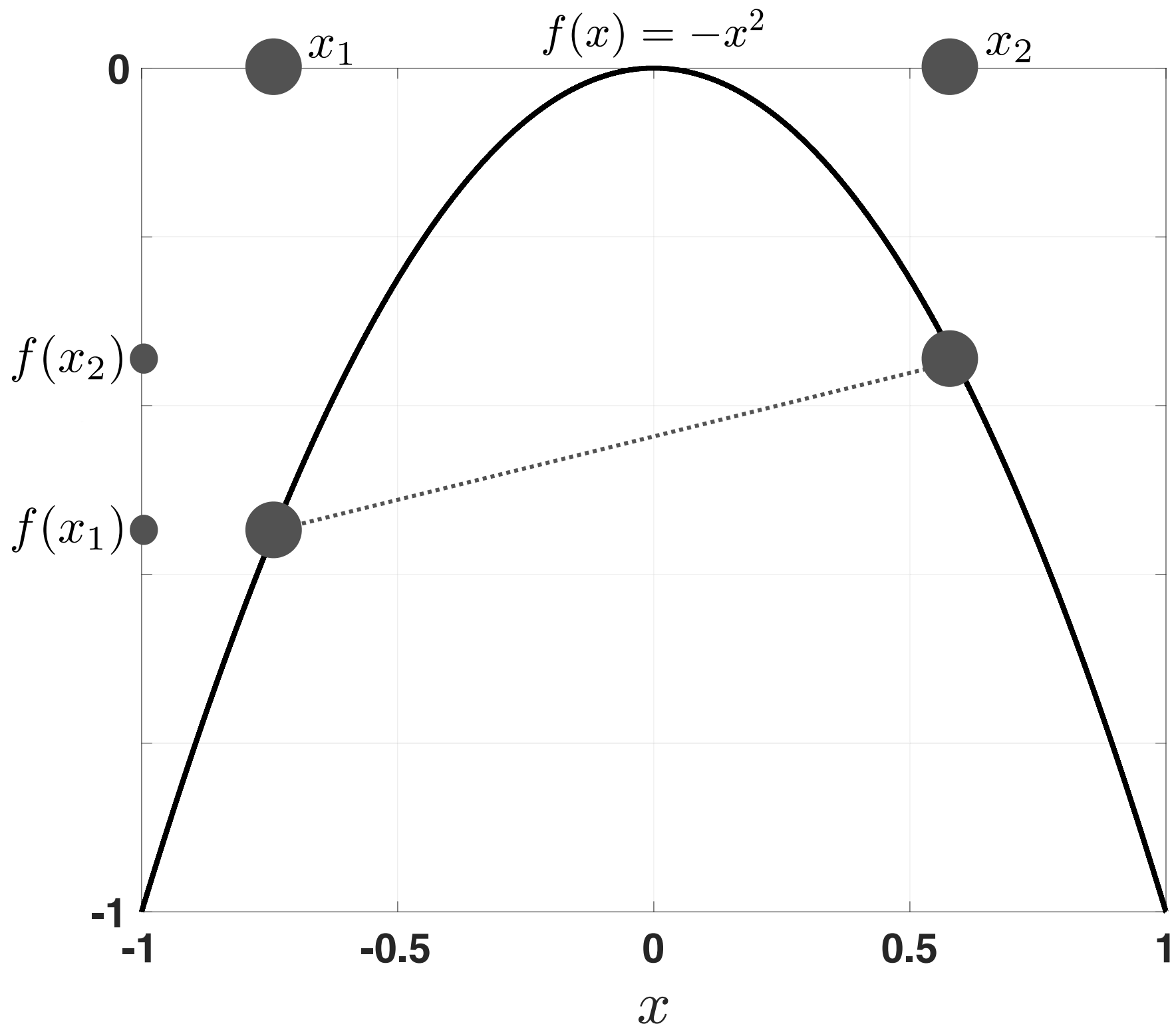
$$f(x) = -x^2$$











The derivative of concave and convex functions

Theorem

Let $f : D \rightarrow \mathbb{R}$ be a function. Assume that f is differentiable on $(a, b) \subseteq D$. Then:

- f is concave on (a, b) if and only if f' is strictly decreasing on (a, b)*

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The have an intuition on this theorem, observe that, if a function is concave, the slope of the line tangent to a point decreases. Instead, if a function is convex, the slope of the line tangent to a point increases.

Higher order derivatives

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Let $f : D \rightarrow \mathbb{R}$ be a differentiable function and let $f'(x)$ denote the derivative of $f(x)$.

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Higher order derivatives, cont'd

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If $f'(x)$ exists, $f''(x)$ is not guaranteed to exist as well. For instance, consider the function:

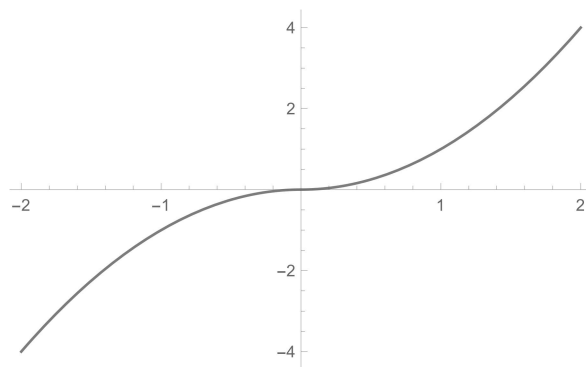
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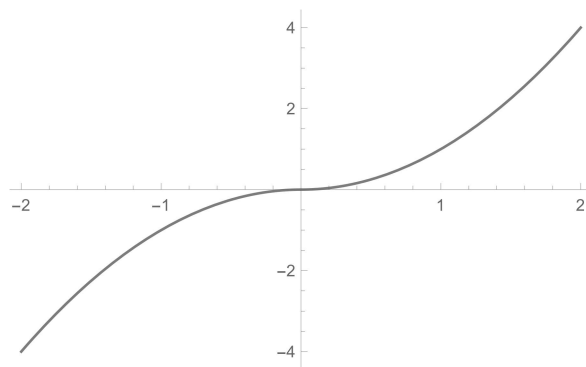
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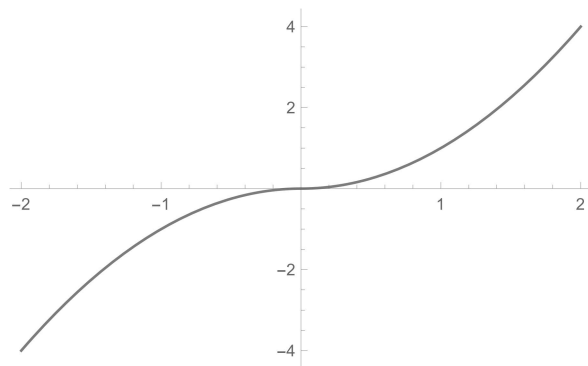


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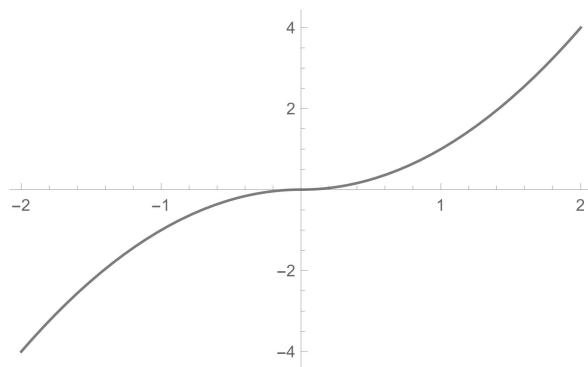


This function is differentiable in $\mathbb{R}$, but in $x = 0$ $f''(x)$ does not exist because the left and right limits of $f''(x)$ are -2 and 2, respectively.

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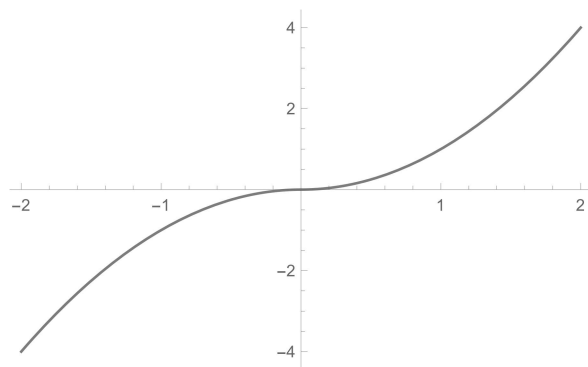


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This function is differentiable in $\mathbb{R}$, but in $x = 0$ $f''(x)$ does not exist because the left and right limits of $f''(x)$ are -2 and 2, respectively. Thus, $f'(x)$ has an angle point in $x = 0$ and is not differentiable.

If $f''(x)$ exists everywhere in the domain of the function, we say that the function is **twice** differentiable.

The derivative of concave and convex functions, cont'd

We have seen that:

The derivative of concave and convex functions, cont'd

We have seen that:

- A differentiable function is concave if and only if

The derivative of concave and convex functions, cont'd

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We have seen that:

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But this implies that:

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But this implies that:

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- A differentiable function is concave if and only if $f'(x)$ is strictly decreasing.
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But this implies that:

- A **twice** differentiable function is concave if and only if $f''(x) < 0$.
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The second derivative test for local maxima and minima

Theorem

Let $f : D \rightarrow \mathbb{R}$ be twice differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$ be such that $f'(x_0) = 0$, that is, x_0 is a critical point. Then:

The second derivative test for local maxima and minima

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- If $f''(x_0) < 0$ then x_0 is a local maximum.*

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- If $f''(x_0) > 0$ then x_0 is a local minimum.*

To have an intuition on this theorem, observe that, if x_0 is a local maximum, the function is concave in a neighborhood of x_0 and therefore $f''(x) < 0$.

The second derivative test for local maxima and minima

Theorem

Let $f : D \rightarrow \mathbb{R}$ be twice differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$ be such that $f'(x_0) = 0$, that is, x_0 is a critical point. Then:

- If $f''(x_0) < 0$ then x_0 is a local maximum.*
- If $f''(x_0) > 0$ then x_0 is a local minimum.*

To have an intuition on this theorem, observe that, if x_0 is a local maximum, the function is concave in a neighborhood of x_0 and therefore $f''(x) < 0$.

Similarly, if x_0 is a local minimum, the function is convex in a neighborhood of x_0 and therefore $f''(x_0) > 0$.

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

Theorem

Let $f : D \rightarrow \mathbb{R}$ be three times differentiable on $(a, b) \subseteq D$

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

Theorem

Let $f : D \rightarrow \mathbb{R}$ be three times differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 is an inflection point.

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What if $f''(x) = 0$?

Theorem

Let $f : D \rightarrow \mathbb{R}$ be three times differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 is an inflection point.

Example: Consider the function $f(x) = x^3$ and note it is three times differentiable with $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$.

The second derivative test for local maxima and minima, cont'd

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Example: Consider the function $f(x) = x^3$ and note it is three times differentiable with $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$. The point $x = 0$ is a critical point because $f'(0) = 0$.

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

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Let $f : D \rightarrow \mathbb{R}$ be three times differentiable on $(a, b) \subseteq D$ and let $x_0 \in (a, b)$. If $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, then x_0 is an inflection point.

Example: Consider the function $f(x) = x^3$ and note it is three times differentiable with $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$. The point $x = 0$ is a critical point because $f'(0) = 0$. Moreover, $f''(0) = 0$.

The second derivative test for local maxima and minima, cont'd

What if $f''(x) = 0$?

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The second derivative test for local maxima and minima: exercises

Find the local maxima and minima of the following functions. Determine also in which intervals the function is convex and/or concave.

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- ⑩ $f(x) = \log(1 + \log(x)) - \log(x)$ (difficult)

Derivative of the inverse function

Theorem

Let $f : X \rightarrow Y$ be differentiable in X .

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Let $f : X \rightarrow Y$ be differentiable in X . Assume f is invertible and call $f^{(-1)} : Y \rightarrow X$ the inverse function. Then $f^{(-1)}$ is differentiable in Y and

$$[f^{(-1)}]'(y) = \frac{1}{f'(f^{(-1)}(y))}$$

for all $y \in Y$ such that $f'(f^{(-1)}(y)) \neq 0$.

Derivative of the inverse function: an example

The function $\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1]$ is injective and surjective and therefore it can be inverted. The inverse is called “arcsin”:

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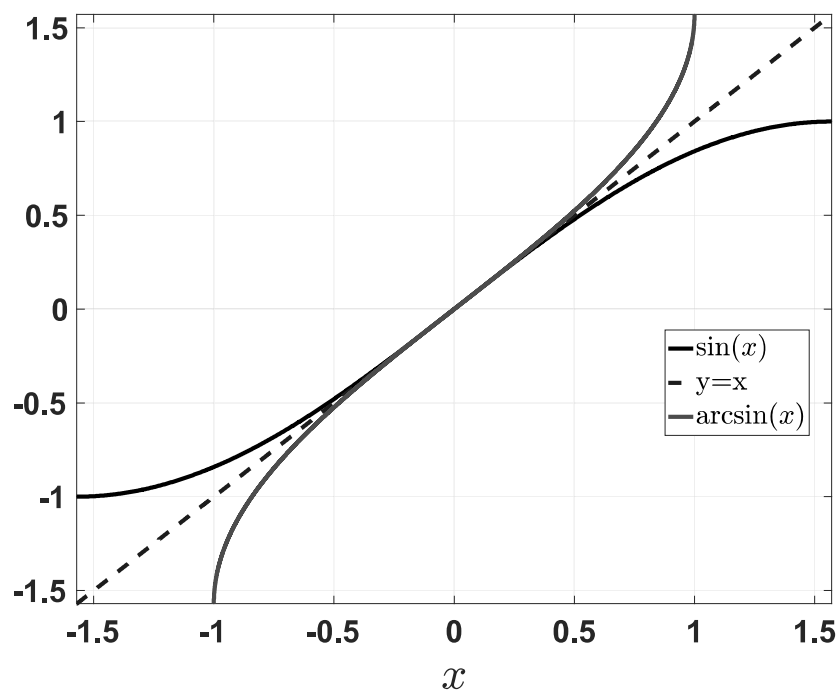
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Then we have:

$$\sin(0) = 0 \Rightarrow \arcsin(0) = 0$$

$$\sin\left(\frac{\pi}{2}\right) = 1 \Rightarrow \arcsin(1) = \frac{\pi}{2}$$

$$\sin\left(-\frac{\pi}{2}\right) = -1 \Rightarrow \arcsin(-1) = -\frac{\pi}{2}$$

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \Rightarrow \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

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De L'Hôpital rule

Theorem

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$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0 \quad \text{AND} \quad \exists \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = L,$$

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The point x_0 can be either finite or $\pm\infty$.

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Compute the following limit:

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We have an indeterminate form $0 \times (-\infty)$. However, applying the De L'Hôpital rule we have:

$$\begin{aligned} \lim_{x \rightarrow 0^+} x \log(x) &= \lim_{x \rightarrow 0^+} \frac{\ln(x)}{\frac{1}{x}} \\ &= \lim_{x \rightarrow 0^+} \frac{[\ln(x)]'}{\left[\frac{1}{x}\right]'} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x \rightarrow 0^+} -\frac{x^2}{x} = \lim_{x \rightarrow 0^+} -x = 0 \end{aligned}$$

De L'Hôpital rule

Compute the following limit:

$$\lim_{x \rightarrow +\infty} \frac{\ln(x)}{\sqrt{x}}$$

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{\ln(x)}{\sqrt{x}} &= \lim_{x \rightarrow +\infty} \frac{[\ln(x)]'}{[\sqrt{x}]'} \\ &= \lim_{x \rightarrow +\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} \\ &= \lim_{x \rightarrow +\infty} \frac{2\sqrt{x}}{x} = \lim_{x \rightarrow +\infty} \frac{2}{\sqrt{x}} = 0 \end{aligned}$$

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and thus:

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\arctan x - \frac{\pi}{2} \right) e^x &= 0 \times (+\infty) = \lim_{x \rightarrow \infty} \left(\frac{\arctan x - \frac{\pi}{2}}{e^{-x}} \right) . \\ &= \frac{0}{0} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1}{-e^{-x}} = - \lim_{x \rightarrow \infty} \frac{e^x}{1+x^2} \\ &= \frac{\infty}{\infty} \stackrel{H}{=} - \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \frac{\infty}{\infty} \stackrel{H}{=} - \lim_{x \rightarrow \infty} \frac{e^x}{2} = -\infty \end{aligned}$$

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$$\lim_{x \rightarrow +\infty} \frac{x + \sin(x)}{x} =$$

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