FATTORIALE: $\forall\, n \in \mathbb{N}$ DEFINISCO

$$0! \equiv 1$$

$$n! = n\cdot(n-1)\cdot(n-2)\cdots 1$$

$2! = 2\cdot 1 = 2$

$3! = 3\cdot 2\cdot 1 = 6$

$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$\binom{n}{k}$ = COEFFICIENTE BINOMIALE = "n SU k" $= \dfrac{n!}{k!\,(n-k)!}$

$n \in \mathbb{N}$ $\boxed{n \geq k}$

$k \in \mathbb{N}$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)\cdot\cancel{(n-k)!}}{k!\ \cancel{(n-k)!}}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

LEMMA: $\forall\, a,b \in \mathbb{R}$, $\forall\, n \in \mathbb{N}$ SI HA CHE

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\cdot b^k \quad (*)$$

DIMOSTRAZIONE: SI DIMOSTRA USANDO IL PRINCIPIO DI INDUZIONE

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$\boxed{n=2}$ APPLICO LA FORMULA $(*)$ E OTTENGO

$$(a+b)^2 = \sum_{k=0}^{2} \binom{2}{k} a^{2-k} b^k = \underbrace{\binom{2}{0} a^2 b^0}_{k=0} + \underbrace{\binom{2}{1} ab}_{k=1} + \underbrace{\binom{2}{2} a^0 b^2}_{k=2}$$

$$= \frac{2!}{\underbrace{0!}_{}\,\underbrace{(2-0)!}_{}} a^2 + \frac{\overset{=2}{2!}}{\underbrace{1!}_{=2}\,\underbrace{(2-1)!}_{=2}} ab + \frac{\overset{=2}{2!}}{\underbrace{2!}_{=2}\,\underbrace{(2-2)!}_{=1}} b^2 = a^2 + 2ab + b^2 = (a+b)^2$$

ESEMPIO: SIA $k \in \mathbb{N}$, $a \in \mathbb{R}$, $a > 1$. VOGLIO CALCOLARE

$$\lim_{n \to +\infty} \frac{n^k}{a^n} = \frac{+\infty}{+\infty}$$

SICCOME $a > 1 \Rightarrow a = 1 + h$, $h > 0$ PER CUI

$$a^n = (1+h)^n = \sum_{q=0}^{n} \binom{n}{k} h^q = \binom{n}{0} h^0 + \underbrace{\binom{n}{1}}_{\geq 0} h + \underbrace{\binom{n}{2}}_{\geq 0} h^2 + \dots + \underbrace{\binom{n}{k+1}}_{\geq 0} h^{k+1} + \dots + \underbrace{\binom{n}{n}}_{\geq 0} h^n$$

$$\geq \binom{n}{k+1} h^{k+1} = \frac{n!}{(k+1)!\,(n-(k+1))!} \cdot h^{k+1} = \frac{n \cdot (n-1) \cdots (n-k)\cancel{(n-k-1)!}}{(k+1)!\,\cancel{(n-k-1)!}} h^{k+1}$$

$$= \boxed{\frac{n(n-1)\cdots(n-k)}{(k+1)!} \cdot h^{k+1}}$$

RIASSUMENDO: $$a^n \geq \frac{n(n-1)\cdots(n-k)}{(k+1)!} \cdot h^{k+1} \Rightarrow$$

$$\frac{1}{a^n} \leq \frac{(k+1)!}{h^{k+1}\; n \cdot (n-1) \cdots (n-k)}$$

$$0 \leq \frac{n^k}{a^n} \leq \frac{(k+1)!}{h^{k+1}} \frac{n^k}{n \cdot (n-1) \cdots (n-k)}$$

$$= \frac{(k+1)!}{h^{k+1}} \frac{\overbrace{n \cdots n}^{k \text{ FATTORI}}}{n \cdot \underbrace{(n-1) \cdots (n-k)}_{k \text{ FATTORI}}}$$

$$= \frac{(k+1)!}{h^{k+1}} \cdot \frac{1}{n} \cdot \frac{1}{\underbrace{\left(1-\frac{1}{n}\right)}_{\to 1} \cdots \underbrace{\left(1-\frac{k}{n}\right)}_{\to 1}} \longrightarrow 0$$

$$\frac{1}{n} \to 0$$

PER IL TEOREMA DEL CONFRONTO SI HA CHE

$$\lim_{n\to+\infty} \frac{n^k}{a^n} = 0 \qquad \forall k \in \mathbb{N} \text{ FISSATO}, \quad \forall a > 1$$

$$\lim_{n\to+\infty} \frac{n^{1000000000}}{(1{,}0000001)^n} = 0$$

ESERCIZIO: $\displaystyle\lim_{n\to+\infty} n^{\frac{1}{n}} = (+\infty)^0 = ??$

$$\lim_{n\to+\infty} n^{\frac{1}{\log_2(n)}} = (+\infty)^0 = 2 \qquad \Bigg| \qquad n^{\frac{1}{\log_2(n)}} = 2^{\log_2\left(n^{\frac{1}{\log_2(n)}}\right)} = 2^{\frac{1}{\log_2(n)}\cdot\log_2(n)} = 2$$

$$n^{\frac{1}{n}} \geq 1 \Rightarrow n^{\frac{1}{n}} = 1 + a_n \qquad a_n \geq 0$$

$$\Rightarrow n = (1+a_n)^n = \sum_{k=0}^{n} \binom{n}{k} a_n^k = 1 + \binom{n}{1} a_n + \binom{n}{2} a_n^2 + \ldots$$

$$\geq 1 + \binom{n}{2} a_n^2$$

$$= 1 + \frac{n!}{2!(n-2)!} a_n^2$$

$$= 1 + \underbrace{\frac{n\cdot(n-1)\cancel{(n-2)!}}{2!\,\cancel{(n-2)!}}\,a_n^2}$$

$$n \geq 1 + \frac{n(n-1)}{2}a_n^2 \Rightarrow \cancel{n-1} \geq \frac{n\cancel{(n-1)}}{2}a_n^2$$

$$\Rightarrow \quad \frac{1}{n} \geq \frac{a_n^2}{2} \geq 0 \quad \Rightarrow a_n \longrightarrow 0$$

$$\Rightarrow \boxed{\lim_{n\to+\infty} n^{\frac{1}{n}} = 1}$$

ESERCIZIO: $\displaystyle \lim_{n\to+\infty} \frac{n^n}{n!} = \frac{+\infty}{+\infty} = +\infty$

$$\frac{n^n}{n!} = \frac{n\cdot n\cdots n}{n\cdot(n-1)\cdots 1} = \underset{\substack{\| \\ 1}}{\frac{n}{n}}\cdot\underset{>1}{\frac{n}{n-1}}\cdot\underset{>1}{\frac{n}{n-2}}\cdots\underset{\substack{\| \\ n}}{\frac{n}{1}} > n \to +\infty$$

$$\lim_{n\to+\infty}\frac{\log_a(n)}{n} = \frac{\infty}{\infty} \qquad a>1$$

$$= \lim_{n\to+\infty} \frac{1}{n}\log_a(n) = \lim_{n\to+\infty}\log_a\left(n^{\frac{1}{n}}\right) = 0$$

$$n^{\frac{1}{n}} \longrightarrow 1$$

$$\log_a\left(n^{\frac{1}{n}}\right) \longrightarrow \log_a(1) = 0.$$

ESERCIZIO: $\lim_{n\to+\infty} \frac{a^n}{n!} = \frac{\infty}{\infty}$ $a>1$

SIA $x_n = \frac{a^n}{n!}$. CONSIDERIAMO $\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{(n+1)!}\cdot\frac{n!}{a^n} = \frac{\cancel{a^n}\cdot a}{(n+1)\cdot \cancel{n!}}\cdot\frac{\cancel{n!}}{\cancel{a^n}} = \frac{a}{n+1}$

QUINDI $\boxed{\frac{x_{n+1}}{x_n} = \frac{a}{n+1} \longrightarrow 0}$

$\forall \varepsilon>0 \ \exists \underset{n_\varepsilon}{\underset{\shortparallel}{N}} : \forall n \geq N \Rightarrow 0 < \frac{x_{n+1}}{x_n} < \varepsilon \Rightarrow 0 < x_{n+1} < \varepsilon x_n$

$$0 < x_{n+1} < \varepsilon x_n < \varepsilon\cdot\varepsilon x_{n-2} = \varepsilon^2 x_{n-2} < \varepsilon^2\cdot\varepsilon x_{n-3} = \varepsilon^3 x_{n-3}$$

$$0 < x_{n+1} < \varepsilon^{n-N} x_N = \left(\frac{1}{2}\right)^{n-N} x_N \longrightarrow 0$$

$$\Rightarrow \lim_{n\to+\infty}\frac{a^n}{n!} = 0$$

LA POTENZA DIVERGE PIÙ VELOCEMENTE DEL LOGARITMO

ORDINAMENTO DELLE DIVERGENZE

SIA $a>1$, $b>0$ 1) $\lim_{n\to+\infty}\frac{\log_a(n)}{n^b} = 0$

L'ESPONENZIALE DIVERGE PIÙ VELOCEMENTE DELLA POTENZA

2) $\lim_{n\to+\infty}\frac{n^b}{a^n} = 0$

IL FATTORIALE DIVERGE PIÙ VELOCEMENTE DELL'ESPONENZIALE

3) $\lim_{n\to+\infty}\frac{a^n}{n!} = 0$

$$a)\ \lim_{n \to +\infty} \frac{n!}{n^n} = 0$$

ESEMPIO:

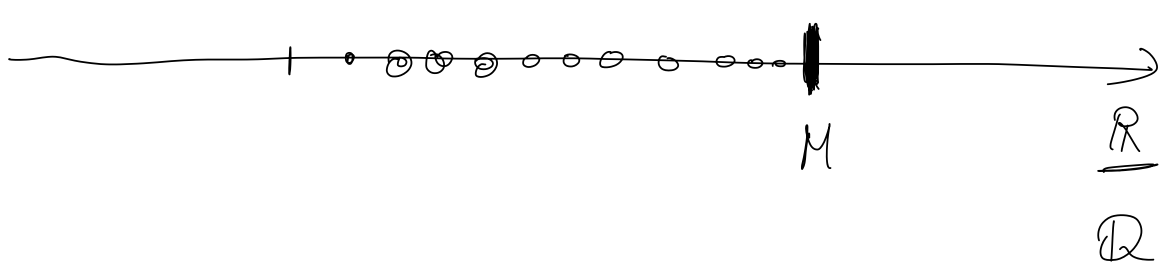
$$\lim_{n \to +\infty} \frac{3^n + \log_2(n)}{n!} = 0$$

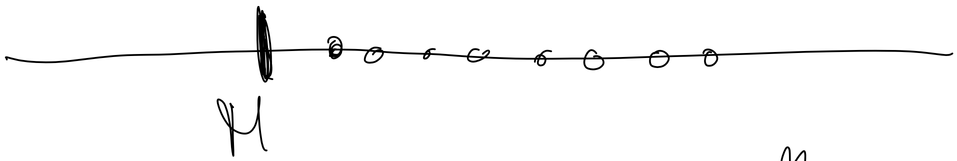
$$\lim_{n \to +\infty} \frac{n^{\frac{1}{100}}}{\log_{20}(n^{100})} = +\infty$$

$$\lim_{n \to +\infty} \frac{n! + n^{2000}}{n^n} = 0$$

TEO: SIA $\{s_n\}$ UNA SUCCESSIONE DI NUMERI NATURALI. ALLORA

- 1) SE s_n È CRESCENTE ALLORA $s_n \to L$ SE E SOLTANTO SE $s_n \leq M \ \forall n$
- 2) SE s_n È DECRESCENTE ALLORA $s_n \to L$ SE E SOLTANTO SE $M \leq s_n \ \forall n$





DEF: LA SUCCESSIONE $S_n = \left(1+\frac{1}{n}\right)^n$ SI CHIAMA SUCCESSIONE DI EULERO

TEO: LA SUCCESSIONE DI EULERO È CRESCENTE

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \frac{1}{n^k} =$$

$$= \sum_{k=0}^{n} \frac{n\cdot(n-1)\cdots(n-k+1)\cancel{(n-k)!}}{k!\,\cancel{(n-k)!}} \frac{1}{n^k} =$$

$$= \sum_{k=0}^{n} \frac{n\cdot\underbrace{(n-1)\cdots(n-k+1)}_{k-1 \text{ FATTORI}}}{n^k} \cdot \frac{1}{k!} =$$

$$= \sum_{k=0}^{n} \frac{n}{n}\,\frac{n-1}{n}\cdots\frac{n-(k-1)}{n}\,\frac{1}{k!} = \sum_{k=0}^{n}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{1}{k!}$$

$$\leq \sum_{k=0}^{n}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)\frac{1}{k!}$$

$$\leq \sum_{k=0}^{n+1}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right)\frac{1}{k!} = \left(1+\frac{1}{n+1}\right)^{n+1} = S_{n+1}$$

$S_n \leq S_{n+1} \Rightarrow S_n$ È CRESCENTE

$$k! = \underset{\geq 2}{k} \cdot \underset{\geq 2}{(k-1)}\underset{\geq 2}{(k-2)} \cdots 1 \geq 2^{k-1} \Rightarrow \boxed{\frac{1}{k!} \leq \frac{1}{2^{k-1}}}$$

$$S_n = 1 + \sum_{k=1}^{n} \underbrace{\left(1-\frac{1}{n}\right)}_{\leq 1} \cdots \underbrace{\left(1-\frac{k-1}{n}\right)}_{\leq 1} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{k!}$$

$$0 \leq S_n \leq 1 + \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k-1} = 1 + \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k} \cdot \left(\frac{1}{2}\right)^{-1} = 1 + 2 \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k}$$

$$\leq 1 + 2 \lim_{n \to +\infty} \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k} = 1 + 2\left(\frac{1}{1-\frac{1}{2}} - 1\right) = 3$$

$$\circ \quad 0 \leq \left(1+\frac{1}{n}\right)^n \leq 3$$

$\circ\circ$ $\left(1+\frac{1}{n}\right)^n$ é CRESCENTE

$$\sum_{k=0}^{+\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}}$$

$$\sum_{k=1}^{+\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} - 1 = 2 - 1 = 1$$

$$\exists \lim_{n \to +\infty} \left(1+\frac{1}{n}\right)^n = e \notin \mathbb{Q} \qquad 0 \leq e \leq 3$$

$$\nexists\, p, q \in \mathbb{N} : e = \frac{p}{q}$$