Indefinite Integral

Suppose you are given a function $g : \mathcal{D} \to \mathbb{R}$ and you want to compute a function F such that

$$F'(x) = g(x), \qquad \forall x \in \mathcal{D}$$

The function F is called an *anti-derivative* of g.

In fact, to compute F we will need to reverse the process of computing a derivative

Indefinite Integral

Remark (The antiderivative is not unique)

Suppose you are given the function

$$g(x) = 2x.$$

An anti-derivative of $g(x)$ is

$$F(x) = x^2$$

Another antiderivative of $g(x)$ is

$$F(x) = x^2 + 5$$

In fact the derivative of any constant is null.

Theorem

Let $g : D \subseteq \mathbb{R} \to \mathbb{R}$ *be a function.*
If $g(x)$ *has an anti-derivative* $F(x)$*, then it has infinitely many anti-derivatives that are given by*

$$F(x) + c$$

Proof. By hypothesis, $F'(x) = g(x)$. We want to show that the derivative of $F(x) + c$ is also $g(x)$. In fact we get that:

$$D[F(x) + c] = F'(x) + 0 = g(x).$$

This concludes the proof.

Definition (Indefinite integral)

Let $g : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. The indefinite integral of g is the class of all antiderivatives of g.

We indicate the integral of g with the symbol

$$\int g(x)\mathrm{d}x.$$

and the function $g(x)$ is called the integrand.

Elementary Indefinite Integral

(i) Let $a \neq -1$.

$$\int x^a \, \mathrm{d}x = \frac{x^{a+1}}{a+1} + c, \qquad \text{for all } c \in \mathbb{R}$$

Proof. We compute the derivative of $\frac{x^{a+1}}{a+1} + c$ and we would like to get x^a.

$$D\left[\frac{x^{a+1}}{a+1} + c\right] = \frac{1}{a+1} D\left[x^{a+1}\right] + 0 = \frac{(a+1)x^a}{a+1} = x^a$$

Elementary Indefinite Integral

(ii) If $a = -1$, then $x^a = \frac{1}{x}$ which is well defined for all $x \neq 0$.

$$\int \frac{1}{x}\,\mathrm{d}x = \lg|x| + c, \qquad \text{for all } c \in \mathbb{R},\ x \neq 0$$

Be careful! You need the ABSOLUTE VALUE of x, since the function $\frac{1}{x}$ is defined for all $x \neq 0$, BUT $\lg x$ is only defined for $x > 0$.

Elementary Indefinite Integral: Examples

1. $$\int x \,\mathrm{d}x = \frac{x^2}{2} + c, \qquad \text{for all } c \in \mathbb{R}$$

2. $$\int x^2 \,\mathrm{d}x = \frac{x^3}{3} + c, \qquad \text{for all } c \in \mathbb{R}$$

3. $$\int \frac{1}{x^4} \,\mathrm{d}x = \int x^{-4} \,\mathrm{d}x = \frac{x^{-4+1}}{-4+1} + c = -\frac{1}{3x^3} + c, \quad \text{for all } c \in \mathbb{R}$$

4. $$\int \sqrt{x} \,\mathrm{d}x = \int x^{\frac{1}{2}} \,\mathrm{d}x = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{2x\sqrt{x}}{3} + c, \quad \text{for all } c \in \mathbb{R}$$

Elementary Indefinite Integral

(iii) $\int e^x \, \mathrm{d}x = e^x + c, \qquad$ for all $c \in \mathbb{R}$.

(iv) $\int e^{ax} \, \mathrm{d}x = \frac{e^{ax}}{a} + c, \qquad$ for all $c \in \mathbb{R}$.

Proof.

$$D\left[\frac{e^{ax}}{a} + c\right] = \frac{1}{a} D[e^{ax}] + 0 = \frac{ae^{ax}}{a} = e^{ax}.$$

(v) Finally, since $a^x = e^{x \lg a}$, for all $a > 0$, we get that

$$\int a^x \, \mathrm{d}x = \int e^{x \lg a} \, \mathrm{d}x = \frac{1}{\lg a} e^{x \lg a} + c = \frac{1}{\lg a} a^x + c, \qquad \text{for all } c \in \mathbb{R}.$$

Elementary Indefinite Integral:Examples

1. $\int e^{5x}\,\mathrm{d}x = \frac{1}{5}e^{5x} + c, \quad \text{for all } c \in \mathbb{R}.$
2. $\int e^{\frac{1}{5}x}\,\mathrm{d}x = 5e^{\frac{1}{5}x} + c, \quad \text{for all } c \in \mathbb{R}.$
3. $\int 7^{x}\,\mathrm{d}x = \frac{1}{\lg 7}7^{x} + c, \quad \text{for all } c \in \mathbb{R}.$

Elementary Indefinite Integral

(vi) $$\int \cos x \, \mathrm{d}x = \sin x + c, \qquad \text{for all } c \in \mathbb{R}.$$

(vii) $$\int \sin x \, \mathrm{d}x = -\cos x + c, \qquad \text{for all } c \in \mathbb{R}.$$

(viii) $$\int \frac{1}{\cos^2 x} \, \mathrm{d}x = \tan x + c, \qquad \text{for all } c \in \mathbb{R}.$$

Proof.

$$D[\tan(x) + c] = \frac{1}{\cos^2 x} + 0$$

(ix) $$\int \frac{1}{1 + x^2} \, \mathrm{d}x = \arctan(x) + c, \qquad \text{for all } c \in \mathbb{R}.$$

(x) $$\int \frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x = \arcsin(x) + c, \qquad \text{for all } c \in \mathbb{R}.$$

Elementary Indefinite Integral: the chain rule

By the chain rule we have that $D[f(g(x))] = f'(g(x)) \cdot f'(x)$. By reversing this rule we can compute the following elementary integrals:

1. For all $a \neq -1$

$$\int \big(g(x)\big)^a g'(x)\ \mathrm{d}x = \frac{\big(g(x)\big)^{a+1}}{a+1} + c, \qquad \text{for all } c \in \mathbb{R}.$$

Proof. Using the chain rule:

$$D\left[\frac{\big(g(x)\big)^{a+1}}{a+1} + c\right] = \frac{1}{a+1} D\left[\big(g(x)\big)^{a+1}\right]$$

$$= \frac{1}{a+1}(a+1)\big(g(x)\big)^a g'(x) = \big(g(x)\big)^a g'(x).$$

Example: $\displaystyle\int \frac{\lg x}{x}\ \mathrm{d}x = \frac{1}{2}\left(\lg(x)\right)^2 + c$, for all $c \in \mathbb{R}$, and $x > 0$.

Elementary Indefinite Integral: the chain rule

2. $\int e^{g(x)} g'(x) \, \mathrm{d}x = e^{g(x)} + c, \qquad$ for all $c \in \mathbb{R}$.

Proof. Using the chain rule:

$$D\left[e^{g(x)} + c\right] = D\left[e^{g(x)}\right] + 0 = e^{g(x)} g'(x)$$

Elementary Indefinite Integral: the chain rule

3. $\displaystyle\int \frac{g'(x)}{g(x)}\,\mathrm{d}x = \ln|g(x)| + c,$ for all $c \in \mathbb{R}$ and $f(x) \neq 0$.

Proof.

CASE 1: $g(x) > 0$. Then $\ln|g(x)| = \ln(g(x))$. By the chain rule,

$$D[\ln(g(x)) + c] = D[\ln(g(x))] + 0 = \frac{g'(x)}{g(x)}.$$

CASE 2: $g(x) < 0$. Then $\ln|g(x)| = \ln(-g(x))$. By the chain rule,

$$D[\ln(-g(x)) + c] = D[\ln(-g(x))] + 0 = \frac{-g'(x)}{-g(x)} = \frac{g'(x)}{g(x)}.$$

Elementary Indefinite Integral:Examples

- $\displaystyle\int \frac{2x+1}{x^2+x}\mathrm{d}x = \lg|x^2+x| + c, \qquad$ for all $c \in \mathbb{R}$, and $x \neq 0, -1$
- $\displaystyle\int 3x^2 e^{x^3}\ \mathrm{d}x = e^{x^3} + c, \qquad$ for all $c \in \mathbb{R}$

Other Elementary integrals

4. $\displaystyle\int g'(x)\sin(g(x))\ \mathrm{d}x = -\cos(g(x)) + c,\ \text{for all } c \in \mathbb{R}$

5. $\displaystyle\int g'(x)\cos(g(x))\ \mathrm{d}x = \sin(g(x)) + c,\ \text{for all } c \in \mathbb{R}$

6. $\displaystyle\int \frac{g'(x)}{\cos^2(g(x))}\ \mathrm{d}x = \tan(g(x)) + c,\ \text{for all } c \in \mathbb{R}$

7. $\displaystyle\int \frac{g'(x)}{1+[g(x)]^2}\ \mathrm{d}x = \arctan(g(x)) + c,\ \text{for all } c \in \mathbb{R}$

8. $\displaystyle\int \frac{g'(x)}{\sqrt{1-[g(x)]^2}}\ \mathrm{d}x = \arcsin(g(x)) + c,\ \text{for all } c \in \mathbb{R}$

Properties of integrals: Linearity

- $\int kf(x)\ \mathrm{d}x = k\int f(x)\ \mathrm{d}x$
- $\int [f(x)+g(x)]\ \mathrm{d}x = \int f(x)\ \mathrm{d}x + \int g(x)\ \mathrm{d}x$
- $\int k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)\ \mathrm{d}x =$
  $k_1 \int f_1(x)\ \mathrm{d}x + k_2 \int f_2(x)\ \mathrm{d}x + \cdots + k_n \int f_n(x)\ \mathrm{d}x$

Properties of integrals: examples

- $\int 3x^4 + 5x^2 + 2 \, \mathrm{d}x = \frac{3}{5}x^5 + \frac{5}{3}x^3 + 2x + c, \quad \text{for all } c \in \mathbb{R}$
- $\int \frac{2}{x} + \frac{4}{\sqrt[3]{x}} + 6e^{2x} \, \mathrm{d}x = 2 \lg |x| + \frac{3}{2}\sqrt[3]{x^2} + 3e^{2x} + c,$

  for all $c \in \mathbb{R}$ and $x \neq 0$

Integration by substitution

We want to compute:

$$\int g(x)\ \mathrm{d}x$$

5 step procedure!

1. Change of variable: Define $x = h(t)$ for a suitable invertible function h
2. Differentiate: By differentiation we get $\mathrm{d}x = h'(t)\mathrm{d}t$
3. Substitute: Substitute x and $\mathrm{d}x$ $\int g(h(t))h'(t)\mathrm{d}t$
4. Solve the integral with variable t: (perhaps simpler!)

$$\int g(h(t))\ h'(t)\ \mathrm{d}t = F(t) + c$$

5. Come back to x: $\int g(x)\ \mathrm{d}x = F(h^{-1}(x)) + c$

Examples

Solve the following integrals:

1. $\int \frac{\lg \sqrt{x}}{x} \mathrm{d}x$ try with $\sqrt{x} = t$
2. $\int \frac{1}{x + \sqrt{x}} \mathrm{d}x$ try with $\sqrt{x} = t$
3. $\int \frac{1}{e^x + e^{-x}} \mathrm{d}x$ try with $e^x = t$

Integration by parts

Recall the product rule:

$$D[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Suppose you want to compute

$$\int f'(x)g(x)\ \mathrm{d}x$$

By the product rule we get

$$\int f'(x)g(x)\ \mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\ \mathrm{d}x$$

This formula is called Integration by parts rule

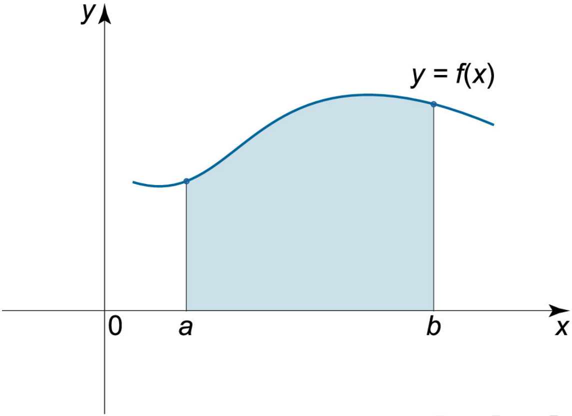
Examples

Compute the following integrals:

1. $$\int x \sin x \, \mathrm{d}x$$
2. $$\int x e^{2x} \, \mathrm{d}x$$
3. $$\int \lg x \, \mathrm{d}x$$

Area under a curve

Suppose you want to compute the area bounded by the graph of the positive and continuous function $f(x)$ and the x-axis for $a \leqslant x \leqslant b$:



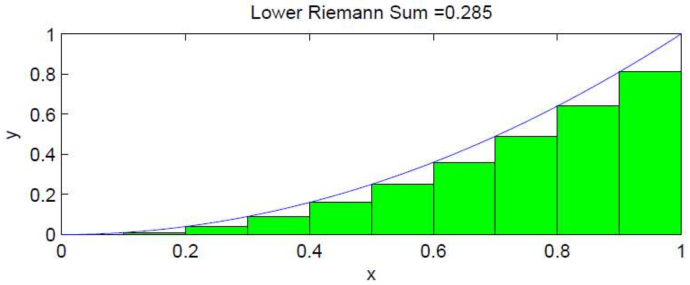
Area under a curve

- Divide the interval $[a, b]$ into n sub-intervals, by choosing points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

  Call the partition P
- Let $\Delta^{(i)} = x_i - x_{i-1}$ be the length of each sub-interval.
- Let m_i be the minimum value of the function f in the interval $[x_{i-1}, x_i]$ That is, there is $c_i \in [x_{i-1}, x_i]$ such that $m_i = f(c_i)$
- Draw, for each interval $[x_{i-1}, x_i]$ a rectangle with dimensions $\Delta^{(i)}, m_i$
- By summing up the area of each single rectangle we approximate from below the area under the curve
- That is

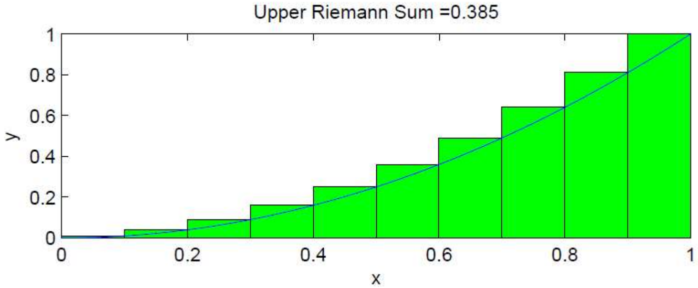
$$\sum_{i=0}^{n-1} m_i \Delta^{(i)} = \sum_{i=0}^{n-1} f(c_i) \Delta^{(i)} < A$$

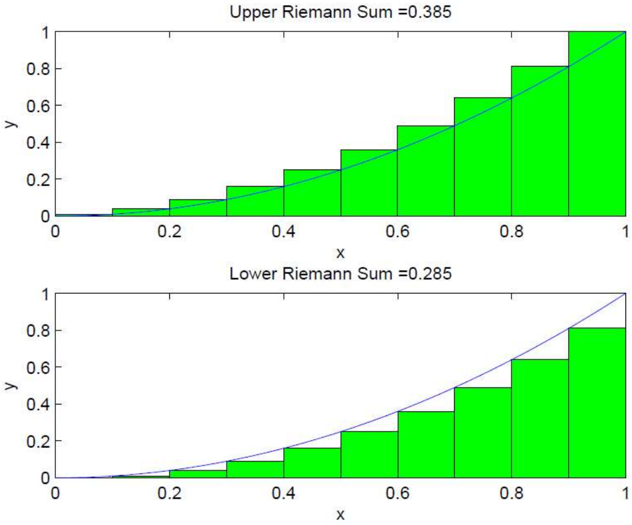

Lower Riemann Sum =0.285
1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x

Area under a curve

- Now choose the maximum value M_i assumed by the function f in the interval $[x_{i-1}, x_i]$ That is, there is $d_i \in [x_{i-1}, x_i]$ such that $M_i = f(d_i)$
- Draw, for every interval $[x_{i-1}, x_i]$ a rectangle with dimensions Δ^i and M_i
- By summing up the area of each single rectangle approximate from above the area under the curve
- That is

$$\sum_{i=0}^{n-1} M_i \Delta^{(i)} = \sum_{i=0}^{n-1} f(d_i)\Delta^{(i)} > A$$

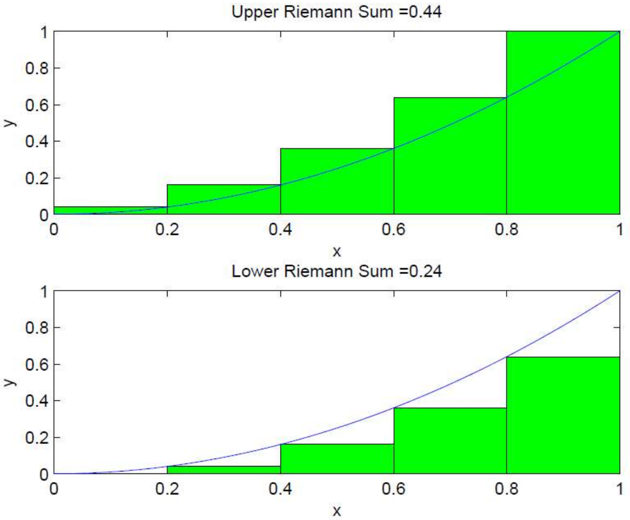

Upper Riemann Sum =0.385
1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x

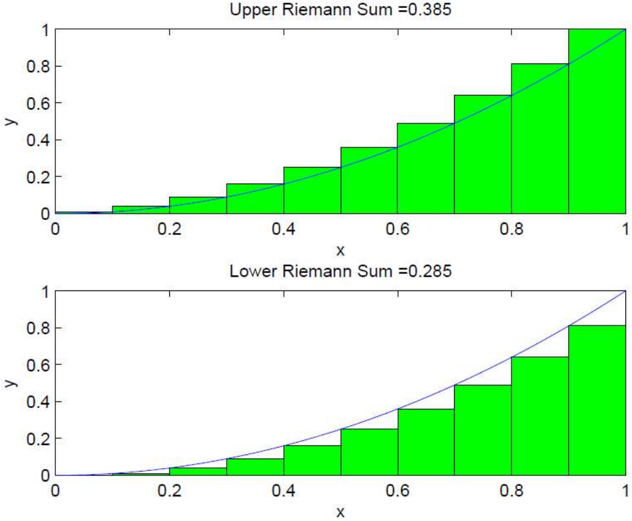

Upper Riemann Sum =0.385
1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x
Lower Riemann Sum =0.285
1
0.8
0.6
0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x

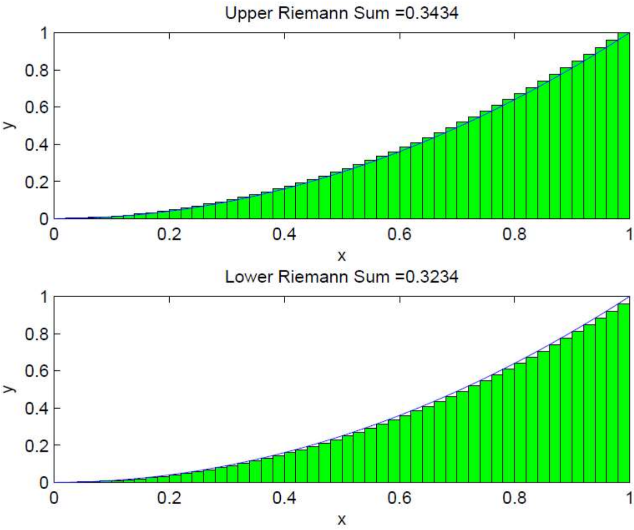
Area under a curve

- By making the partition of the interval $[a, b]$ finer and finer, the upper and lower approximations get better
- In the limit (that is when taking $\Delta^{(i)} \to 0$) the upper and the lower approximations coincide and they also coincide with the Area under the curve from a to b
- That is

$$A = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(c_i)\Delta^{(i)} = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(d_i)\Delta^{(i)} = \int_a^b f(x)\, \mathrm{d}x$$


Upper Riemann Sum =0.44
y
x
0
0.2
0.4
0.6
0.8
1
Lower Riemann Sum =0.24
y
x


Upper Riemann Sum =0.385
Lower Riemann Sum =0.285
y
x
0
0.2
0.4
0.6
0.8
1


Upper Riemann Sum =0.3434
y
x
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
Lower Riemann Sum =0.3234
y
x
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

Riemann Integral

- Let $P = [a = x_0, x_1, \ldots, x_n = b]$ be a partition of the interval $[a, b]$
- Define the *Upper Riemann sum of f with respect to the partition P* as

$$U(f, P) = \sum_{i=0}^{n-1} M_i \Delta^{(i)}$$

- Define the *Lower Riemann sum of f with respect to the partition P* as

$$L(f, P) = \sum_{i=0}^{n-1} m_i \Delta^{(i)}$$

Riemann Integral

Let Π the set of all possible partitions of the interval $[a, b]$ and define the **Upper Riemann sum** as

$$U(f) = inf_{P \in \Pi} U(f, P)$$

and the **Lower Riemann sum** as

$$L(f) = sup_{P \in \Pi} L(f, P)$$

Definition

A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if the Upper Riemann sum $U(f)$ and Lower Riemann sum $L(f)$ are equal.
The Riemann integral, denoted by

$$\int_a^b f(x)\,\mathrm{d}x$$

is the value $U(f)$ (or $L(f)$).

Properties of Riemann Integral

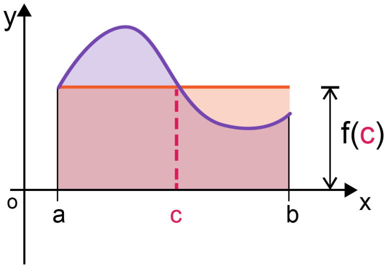
1. The Riemann Integral is defined for every continuous function (positive and negative).
2. $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$ if $a > b$.
3. $\int_a^a f(x)\,\mathrm{d}x = 0$.
4. $\int_a^b [\alpha f(x) + \beta g(x)]\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x, \quad \alpha, \beta \in \mathbb{R}$.
5. $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$.
6. If $f(x) \leqslant g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x$.

Theorem (The Mean Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists a point $c \in [a, b]$ such that

$$\int_a^b f(x)\mathrm{d}x = f(c)(b - a)$$

Meaning: If $f(x) > 0$ in the interval $[a, b]$, the area under the graph of $f(x)$ is equivalent to the area of a rectangle with width $(b - a)$ and height $f(c)$.



Let $f : [a, b] \to \mathbb{R}$ be a continuous function. We define the integral function $F : [a, b] \to \mathbb{R}$ as

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

Theorem (The Fundamental Theorem of Calculus)

The function F s differentiable for all $x \in [a, b]$ and satisfies

1. $F'(x) = f(x)$
2. $F(a) = 0$

The importance of the *Fundamental Theorem of Calculus*

1. F is an antiderivative of f
2. For any other antiderivative G of f we have that

$$F(x) = G(x) - G(a)$$

In particular

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a)$$

Example

Compute

$$\int_{-2}^{1} x^2 + 1 \, \mathrm{d}x$$

An antiderivative is $G(x) = \frac{x^3}{3} + x$. Then

$$G(1) - G(-2) = \left(\frac{1}{3} + 1\right) - \left(\frac{-8}{3} - 2\right) = \frac{9}{3} + 3 = 6$$

In short we write

$$\int_{-2}^{1} x^2 + 1 \, \mathrm{d}x = \left[\frac{x^3}{3} + x\right]_{-2}^{1} = \left(\frac{1}{3} + 1\right) - \left(\frac{-8}{3} - 2\right) = 6$$

Example

$$\int_0^2 e^{x+2}\,\mathrm{d}x = \left[e^{x+2}\right]_0^2 = e^4 - e^2$$

Example (IMPORTANT EXAMPLE!)

Compute the derivative of the function

$$F(x) = \int_1^x (1 + 4t)\mathrm{d}t$$

By the Fundamental Theorem of Calculus:

$$F'(x) = f(x) = 1 + 4x$$

Example (IMPORTANT EXAMPLE!)

Compute the derivative of the function

$$F(x) = \int_1^{x^2+x} (1+4t)\mathrm{d}t$$

We cannot apply the Fundamental Theorem of Calculus directly!

Call $g(x) = x^2 + x$ and $h(u) = \int_1^u (1+4t)\ \mathrm{d}t$. Then

$$F(x) = h(g(x))$$

To compute $F'(x)$ we apply the differentiation rule

$$F'(x) = h'(g(x)) \cdot g'(x) = (1 + 4g(x))g'(x) = (1 + 4(x^2 + x)) \cdot (2x + 1)$$

Again on the area under a curve

- Suppose that the function f is always positive on the interval $[a, b]$. Then the quantity
$$A = \int_a^b f(x)\,\mathrm{d}x$$
indicates the area between the graph of the function f and the x-axis.
- If the function f is always negative on the interval $[a, b]$, then the area between the graph of the function f and the x-axis is
$$A = -\int_a^b f(x)\,\mathrm{d}x.$$
- If f changes its sign we need to detect all points where f vanishes and divide the interval $[a, b]$ so that, when f is positive, we keep the sign $+$, when f is negative we put a $-$ in front of the integral

Improper Integrals

In statistics and in economics it is common to consider definite integrals over an infinite interval, that is either $a = -\infty$, or $b = +\infty$ or both, or over intervals where the function is not well defined.

An improper integral is the limit, **if it exists**, of a definite integral when an endpoint of the integration interval

- goes to ∞
- is a point of discontinuity of the integrand $f(x)$

We write, for instance

$$\int_a^{\infty} f(x)\ \mathrm{d}x = \lim_{b\to\infty} \int_a^b f(x)\ \mathrm{d}x$$

Example

For any $\lambda > 0$, *show that the following improper integral satisfies*

$$\int_0^\infty \lambda e^{-\lambda x}\,\mathrm{d}x = 1$$

The function $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ is very well known in statistics and it is called the *density of the exponential distribution*.

Exercises

Prove that the following integrals converge and compute their values:

1. For $c > 0$

$$\int_{-\infty}^{\infty} xe^{-cx^2}\, \mathrm{d}x$$

2. For $\alpha > 1$

$$\int_{1}^{\infty} \frac{1}{x^{\alpha}}\, \mathrm{d}x$$