DEF: $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \lim_{m\to+\infty} \sum_{j=1}^{m} f(x_{j-1})(x_j - x_{j-1})$

DOVE $a = x_0 < x_1 < x_2 < \dots < x_m = b$ È UNA QUALSIASI PARTIZIONE DI $[a,b]$

TEO 1: f È CONTINUA IN $x_0 \in (a,b)$ ALLORA

$$F(x) = \int_a^x f(t)\,dt \qquad x \in [a,b]$$

È DERIVABILE IN x_0 E $F'(x_0) = f(x_0)$

TEO 2: SIA f UNA FUNZIONE RIEMANN-INTEGRABILE IN $[a,b]$.
SI SUPPONGA CHE $\exists\, F$ TALE CHE

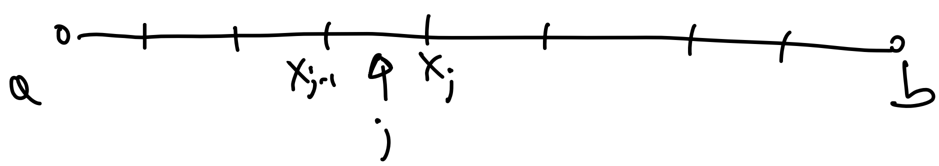
$$F'(x) = f(x) \qquad \forall\, x \in [a,b]$$

(TALE FUNZIONE F SI CHIAMA FUNZIONE PRIMITIVA DI f)

ALLORA

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

DIM: SIA $a = x_0 < x_1 < \dots < x_m = b$ UNA PARTIZIONE DI $[a,b]$. SIA $j = 1, \dots, m$ (OVVERO SCEGLIAMO UN INTERVALLINO $[x_{j-1}, x_j]$)



APPLICO IL TEOREMA DI LAGRANGE A F IN $[x_{j-1}, x_j]$

$\exists\, t_j \in (x_{j-1}, x_j)$ TALE CHE

$F' = f$

$$F(x_j) - F(x_{j-1}) = F'(t_j)\cdot(x_j - x_{j-1})$$

$$F(x_j) - F(x_{j-1}) = f(t_j)\,(x_j - x_{j-1})$$

PRENDO LA SOMMA SU j IN AMBO I LATI DELL' EQUAZIONE

$$\underbrace{\sum_{j=1}^{m}\left[F(x_j) - F(x_{j-1})\right]}_{\text{TELESCOPICA}} = \sum_{j=1}^{m} f(t_j)(x_j - x_{j-1})$$

$$= \underbrace{\cancel{F(x_1)} - F(x_0)}_{j=1} + \underbrace{\cancel{F(x_2)} - \cancel{F(x_1)}}_{j=2} + \underbrace{\cancel{F(x_3)} - \cancel{F(x_2)}}_{j=3} + \cancel{\ldots} + \underbrace{F(x_m) - \cancel{F(x_{m-1})}}_{j=m}$$

$$= F(x_m) - F(x_0) = \underline{F(b) - F(a)} \quad \forall m$$

$\forall m$

$$F(b) - F(a) = \sum_{j=1}^{m} f(t_j)(x_j - x_{j-1})$$

$$t_j \in (x_{j-1}, x_j)$$

CONSIDERO SIA A SX CHE A DX IL LIMITE $m \to +\infty$

MA LA QUANTITÀ A SX NON DIPENDE DA m

QUINDI:

$$F(b)-F(a)=\lim_{n\to+\infty}\sum_{j=1}^{n} f(t_j)(x_j-x_{j-1})=\int_a^b f(t)\,dt$$

ESERCIZI: SIA $F(x)=\int_0^{x^3}(\sin(t))^3\,dt$ $\qquad x\geq 0$

CALCOLARE $F'(x)$.

$$g(x)=x^3 \qquad f(g)=\int_0^{g}(\sin(t))^3\,dt \Rightarrow f'(g)=(\sin(g))^3$$

$$F(x)=f(g(x)) \Rightarrow F'(x)=f'(g(x))\cdot g'(x)$$

$$F'(x)=(\sin(g(x)))^3\cdot g'(x)=$$

$$=(\sin(x^3))^3\cdot 3x^2=\underbrace{3(\sin(x^3))^3x^2}$$

SIA: $F(x)=\int_x^0 \frac{1}{1+t^2+\sin(t)}\,dt$ $\qquad x\geq 0$

$$=-\int_0^x \frac{1}{1+t^2+\sin(t)}\,dt$$

$$F'(x)=-\frac{1}{1+x^2+\sin(x)}$$

ESERCIZIO: CALCOLARE:

$$\frac{1}{2} = \lim_{x\to 0^+} \frac{\int_0^x \sin(t)\,dt}{x^2} = \frac{0}{0} \overset{H}{=}$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{\sin(x)}{2x} = \frac{1}{2}$$

DEF: SIA f UNA FUNZIONE RIEMANN- INTEGRABILE ALLORA INDICO CON

$\int f(x)dx$ LA (INSIEME) CLASSE DI TUTTE LE FUNZIONI PRIMITIVE DI f. TALE INSIEME SI CHIAMA INTEGRALE INDEFINITO.

TEO: L' INSIEME DELLE PRIMITIVE DI f È DEFINITO A MENO DI UNA COSTANTE, OVVERO SE F È UNA PRIMITIVA DI f ALLORA $F+c$ CON $c\in\mathbb{R}$ È ANCH' ESSA UNA PRIMITIVA DI f.

DIM: SE $F' = f$ ALLORA $(F+c)' = F' = f$

Caso 1: $\int x^{\alpha}\, dx \qquad \alpha \neq -1$

$$\boxed{\int x^{\alpha}\, dx = \frac{1}{1+\alpha} x^{1+\alpha} + c} \qquad \alpha = 2$$

$$\left(\frac{1}{1+\alpha} x^{1+\alpha} + c\right)' = \frac{1}{1+\alpha}\left(x^{1+\alpha}\right)' = \frac{1+\alpha}{1+\alpha} x^{1+\alpha-1} = x^{\alpha}$$

Caso 2: $\int \frac{1}{x}\, dx = \log(|x|) + c$

$$\left(\log(|x|) + c\right)' = \left(\log(|x|)\right)' = \frac{1}{|x|}\left(|x|\right)' = \frac{1}{\cancel{|x|}}\frac{\cancel{|x|}}{x}$$

$$= \frac{1}{x}$$

$$\int x^2\, dx = \frac{1}{3} x^3 + c$$

$$\int \frac{1}{\sqrt{x}}\, dx = \frac{1}{1-\frac{1}{2}} x^{1-\frac{1}{2}} + c = 2\sqrt{x} + c$$

Caso 3: $\int e^x\, dx = e^x + c$

$$\left[\int e^{\alpha x}\, dx = \frac{1}{\alpha} e^{\alpha x} + c\right]$$

$$\left(\frac{1}{\alpha} e^{\alpha x}\right)' = \frac{1}{\alpha}\left(e^{\alpha x}\right)' = \frac{\alpha}{\alpha} e^{\alpha x} = e^{\alpha x}$$

$$\int e^{-x}\,dx = -e^{-x} + c \qquad (a^x)' = a^x \cdot \log(a)$$

$$\int a^x\,dx = \frac{a^x}{\log(a)} + c$$

$$\left(\frac{a^x}{\log(a)}\right)' = \frac{1}{\log(a)}\left(a^x\right)' = \frac{\log(a)}{\log(a)} \cdot a^x = a^x$$

$$\int 2^x\,dx = \frac{2^x}{\log(2)} + c$$

$\boxed{\text{Aso4}}$:

$$\int \cos(x)\,dx = \sin(x) + c$$

$$\int \sin(x)\,dx = -\cos(x) + c$$

$$(\tan(x))' = \frac{1}{\cos^2(x)}$$

$$\int \frac{1}{\cos^2(x)}\,dx = \tan(x) + c$$

$$(\operatorname{ATAN}(x))' = \frac{1}{1+x^2} \Rightarrow \int \frac{1}{1+x^2}\,dx = \operatorname{ATAN}(x) + c$$

$$(\operatorname{ASIN}(x))' = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{1}{\sqrt{1-x^2}}\,dx = (\operatorname{ASIN}(x)) + c$$

$\boxed{\text{CASO 5}}$: $\displaystyle\int (g(x))^{\alpha} \cdot g'(x)\, dx = \frac{(g(x))^{\alpha+1}}{\alpha+1} + c \quad \Big| \quad \alpha \neq -1$

$$\left(\frac{g(x)^{1+\alpha}}{1+\alpha} + c\right)' = \frac{1}{1+\alpha}\left(g(x)^{1+\alpha}\right)' = \frac{\cancel{1+\alpha}}{\cancel{1+\alpha}} g(x)^{1+\alpha-1} \cdot g'(x)$$

$$= g(x)^{\alpha} \cdot g'(x)$$

$$\int \frac{\log(x)}{x}\, dx = \int (\log(x)) \cdot \frac{1}{x}\, dx = \int (\log(x)) \cdot (\log(x))'\, dx$$

$g(x) = \log(x) \quad \alpha = 1 \qquad = \dfrac{(\log(x))^2}{2} + c$

$$\left(\frac{(\log(x))^2}{2}\right)' = \frac{1}{2} 2 \log(x) \cdot (\log(x))' = \log(x) \cdot \frac{1}{x}$$

$\boxed{\text{CASO 6}}$ $\displaystyle\int e^{g(x)} \cdot g'(x)\, dx = e^{g(x)} + c$

$\boxed{\text{CASO 7}}$: $\displaystyle\int \frac{g'(x)}{g(x)}\, dx = \log(|g(x)|) + c$

1) $g(x) > 0 \Rightarrow \log(|g(x)|) = \log(g(x))$

$$\left(\log(g(x))\right)' = \frac{g'(x)}{g(x)}$$

2) $g(x) < 0 \Rightarrow \log(|g(x)|) = \log(-g(x))$

$$\left(\log(-g(x))\right)' = \frac{-g'(x)}{-g(x)} = \frac{g'(x)}{g(x)}$$

$$\int \frac{6x}{3x^2+4}\,dx = \int \frac{g'(x)}{g(x)}\,dx$$

$g(x) = 3x^2+4$

$$\int \frac{6x}{3x^2+4}\,dx = \log(|3x^2+4|) + c$$

$g(x) = x^3$

$$\int 3x^2 e^{x^3}\,dx = \int g'(x) e^{g(x)}\,dx = e^{x^3} +$$

CASO: $\int g'(x)\sin(g(x))\,dx = -\cos(g(x)) + c$

$$\int g'(x)\cos(g(x))\,dx = \sin(g(x)) + c$$

$$\int \frac{g'(x)}{\cos^2(g(x))}\,dx = \tan(g(x)) + c$$

$$\int (3x^2 + x + 1)\,dx = 3\int x^2\,dx + \int x\,dx + \int dx =$$

$$= 3\,\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c =$$

$$= x^3 + \frac{1}{2}x^2 + x + c$$

CAMBIO DI VARIABILE:

$$\int \frac{\log(\sqrt{x})}{x}\,dx$$

$$t = \sqrt{x} \Rightarrow t^2 = x$$

$$dt = (\sqrt{x})'\,dx = \frac{1}{2\sqrt{x}}\,dx$$

$$\Rightarrow dx = 2\sqrt{x}\,dt = 2t\,dt$$

$$\|$$

$$\int \frac{\log(t)}{t^2}\,2t\,dt = 2\int \frac{\log(t)}{t}\,dt = 2\cdot\frac{1}{2}\left(\log(t)\right)^2 + c$$

$$= \left(\log(t)\right)^2 + c$$

$$= \left(\log(\sqrt{x})\right)^2 + c$$

$$\boxed{\int \frac{1}{x+\sqrt{x}}\,dx}$$

$$\boxed{\sqrt{x} = t} \Rightarrow x = t^2$$

$$\frac{1}{\phantom{2\sqrt{x}}}\,dx = dt \Rightarrow dx = 2\sqrt{x}\,dt = 2t\,dt$$

$$2\sqrt{x}$$

$$\parallel$$

$$\int \frac{1}{t^{\cancel{2}}+\cancel{t}}\, 2\cancel{t}\cdot dt = \int \frac{2}{1+t}\, dt$$

$$\int \frac{2t}{t^2+t}\, dt = \int \frac{2}{t+1} = \int \frac{2}{1+t}\, dt = 2\log(|1+t|)$$

$$= 2\log(|1+\sqrt{x}|)$$

$$\int \frac{1}{x+\sqrt{x}}\, dx = 2\log(|1+\sqrt{x}|)$$

$$\int \frac{1}{e^x+e^{-x}}\, dx \qquad e^x = t \Rightarrow e^x dx = dt \Rightarrow$$

$$\Rightarrow dx = e^{-x} dt \Rightarrow$$

$$\parallel \qquad \Rightarrow dx = \frac{dt}{e^x} = \frac{dt}{t}$$

$$\int \left(\frac{1}{t+\frac{1}{t}}\right)\cdot \frac{dt}{t} =$$

$$= \int \frac{1}{1+t^2}\, dt = \text{ATAN}(t) + c$$

$$= \text{ATAN}(e^x)$$