Microeconomics I - MSc Economics - 2023-24

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Introduction

- Microeconomics deals with the analysis of economic choices. Economic agents are consumers, firms, financial intermediaries, who operate in different contexts.
- Consumers typically take consumption and saving decisions, firms deal with investment and production decisions, financial intermediaries handle financial decisions of consumers and firms and manage their own portfolios... all of them interact in (complex) economic systems.
- As a result of their choices, (individual and market) demand and supply curves emerge, to which economists combine the notion of equilibrium, typically in terms of price and quantity.

Syllabus

- This course offers a formal (logico-mathematical) approach to the analysis of the choices of consumers and firms operating in market economies, mostly, and deals with the notion of equilibrium in competitive markets.
- When examining choices, we will pay attention to individual decision making in setting without and with risky alternatives.
- Eventually, we will discuss the two fundamental Theorems of Welfare Economics for competitive market systems which are embedded in a general equilibrium perspective.

Introduction

- References:
  - Mas Colell A., Whinston M. and J. Green, Microeconomic Theory, Oxford University Press
  - Varian H. R., Microeconomic Analysis, Norton & Co

- Lecturer and TA
  - **Eloisa Campioni** (eloisa.campioni@uniroma2.it) Office hours to be arranged via e-mail.
  - **Lorenzo Bozzoli** (lor.bozzoli@gmail.com) Office hours to be arranged via e-mail.

Structure of the course

- Lectures and practices. Practical classes will be held every week on Fridays.
- Problem sets will be assigned on a regular basis before the practices and corrected in class.
- Lectures are in presence for the lecturers and for the students!!

In case one needs to activate the remote participation, the student is required to send an e-mail (to eloisa.campioni@uniroma2.it) to alert about his/her situation and the reason why she/he is exceptionally asking to participate remotely. Remote participation is subject to approval.

Evaluation

- Each student's final evaluation consists of the combination of the ongoing assessment and of the final (written) exam.
  - *Problem sets.* Each student will hand in her/his solutions to the assigned problem sets, cooperative work is encouraged. The student's solutions to the weekly problem sets will be evaluated/graded and will attribute 40% of the final grade. The exercises/questions will be then corrected during the practices.
  - *Final written exam.* Written closed-book exam (questions and exercises), yields the remaining 60% of the final grade.
  - *In class participation.* Active participation during lectures and open discussions will also be part of the evaluation. During the practical classes, students will be randomly asked to present their solutions to the assigned problem sets.

Primitives

- In most of this course, we focus on individual economic agents, and make two assumptions about these agents:

  1. *Atomistic:* the agents are small enough compared to the size of the market that their choices do not affect the market price.

  2. *Non–strategic:* agents do not interact when making their choices.

- We start by examining the choice problem of a consumer.

Primitives

- There are four building blocks in modeling consumer choice:
    - *Consumption (Choice) Set:* The set X of all alternatives (complete consumption plans) that the consumer can conceive;
    - *Feasible Set:* The subset B of X that is achievable given the constraints the consumer faces;
    - *Consumer's Preferences:* A rule specifying how the consumer ranks different alternatives;
    - *Behavioral Assumption:* The consumer seeks to identify a feasible alternative that is preferred to all other feasible alternatives.

Consumption set

- We assume that the consumption set X satisfies:

  i) X is non-empty, specifically $X = \mathbb{R}^L_+$;

  ii) X is closed;

  iii) X is convex;

  iv) $\mathbf{0} \in X$;

  v) consumption of larger quantities is always feasible, i.e. if $x \in X$ and $y \geq x$, then $y \in X$.

- A typical element of X is denoted by $x = (x_1, \ldots, x_L)$, where $x_i \geq 0$ is the amount consumed of good $i = 1, 2, \ldots, L$.

Walrasian/competitive budget set

- Economic constraints on alternatives: consumer cannot achieve what she cannot afford. Some alternatives may not be (economically) feasible. The *Budget set* identifies the set of economically feasible alternatives.

- For economic decisions, feasibility concerns prices and wealth.

Walrasian/competitive budget set

- Let $p = (p_1, \ldots, p_L) \gg 0$ be the vector of prices of L commoditites, and $w > 0$ the consumer's wealth.

- The *budget set* is given by

$$\mathcal{B}(p, w) = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}.$$

  with $p \cdot x \ = p_1 x_1 + \ldots + p_L x_L$.

- The consumer's problem, given price vector p and wealth w, is then to choose $x \in \mathcal{B}(p, w)$ according to some choice criterion, to be specified.

Consumers' choice problem

- Preference approach: the tastes of the decision maker are primitives (given) and embodied in her preferences over alternatives. Axioms of rationality imposed on preferences, then examine behavior.
- Revealed Preference approach: individual's choice behavior first, impose assumptions on choices then reconstruct underlying consistent preferences.
- The two approaches can be reconciled. The first one prevails in courses.

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- Consumer preferences are represented by a binary relation $\succsim$ on consumption set X,

$$x \succsim y \qquad\qquad x \text{ is at least as good as } y.$$

- Define:
  - Strict preference: $x \succ y$ if, and only if, $x \succsim y$ but $y \not\succsim x$;
  - Indifference: $x \sim y$ if, and only if, $x \succsim y$ and $y \succsim x$.

- The binary relation $\succsim$ compares *two* consumption plans at a time.
    - The same is true for strict preference $\succ$ and indifference $\sim$.
- The following axioms determine basic criteria these binary comparisons must adhere to.

- **Axiom 1 (Completeness):** The binary relation $\succsim$ is *complete* if for all $x, y \in X$, we have that

$$x \succsim y \qquad \text{or} \qquad y \succsim x \text{ (or both).}$$

Remark: if $\succsim$ is complete, then $\succsim$ is *reflexive*, i.e., $x \succsim x$ for all $x \in X$.

- **Axiom 2 (Transitivity):** The binary relation $\succsim$ is *transitive* if for all $x, y, z \in X$,

$$\text{if } x \succsim y \text{ and } y \succsim z, \qquad \text{then } x \succsim z.$$

Definition

A *preference relation* is a complete and transitive binary relation.

- Preference relations that satisfy Axiom 1 (completeness) and Axiom 2 (transitivity) are *rational*.
- In this course we will focus on rational preference relations.

- Given the preference relation $\succsim$ and a consumption bundle x, we define the following subsets of X:

1. the set of bundles that are *at least as good as* x,
$\succsim(x) = \{x' \in X : x' \succsim x\}$, i.e. the upper contour set of x;

2. the set of bundles that are *no better than* x,
$\precsim(x) = \{x' \in X : x \succsim x'\}$, i.e. the lower contour set of x;

3. the set of bundles that are *indifferent* to x,
$\sim(x) = \{x' \in X : x \sim x'\}$, i.e. the indifference set of x.

Preferences

Axioms of choice: Monotonicity

- **Axiom 3 (Monotonicity):** The preference relation $\succsim$ is *monotone* if for each $x \in X$ and $y \gg x$, then $y \succ x$.

- **Axiom** $3'$ **(Strong Monotonicity):** The preference relation $\succsim$ is *strongly monotone* if for each $x \in X$ and $y \geq x$ and $y \neq x$, then $y \succ x$.

- **Axiom** $3''$ **(Local Non–Satiation):** The preference relation $\succsim$ is *locally non–satiated* if for each $x \in X$ and for each $\varepsilon > 0$, there exists $y \in X$ such that $||y - x|| \leq \epsilon$ and $y \succ x$.

$|| \cdot ||$ is the Euclidean distance, defined as $\left[\sum_{l=1}^{L} (y_l - x_l)^2\right]^{1/2}$

Preferences

- Monotonicity has implications on how upper contour sets and lower contour sets of $x \in X$...
- Local Non-Satiation implies that the Indifference set of $x \in X$ is not thick!!

- **Axiom 4 (Convexity):** The preference relation is *convex* if for every $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ is convex, that is take two bundles $y \succsim x$ and $z \succsim x$, then their convex combination $\alpha y + (1 - \alpha)z \succsim x$ for any $\alpha \in [0, 1]$.

- **Axiom** $4'$ **(Strict Convexity):** The preference relation is *strictly convex* if for every $x \in X$, we have that $y \succsim x$ and $z \succsim x$ with $y \neq z$ imply $\alpha y + (1 - \alpha)z \succ x$ for any $\alpha \in [0, 1]$.

- **Axiom 5 (Continuity):** The preference relation $\succsim$ is *continuous* if for each sequence of pairs of bundles $\{x^n, y^n\}$,

  that verify $x^n \succsim y^n$ for each n, and

  that are converging $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = y$,

  we have that $x \succsim y$.

- The preference relation is preserved under the limit.

- An equivalent statement of *continuity* of $\succsim$ is that for each $x \in X$ the upper contour set $\succsim(x)$ and the lower contour set $\precsim(x)$ are closed subsets of X.

- Since $\sim(x) = \succsim(x) \bigcap \precsim(x)$, $\sim(x)$ is also closed if continuity holds. Indeed, the intersection of closed sets is closed.

Definition

The consumption bundle $x^* \in X$ is a *satiation point* of $\succsim$ if $x^* \succsim x$ for all $x \in X$.

If $\succsim$ is locally non–satiated, then $\succsim$ has no satiation point.

Utility Functions

- Preference relations satisfying Axioms 1- 5 (and their variants) discipline consumer behavior, but are difficult to work with.
- Microeconomic theory has developed a more suitable approach to represent consumer's preferences.
- First, we establish the existence of a (ordinal) function that represents well–behaved preference relations, i.e. the *utility function*.
- Then we move on to study the properties of such function when the consumer must choose her most preferred alternative.

Utility Functions

Definition

An utility function $u : X \to \mathbb{R}$ represents the binary relation $\succsim$ on X if for all $x, x' \in X$,

$$u(x') \geq u(x) \ \text{ if and only if } x' \succsim x.$$

Lemma 2

Suppose that $u : X \to \mathbb{R}$ represents the binary relation $\succsim$ on X. Then $\succsim$ is a rational preference relation.

- *Completeness:* For any $x, y \in X$, either $u(x) \geq u(y)$ or $u(x) \leq u(y)$.

  Since u represents $\succsim$, we then have that either $x \succsim y$ or $y \succsim x$, that is $\succsim$ is complete.

- *Transitivity:* Take any $x, y, z \in X$ such that $u(x) \geq u(y)$ and $u(y) \geq u(z)$, then $u(x) \geq u(z)$.

  Since u represents $\succsim$, we have that $x \succsim y$ and $y \succsim z$ imply $x \succsim z$, that is $\succsim$ is transitive. $\blacksquare$

Utility Functions

- Lemma 2 says that if a binary relation is represented by a utility function, then it is complete and transitive, which qualifies a (rational) preference relation.
- In general, is the converse also true? That is, can every preference relation be represented by a utility function?
- The answer is negative, let me show you why by means of an example.

- Consider a particular preference order: the lexicographic order in $\mathbb{R}^2_+$, that is the binary relation $\succsim_\ell$ such that $x \succsim_\ell y$ is defined by looking at the ordered components of the bundles. That is,

$$(x_1, x_2) \succsim_\ell (y_1, y_2) \text{ if, and only if, } x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 \geq y_2.$$

- Rank elements by comparing the quantities of each good in turn.

- **We show that $\succsim_\ell$ is a preference order, but it cannot be represented by an utility function.**

- The lexicographic order is a preference order, i.e. $\succsim_\ell$ is **complete** and transitive.

- **Completeness.** Take two bundles $x, y \in \mathbb{R}^2_+$. Focus on the first commodity, it must be that either $x_1 > y_1$ or that $x_1 \leq y_1$.

    - If $x_1 > y_1$, then $x \succsim_\ell y$.

    - If $x_1 = y_1$, it could be one of two possibilities: either $x_2 \geq y_2$ or $y_2 \geq x_2$, which respectively lead to either $x \succsim_\ell y$ or $y \succsim_\ell x$.

    - If $x_1 < y_1$, then $y \succsim_\ell x$.

- The lexicographic order is a preference order, i.e. $\succsim_\ell$ is complete and **transitive**.
- **Transitivity.** Take three bundles $x, y, z \in \mathbb{R}^2_+$ such that $x \succsim_\ell y$ and $y \succsim_\ell z$. By the lexicographic order, it could be that
  - $x_1 > y_1 > z_1$, then $x \succsim_\ell z$ is an immediate implication;
  - $x_1 = y_1$ and $x_2 \geq y_2$ and $y_1 > z_1$, in which case again $x \succsim_\ell z$ is implied;
  - $x_1 = y_1$ and $x_2 \geq y_2$, and $y_1 = z_1$ and $y_2 \geq z_2$, in which case again $x \succsim_\ell z$ is implied.

The lexicographic preference order $\succsim_\ell$ cannot be represented by an utility function!!

The lexicographic preference order $\succsim_\ell$ cannot be represented by an utility function!!

- The proof goes by contradiction. Suppose that $u : \mathbb{R}^2_+ \to \mathbb{R}$ represents $\succsim_\ell$.
- Since $(x, 1) \succ_\ell (x, 0)$ for all $x \in \mathbb{R}_+$, then $u(x, 1) > u(x, 0)$ for all $x \in \mathbb{R}_+$
- For each $x \in \mathbb{R}_+$, pick a $q_x \in \mathbb{Q}$, such that $u(x, 1) > q_x > u(x, 0)$.
- If $x' > x$, then $(x', 0) \succ_\ell (x, 1)$, it is also true that $q_{x'} > u(x', 0) > u(x, 1) > q_x$.
- Thus, $q(.)$ is a one-to-one function that associates to each real number a rational number.
- This is impossible!! since the set of rational number is a countable set, while the set of reals is uncountable. $\blacksquare$

- The big issue with the lexicographic order is that it is not continuous. This discontinuity allows for sudden reversals of preferences.

- Continuity is a crucial property for the existence of an utility representation of a preference relation.

Theorem 1

Suppose $\succsim$ is a continuous and rational preference relation on X. Then, there exists a continuous function $u : X \to \mathbb{R}$ that represents $\succsim$.

- We establish the result when $X = \mathbb{R}^L_+$ and $\succsim$ is monotone.
- Let $e = (1, \ldots, 1)$ denote the L-vector with all elements equal to 1. For each $x \in \mathbb{R}^L_+$, let $A^-(x) = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$ and $A^+(x) = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$.
- Since $\succsim$ is assumed to be monotone, $x \succsim \mathbf{0}$, and therefore $\mathbf{0} \in A^-(x)$ for all $x \in \mathbb{R}^n_+$, i.e. $A^-(x)$ is non–empty.
- Monotonicity of $\succsim$ implies that $A^+(x)$ is also non–empty: indeed, given x we can find $\overline{\alpha}$ such that $\overline{\alpha} e \gg x$ which belongs to $A^+(x)$.

- Continuity of $\succsim$ implies the upper contour set and the lower contour set of x are closed. Hence, also $A^-(x)$ and $A^+(x)$ are non-empty and closed for every $x \in \mathbb{R}^L_+$.
- Fix $x \in \mathbb{R}^L_+$. Since $\succsim$ is complete, $\mathbb{R}_+ = A^+(x) \bigcup A^-(x)$.
- Thus, $A^+(x) \bigcap A^-(x) \neq \emptyset$, otherwise $\mathbb{R}_+$ would be the union of two disjoint sets, which is not possible since $\mathbb{R}_+$ is connected.
- Hence, there exists a scalar $\hat{\alpha} \in A^+(x) \bigcap A^-(x)$ such that $\hat{\alpha} e \sim x$.
- Since $\succsim$ is monotone, such scalar is unique. Indeed, by monotonicity $\alpha_1 e \succ \alpha_2 e$ whenever $\alpha_1 > \alpha_2$. Let $\hat{\alpha}(x)$ denote the unique scalar satisfying $\hat{\alpha}(x) e \sim x$.

- Let our utility function, $u : X \rightarrow \mathbb{R}$, be such that $u(x) = \hat{\alpha}(x)$ for every $x \in X$, with $\hat{\alpha}(x)$ being the unique real number in $A^{+}(x) \bigcap A^{-}(x)$.

- We need to check that:

  a.) u represents $\succsim$, that is for all $x, x' \in X$,

$$u(x') \geq u(x) \text{ if and only if } x' \succsim x.$$

  b.) u is continuous.

Utility Functions

Proof of Theorem 1 - u represents $\succsim$

a.) We want to prove that for all $x, x' \in X$,

$$u(x) \geq u(x') \text{ if and only if } x \succsim x'.$$

- *[If]* Consider, $\hat{\alpha}(x)$ and $\hat{\alpha}(x')$, such that $\hat{\alpha}(x) \geq \hat{\alpha}(x')$. By construction, since $\succsim$ are monotone, $\hat{\alpha}(x)e \succsim \hat{\alpha}(x')e$, that implies $x \succsim x'$.

- *[Only if]* Suppose alternatively that $x \succsim x'$, then $\hat{\alpha}(x)e \succsim \hat{\alpha}(x')e$, which implies that $\hat{\alpha}(x) \geq \hat{\alpha}(x')$.

b.) we omit the proof that u is a continuous function: very technical!$\blacksquare$

- The idea behind the proof of Theorem 1 is to construct an utility function that selects the value of α that makes the individual indifferent between the bundles x and αe.

- Continuity and monotonicity of preferences are indispensable here to guarantee the uniqueness of such value, in particular when the consumption set is infinite, as $X = \mathbb{R}^L_+$.

- From now on, we focus on continuous preferences $\succsim$, hence representable by a continuous utility function.

Definition. Let $u : X \to \mathbb{R}$ and denote the image of u by $\mathcal{U}$. Consider a strictly increasing function, $\tau : \mathcal{U} \to \mathbb{R}$, we call the function v which is the composition of τ and u, so that $v(x) = \tau(u(x))$, a *monotone transformation* of u.

Notice that $v : X \to \mathbb{R}$ is itself a function from X to $\mathbb{R}$.

Theorem 2

Let $\succsim$ be a preference relation on X and let $u : X \to \mathbb{R}$ be a utility function that represents $\succsim$. Then $v : X \to \mathbb{R}$ also represents $\succsim$ if, and only if, v is a monotone transformation of u.

- If $u : X \to \mathbb{R}$ represents $\succsim$, then monotonicity of preferences implies that the utility function u is increasing.

Utility Functions

Other Properties

- Suppose $u : X \to \mathbb{R}$ represents $\succsim$, convexity of the preference relation implies that u is quasi–concave.

- If $u : X \to \mathbb{R}$ represents $\succsim$, then convexity of $\succsim$ implies that the utility function u is quasi–concave.

  The function $u : X \to \mathbb{R}_+$ is quasi-concave if :

  i.) for every $x \in X$ the set $\{y \in X : u(y) \geq u(x)\}$ is convex,

  ii.) or, equivalently, for every $x, y \in X$, $u(\alpha x + (1 - \alpha)y) \geq Min\{u(x), u(y)\}$ for every $\alpha \in [0, 1]$.

- A preference relation $\succsim$ that is *strictly convex* implies that u is *strictly quasi–concave*.

Utility Function: quasi-concavity

Lemma

If $u : X \to \mathbb{R}$ represents $\succsim$, then convexity of $\succsim$ implies that the utility function u is quasi–concave.

Proof.

- By definition, if $\succsim$ is convex then $\{x' \in X : u(x') \geq u(x)\}$ is convex for all $x \in X$. Thus, u is quasi–concave.

- Suppose now that u is quasi–concave and let $x^1, x^2 \in X$ be such that $x^1 \succsim x^2$.

  Then $u(x^1) \geq u(x^2)$ and, since u is quasi–concave, $u(tx^1 + (1-t)x^2) \geq u(x^2)$ for all $t \in [0, 1]$.

  Thus $tx^1 + (1-t)x^2 \succsim x^2$ for all $t \in [0, 1]$; that is $\succsim$ is convex.

- Take $L = 2$ and consider an utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ that represents a convex preference relation $\succsim$ in $X = \mathbb{R}^2_+$.

- Assume u is twice continuously differentiable $\mathcal{C}^2$.

- Let $c \in \mathcal{U}$ be an element in the image of u. We construct an *indifference curve*

$$u(x_1, x_2) = c. \tag{1}$$

that is the locus of all pairs $(x_1, x_2) \in \mathbb{R}^2_+$ that yield the same utility level c to the consumer.

Indifference Curve

- Let $X_1 = \{x_1 \in \mathbb{R}_+ : \exists x_2 \in \mathbb{R}_+ \text{ s.t. } u(x_1, x_2) = c\}$. By construction,
  - X_1 is non–empty;
  - for each $x_1 \in X_1$, since u is quasi-concave there exists a *unique* $x_2 \in \mathbb{R}_+$ such that $u(x_1, x_2) = c$.
- Equation (1) then defines a function $f : X_1 \to \mathbb{R}_+$ such that $u(x_1, f(x_1)) = c$ for all $x_1 \in X_1$.

Indifference Curve and Marginal Rate of Substitution

- Since all pairs of bundles (x_1, x_2) which belong to a given indifference curve yield to the consumer the same level of utility, say c. Then, by totally differentiating (1), we derive that

$$\frac{dx_2}{dx_1} = -\frac{\dfrac{\partial u}{\partial x_1}(x_1, x_2)}{\dfrac{\partial u}{\partial x_2}(x_1, x_2)} \equiv -\mathrm{MRS}_{1,2}(x_1, x_2) \tag{2}$$

The Consumer's Problem

- Assume $\succsim$ is a rational, continuous and locally non-satiated preference relation, and therefore represented by a continuous utility function u (Theorem 1).

- The consumer's problem (henceforth, *UMP*) is then given by

$$\begin{array}{ll} \max & u(x) \\ s.t. & w - p \cdot x \geq 0 \\ & x \geq 0 \end{array}$$

- Does this problem have a solution???

- For all $p \gg 0$ and $w > 0$, $\mathcal{B}(p, w)$ is closed and bounded.

  - Bounded: if $x \in \mathcal{B}(p, w)$, then $x_i \geq 0$ and $x_i \leq w/p_i$ for each $i \in \{1, \ldots, n\}$.

  - Closed: let $\{x^k\}$ be a converging sequence in $\mathcal{B}(p, w)$. Since $x^k \geq 0$ for all $k \geq 1$, we have that $\lim x^k = x \geq 0$ as well.

    Consider $g_0(x) = w - p \cdot x$, which is a continuous function in x, that is $g_0(x^k)$ converges to $g_0(x)$, notice that $g_0(x^k) \geq 0$ for all k implying that $g_0(x) \geq 0$. Thus, $x \in \mathcal{B}(p, w)$.

Since u is a continuous function and the set $\mathcal{B}(p, w)$ is closed and bounded, by Weierstrass Theorem: *UMP* has a solution for all $p \gg 0$ and $w > 0$.

The Utility Maximization Problem - UMP

- Given *UMP*,

$$\begin{aligned} \max \quad & u(x) \\ s.t. \quad & w - p \cdot x \geq 0 \\ & x \geq 0 \end{aligned}$$

- Let $x(p, w) \subseteq \mathcal{B}(p, w)$ denote the set of solutions for given p and w

- Let $u(x(p, w))$ be the associated maximum value for the utility

The Theorem of Maximum

Theorem of Maximum

Assume $u(.)$ is a continuous utility function and that the constraint (budget set) is a continuous correspondence $\mathcal{B} : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_+$.
Then the maximizer,

$$x : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_+$$

is an upper-hemicontinuous correspondence;
and the value function $v(p, w) \equiv u(x(p, w))$ is continuous.

▸ Upper-hemicontinuity

The Utility Maximization Problem - UMP

$$\begin{aligned} \max \quad & u(x) \\ s.t. \quad & w - p \cdot x \geq 0 \\ & x \geq 0 \end{aligned}$$

- We call $x(p, w)$ the *Walrasian demand correspondence (function)*.
- We call $v(p, w) = u(x(p, w))$ the *indirect utility function*.

Theorem 3

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the Walrasian demand correspondence has the following properties:

i) $x(p, w)$ is homogeneous of degree zero in (p, w), i.e. $x(\alpha p, \alpha w) = x(p, w)$ for every $\alpha > 0$;

ii) $x(p, w)$ satisfies Walras' law, i.e. $p \cdot x = w$ for every $x \in x(p, w)$;

iii) if $\succsim$ is convex, and $u(.)$ is quasi-concave, then $x(p, w)$ is a convex set. If $\succsim$ is strictly convex, and $u(.)$ is strictly quasi-concave, then $x(p, w)$ is a singleton.

Let us prove all three properties.

i) Follows immediately from the fact that $\mathcal{B}(p, w) = \mathcal{B}(\alpha p, \alpha w)$ for all $\alpha > 0$.

ii) $x(p, w)$ satisfies Walras' law, i.e. $p \cdot x = w$ for every $x \in x(p, w)$. It follows from LNS.
Assume by contradiction that $p \cdot x < w$ at an $x \in x(p, w)$.
By LNS, there exists an $\epsilon > 0$ small enough and a bundle y in an ϵ-neighborhood of x, $||y - x|| \leq \epsilon$, such that $y \succ x$ and $p \cdot y \leq w$.
This contradicts that $x \in x(p, w)$.

iii-a) Take $u(.)$ is quasi-concave ($\succsim$ convex) and let $x \in x(p, w)$ and $x' \in x(p, w)$ solve UMP.

We have to show that $\alpha x + (1 - \alpha)x' \equiv x'' \in x(p, w)$ for every $\alpha \in [0, 1]$.

Since x and x' solve UMP, it must be $u(x) = u(x')$, denote it u^*.

Since $u(.)$ is quasi-concave, $u(x'') = u(\alpha x + (1 - \alpha)x') \geq u^*$.

Since $\mathcal{B}(p, w)$ is a convex set, $x'' \in \mathcal{B}(p, w)$. Indeed both x and x' are in $\mathcal{B}(p, w)$, and since $x'' \equiv \alpha x + (1 - \alpha)x'$, it also satisfies $p(\alpha x + (1 - \alpha)x') \leq w$.

Hence, x'' is a budget-feasible bundle which yields utility $u(x'') \geq u^*$, therefore is must be $x'' \in x(p, w)$. Therefore, $x(p, w)$ is a convex set.

iii-b) Take $u(.)$ is strictly quasi-concave ($\succsim$ strictly convex) and assume by contradiction that x, x' with $x \neq x'$ are two solutions to UMP, i.e. $x \in x(p, w)$ and $x' \in x(p, w)$.

Consider $\alpha x + (1-\alpha)x' \equiv x''$ for every $\alpha \in (0, 1)$. Since x and x' solve UMP, it must be $u(x) = u(x')$, denote it u^*.

Since $u(.)$ is strictly quasi-concave, $u(x'') = u(\alpha x + (1-\alpha)x') > u^*$.

Since $\mathcal{B}(p, w)$ is a convex set, again it holds that $x'' \in \mathcal{B}(p, w)$.

Hence, x'' is a budget-feasible bundle which yields utility $u(x'') > u^*$, hence neither x nor x' can solve UMP. Therefore, $x(p, w)$ must contain only one element. ■

Theorem 3

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the Walrasian demand correspondence has the following properties:

i) $x(p, w)$ is homogeneous of degree zero in (p, w);

ii) $x(p, w)$ satisfies Walras' law;

iii) if $\succsim$ is convex, $x(p, w)$ is a convex set. If $\succsim$ is strictly convex, $x(p, w)$ is a singleton.

The Walrasian demand function

- When $\succsim$ is strictly convex, the solution of UPM is called the Walrasian demand function, denote it $x^*(p, w)$.

- Let us now focus on $x^*(p, w)$ for some comparative-statics exercises.

- We discuss wealth effects and price effects.

- Fix the price level at $\bar{p}$, and consider $x^*(\bar{p}, w)$ as a function of w, this is the *Engel curve*.

- Consider how the demand function $x^*(\bar{p}, w)$ changes for different values of wealth, the set of all the values $\{x^*(\bar{p}, w) : w > 0\}$ is the wealth expansion path.

The Walrasian demand function

Comparative statics: wealth effects

- Holding the price level fixed at $\bar{p}$, take $x^*(\bar{p}, w)$ differentiable. We can compute for each commodity k,

$$\frac{\partial x_k^*(\bar{p}, w)}{\partial w}$$

  this is the wealth effect on the demand of good k.

- If $\dfrac{\partial x_k^*(\bar{p}, w)}{\partial w} \geq 0$, good k is a normal good;

- if $\dfrac{\partial x_k^*(\bar{p}, w)}{\partial w} < 0$, good k is an inferior good.

- How would the wealth expansion path of a normal good look like? and of an inferior good?

The Walrasian demand function

Comparative statics: price effects

- Starting from the Walrasian demand, consider $x^*(p, w)$ as a function of the price vector $p = (p_1, \ldots, p_k, \ldots, p_L)$.

- Consider the demand for commodity k, $x_k^*(p_1, \ldots, p_k, \ldots, p_L, w)$.

  Fix the wealth at $\bar{w}$ and the prices of all commodities except k.

  It is customary to write,
  $p = (p_k, \bar{p}_{-k})$ with $\bar{p}_{-k} = (\bar{p}_1, \ldots, \bar{p}_{k-1}, \bar{p}_{k+1}, \ldots, \bar{p}_L) \in \mathbb{R}^{L-1}_{++}$.

- The set of all values $\{x_k^*(p_k, \bar{p}_{-k}, \bar{w}) : p_k > 0\}$ is the *offer curve* for commodity k.

- Let $x^*(p, w)$ be differentiable. In general,

$$\frac{\partial x_k^*(p, \bar{w})}{\partial p_k} < 0,$$

  i.e. demand and price of a commodity are inversely related.

- If $\dfrac{\partial x_k^*(p, \bar{w})}{\partial p_k} > 0$ commodity k is a *Giffen good*.

- Think about how would the offer curve for a Giffen good look like. *[Hint: use $X = \mathbb{R}^2_+$]*

The Walrasian demand function

Comparative statics: price effects

- We can also evaluate the effect of a change in the price of commodity j, p_j, on the demand for commodity k, $x_k^*(p, \bar{w})$, that is

$$\frac{\partial x_k^*(p, \bar{w})}{\partial p_j};$$

- if commodity j is a complement for commodity k, the cross-price effect will be negative

$$\frac{\partial x_k^*(p, \bar{w})}{\partial p_j} < 0;$$

- if commodity j is a substitute for commodity k, the cross-price effect will be positive

$$\frac{\partial x_k^*(p, \bar{w})}{\partial p_j} > 0$$

.

Comparative statics and homogeneity of degree zero

- Consider property i) of Theorem 3, for the Walrasian demand function: $x^*(\alpha p, \alpha w) - x^*(p, w) = 0$. For each commodity j

$$x_j^*(\alpha p_1, \ldots, \alpha p_L, \alpha w) - x_j^*(p_1, \ldots, p_L, w) = 0$$

- Differentiate it w.r.t. α and evaluate at $\alpha = 1$. We get the following result.

Homogeneity of degree zero of the Walrasian demand implies that for all p and w,

$$\sum_{k=1}^{L} \frac{\partial x_j^*(p, w)}{\partial p_k} p_k + \frac{\partial x_j^*(p, w)}{\partial w} w = 0 \qquad \text{for } j = 1, \ldots, L \tag{3}$$

Comparative statics and homogeneity of degree zero

- Take equation (3) and commodity j, divide each addend by $x_j^*(p,w)$, we get

$$\sum_{k=1}^{L} \frac{\partial x_j^*(p,w)}{\partial p_k} \frac{p_k}{x_j^*(p,w)} + \frac{\partial x_j^*(p,w)}{\partial w} \frac{w}{x_j^*(p,w)} = 0$$

- Recall that $\epsilon_{jk} = \frac{\partial x_j^*(p,w)}{\partial p_k} \frac{p_k}{x_j^*(p,w)}$ is the elasticity of the demand for commodity j to the price of commodity k,

  and $\epsilon_{jw} = \frac{\partial x_j^*(p,w)}{\partial w} \frac{w}{x_j^*(p,w)}$ is the wealth elasticity of the demand for commodity j.

Comparative statics and homogeneity of degree zero

- Then, equation (3)

$$\sum_{k=1}^{L} \frac{\partial x_j^*(p,w)}{\partial p_k} p_k + \frac{\partial x_j^*(p,w)}{\partial w} w = 0$$

rewritten in terms of price and wealth elasticities, yields

$$\sum_{k=1}^{L} \epsilon_{jk} + \epsilon_{jw} = 0 \qquad \text{for } j = 1, \ldots, L$$

- When all prices and wealth change by an equal percentage, this leads to no change in demand of commodity j.

Comparative statics and Walras' law

- By Walras law, the Walrasian demand is such that $p \cdot x^*(p, w) = w$, which rewrites as

$$p_1 x_1^*(p, w) + p_2 x_2^*(p, w) + \cdots + p_L x_L^*(p, w) = w. \tag{4}$$

a.) For each commodity evaluate the differential change in (4) w.r.t. p.

By Walras' law, for all p and w, for each commodity j,

$$\sum_{k=1}^{L} \frac{\partial x_j^*(p, w)}{\partial p_k} p_j + x_j^*(p, w) = 0 \qquad \text{for } j = 1, \ldots, L$$

For each commodity, the effect of a change in prices on its expenditure must be zero. Overall, the total expenditure cannot change if prices change.

b.) For each commodity evaluate the differential change in (4) w.r.t. w.

By Walras' law, for all p and w,

$$\sum_{j=1}^{L} \frac{\partial x_j^*(p,w)}{\partial w} p_j = 1 \qquad \text{for } j = 1, \ldots, L$$

If wealth changes, total expenditure must change so to absorb entirely the change in w.

The Walrasian demand and the Weak Axiom

Lemma

The Walrasian demand function $x(p,w)$ satisfies the following property: for every two pairs (p,w), (p',w')

$$\text{if } p\cdot x(p',w')\leq w \text{ and } x(p',w')\neq x(p,w) \qquad \text{then } p'\cdot x(p,w)>w'.$$

This is a property of consistency in the choices of the consumer. Indeed, since both $x(p,w)$ and $x(p',w')$ solve UMP at the respective prices and wealth; if $x(p',w')$ is feasible at (p,w) but is not chosen and the two bundles are different, then it has to be that $x(p,w)$ must not be feasible for the consumer at (p',w')...

Otherwise, one would expect the consumer to keep preferring $x(p,w)$ over $x(p',w')$ also at price (p',w')... differently, the consumer should have an inconsistent demand behavior!!

This property is the weak axiom of revealed preferences (WARP).

Implications of WARP on the price effects of the Walrasian demand

- A price change alters the relative cost of a commodity w.r.t. the other commodities in the UMP. (Substitution effect).
- Consider a change in prices accompanied by the (specific) change in w that maintains the initial consumption bundle, just affordable at the new prices.
- Start with (p, w) and the consumer optimal choice $x(p, w)$. Consider a price change to $p' \neq p$, and the change in wealth s.t. $w' = p' \cdot x(p, w)$.
- Then, $\Delta w \equiv w' - w = (p' - p) \cdot x(p, w) \rightarrow$ this is the Slutsky wealth compensation.
- The price changes accompanied by such wealth compensation are labelled as (Slutsky) compensated price changes.

WARP and the law of demand in UMP

Proposition (WARP)

The Walrasian demand function $x(p, w)$ satisfies WARP if and only if, for any compensated price change from (p, w) to $(p', w') = (p', p' \cdot x(p, w))$, we have

$$(p' - p) \cdot [x(p', w') - x(p, w)] \leq 0$$

with strict inequality whenever $x(p', w') \neq x(p, w)$.

- The Weak Axiom imposes not only a certain consistency on the Walrasian demand, but also a form of the law of demand, in that the change in prices and in Walrasian demands move in opposite directions... at least for compensated price changes!!

- Read the differential version of Proposition (WARP) - MWG chapter 2, p. 33-34.

Compensated change in prices and Walrasian demand

- Consider commodity l and the effect on $x_l(p, w)$ of a compensated change in the price of p_l, only.

- $\Delta p = p' - p = (0, \ldots, \Delta p_l, \ldots, 0)$. We want to measure Δx_l, Proposition WARP implies that if $\Delta p_l > 0$ then $\Delta x_l < 0$.

- We cannot say much about the effect of a price change that is not compensated!

- Consider the differential version of Proposition WARP

$$dp \cdot dx \leq 0$$

for a compensated change in prices.

Substitution effect and Walrasian demand

- In $dp \cdot dx \leq 0$, dx measures the total variation of the Walrasian demand $x(p', w' = p \cdot x(p, w))$ induced by the change in price and the compensation in wealth.

$$dx = D_p x(p, w) dp + D_w x(p, w) dw$$

with $D_p x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \dots & \frac{\partial x_1(p,w)}{\partial p_L} \\ & \dots & \\ \frac{\partial x_L(p,w)}{\partial p_1} & \dots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}$

and $D_w x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \dots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix}$

Substitution effect and Walrasian demand

- Since we deal with compensated price changes, $dw = dp \cdot x(p,w)$. Hence,

$$dx = D_p x(p,w)dp + D_w x(p,w)[dp \cdot x(p,w)]$$

or

$$dx = [D_p x(p,w) + D_w x(p,w)x(p,w)^T]dp.$$

Finally,

$$dp \cdot dx = dp \cdot [D_p x(p,w) + D_w x(p,w)x(p,w)^T]dp \leq 0$$

Slutsky matrix and substitution effects

$$[D_p x(p,w) + D_w x(p,w) x(p,w)^T] \equiv S(p,w)$$

is an $(L \text{ x } L)$ matrix, called Slutsky matrix, $S(p,w)$, with generic element of row l and column k equal to

$$s_{lk}(p,w) = \left[\frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)\right]$$

which is called the substitution effect.

Slutsky matrix and substitution effects

$$s_{lk}(p,w) = \left[\frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)\right]$$

The substitution effects captures the (differential) change in demand of good l due to a differential change in the price of good k, when wealth is compensated so that the consumer can just afford his original bundle ... hence induced by change in relative prices only.

$\frac{\partial x_l(p,w)}{\partial p_k} dp_k$ measures change in demand of good l if w is unchanged;

$x_k(p,w)dp_k$ measures the compensated change in wealth;

$\frac{\partial x_l(p,w)}{\partial w}[x_k(p,w)dp_k]$ measures the change in demand of good l due to the compensated change in wealth.

Slutsky matrix and substitution effects

- To summarize, $dp \cdot dx \leq 0$ is equivalent to

$$dp \cdot [D_p x(p,w) + D_w x(p,w) x(p,w)^T] dp \leq 0$$

- Since the Walrasian demands satisfy the weak axiom, the Slutsky matrix is negative semi-definite for every (p,w).

- Negative semi-definiteness of $S(p,w)$ implies that $s_{ll}(p,w) \leq 0$ for every $l = 1, 2, ..., L$, own substitution effects are non-positive.

- However, we know that $\frac{\partial x_l(p,w)}{\partial p_l} > 0$ (for Giffen goods), hence for $s_{ll}(p,w) \leq 0$ it has to be $\frac{\partial x_l(p,w)}{\partial w} < 0$. That is, a good can be a Giffen good at some (p,w) only if it is inferior.

Constrained Optimization: Lagrange and Kuhn-Tucker approach

Let us now examine how to solve a constrained maximization problem.

The indirect utility function $v(p, w)$

Theorem 4

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the indirect utility function has the following properties:

i) $v(p, w)$ is homogeneous of degree zero in (p, w);

ii) $v(p, w)$ is strictly increasing in w and non-increasing in p;

iii) $v(p, w)$ is quasi-convex, that is the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$;

iv) $v(p, w)$ is continuous in p and w.

The indirect utility function

Proof of Theorem 4 - i)

Let us prove properties $i) - iii)$ and discuss $iv)$.

i) Follows immediately from the fact that $x(p, w)$ is homogeneous of degree zero in (p, w).

   Indeed since $x(\alpha p, \alpha w) = x(p, w)$ for all $\alpha > 0$, then also $u(x(\alpha p, \alpha w)) = u(x(p, w))$.

The indirect utility function

Proof of Theorem 4 - ii)

ii) We prove that $v(p, w)$ is increasing in w.

Take $w' > w$, then $\mathcal{B}(p, w) \subseteq \mathcal{B}(p, w')$.

In particular, if x^* is the optimal bundle at wealth w, then x^* is feasible when wealth is w'. Hence, $v(p, w') \geq u(x^*) = v(p, w)$.

Since $\succsim$ is LNS, there exists x' that $||x' - x^*|| \leq \epsilon$ such that $x' \succ x^*$ and $x' \in \mathcal{B}(p, w')$ when ϵ is small enough, hence $u(x') > u(x^*)$.

Thus, $v(p, w') \geq u(x') > u(x^*) = v(p, w)$.

Using a similar reasoning, show that $v(p, w)$ is non-increasing in p.

We have already proved properties $i) - ii)$, let us proceed with $iii)$ and $iv)$.

iii) We prove that the set $LCS_v = \{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.

Take two elements of LCS_v, $v(p, w) \leq \bar{v}$ and $v(p', w') \leq \bar{v}$.

Consider now the pair $(p'', w'') = (\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w')$ with $\alpha \in [0, 1]$: we need to show that $v(p'', w'') \leq \bar{v}$.

To do so, we show that for every bundle y such that $p'' \cdot y \leq w''$, i.e. $y \in \mathcal{B}(p'', w'')$, we must have $u(y) \leq \bar{v}$.

Indeed,

$$p'' \cdot y \leq w'' \ \rightarrow (\alpha p + (1-\alpha)p') \cdot y \leq \alpha w + (1-\alpha)w',$$

which holds if either $p \cdot y \leq w$ or $p' \cdot y \leq w'$ or both.

If $p \cdot y \leq w$, then $u(y) \leq v(p, w) \leq \bar{v}$.

If $p' \cdot y \leq w'$, then $u(y) \leq v(p', w') \leq \bar{v}$, hence the result.

iv) The continuity of $v(p,w) = u(x(p,w))$ depends on the continuity of $x(p,w)$, since $u(.)$ is a continuous function by assumption.

Under conditions stated in Theorem 4 for $u(.)$ and $\succsim$ the Walrasian demand correspondence is upper-hemicontinuos, an extension of the notion of continuity for correspondences.

If, in addition, $\succsim$ are also strictly convex so that $x(p,w)$ contains a single element, the Walrasian demand function is continuous hence $v(p,w)$ is also continuous.