The Expenditure Minimization Problem

The Expenditure Minimization Problem

- The basic idea behind the consumer's problem is to choose a bundle that maximizes utility without violating feasibility.

- There is a second way for the consumer to choose a consumption bundle: pick the least costly bundle that yields him a desired utility level.

- This second form of choice is the one we explore now.

The Expenditure Minimization Problem

- Let $\mathcal{U} = \{u(x) : x \in \mathbb{R}^L_+\}$ denote the set of attainable utility levels.

- For each $\overline{u} \in \mathcal{U}$ and $p \gg 0$, the *expenditure minimization problem (EMP)* is

$$\begin{array}{ll} \min & p \cdot x \\ \text{s.t.} & u(x) \geq \overline{u} \\ & x \geq 0 \end{array}$$

- We denote $h(p, \overline{u})$ the solution to this problem, this is the *Hicksian (or compensated) demand correspondence*.

- The value function $e(p, \overline{u}) = p \cdot h(p, \overline{u})$ is called the *expenditure function*.

- Existence and uniqueness of a solution to *EMP* are guaranteed by $p \gg 0$ and continuity and strict convexity of $\succsim$.

- These are the same conditions that guarantee existence and uniqueness of a solution to *UMP*.

- In what follows, we assume these conditions are always satisfied.

Duality

- In neoclassical theory, *UMP* and *EMP* are two mirroring ways to look at the same problem.

- On one hand, in *UMP* the consumer seeks to maximize utility given a fixed wealth/expenditure, namely w.

- On the other hand, in *EMP* the consumer seeks to minimize the expenditure necessary to reach a certain utility level, $\overline{u}$.

- We can formally state this intuition.

Duality: implications for the value functions

Theorem 8

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $p \gg 0$. Then,

i) if x^* is optimal in *UMP* when wealth is $w > 0$, then x^* is optimal in *EMP* when the required utility is $u(x^*)$. Moreover, the minimized expenditure level in *EMP* is exactly w, that is $p \cdot x^* = w$;

ii) if x^* is optimal in *EMP* at utility $\bar{u} > u(0)$, then x^* is optimal in *UMP* when wealth is equal to $p \cdot x^*$. Moreover, the maximized level of utility in *UMP* is exactly $\bar{u}$, that is $u(x^*) = \bar{u}$.

We prove i).

i) By contradiction, assume x^* solves UMP, but is not optimal in *EMP* when the required utility is $u(x^*)$.

   Then, there must exist an x' such that $p \cdot x' < p \cdot x^*$ and $u(x') \geq u(x^*)$. Since x^* solves *UMP*, $p \cdot x^* \leq w$.

   By LNS of $\succsim$, we can find x'' in an ϵ-ball around x', i.e. $||x'' - x'|| \leq \epsilon$, that satisfies $p \cdot x'' < w$ and $x'' \succ x' \; \leftrightarrow u(x'') > u(x')$.

   This implies that $x'' \in \mathcal{B}(p, w)$ and that x^* is not optimal in *UMP*. A contradiction.

   Hence, x^* is optimal in *EMP* and the minimized expenditure is $p \cdot x^*$. Since x^* solves *UMP*, it satisfies Walras' law, i.e. $p \cdot x^* = w$.

Let us prove part *ii*) of Theorem 8.

ii) Observe that since $\overline{u} \geq u(0)$, then $x^* \neq 0$ and $p \cdot x^* > 0$.

By contradiction, assume x^* solves *EMP* but it is not optimal in *UMP* at wealth $p \cdot x^*$.

Then, there must exist an x' such that $p \cdot x' \leq p \cdot x^*$ and $u(x') > u(x^*)$.

Take $x'' = \alpha x'$ with $\alpha \in (0, 1)$. By continuity of $u(.)$, when α is close to 1, $u(x'') > u(x^*)$ and $p \cdot x'' < p \cdot x^*$.

This implies that x'' is preferred to x^* in *EMP*, since it guarantees the desired utility at lower expenditure. A contradiction.

Hence, x^* is optimal in *UMP* and the maximized utility is $u(x^*)$. Since x^* solves *EMP*, it satisfies no-excess utility, i.e. $u(x^*) = \overline{u}$. ■

Expenditure Minimization Problem: Summary

- For each $\overline{u} \in \mathcal{U} = \{u(x) : x \in \mathbb{R}^L_+\}$ and $p \gg 0$, the *expenditure minimization problem (EMP)* is

$$\begin{aligned} \min \quad & p \cdot x \\ \text{s.t.} \quad & u(x) \geq \overline{u} \\ & x \geq 0 \end{aligned}$$

- The solution to EMP is the *Hicksian (or compensated) demand correspondence*, $h(p, \overline{u})$;

- The value function of EMP is the *expenditure function*, $e(p, \overline{u}) = p \cdot h(p, \overline{u})$.

- EMP is dual to UMP.

Properties of the Hicksian demand $h(p, \overline{u})$

Parallel to what we did for *UMP*, let us now examine the properties of the Hicksian demand and of the expenditure function.

Theorem 5

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the Hicksian demand correspondence has the following properties:

i) it is homogeneous of degree zero in prices, i.e. $h(\alpha p, \overline{u}) = h(p, \overline{u}) \; \forall \; \alpha > 0$;

ii) it satisfies no excess utility, i.e. $u(x) = \overline{u}$ for every $x \in h(p, \overline{u})$;

iii) if $\succsim$ is convex, then $h(p, \overline{u})$ is a convex set. If $\succsim$ is strictly convex, then $h(p, \overline{u})$ is single-valued.

Let us prove all three properties.

i) Immediate: the optimal bundle x that minimizes $p \cdot x$ also minimizes $\alpha p \cdot x$ for every $\alpha > 0$, subject to the same constraint $u(x) \geq \overline{u}$.

The Hicksian demand correspondence

Proof of Theorem 5 - ii.)

ii) $h(p, w)$ satisfies no excess utility, i.e. $u(x) = \overline{u}$ for every $x \in h(p, \overline{u})$.

Follows from continuity of $u(.)$. Assume by contradiction that $x \in h(p, \overline{u})$ is such that $u(x) > \overline{u}$.

Consider a bundle $x' = \beta x$, with $\beta \in (0, 1)$.

By continuity of $u(.)$, when β is close enough to 1, $u(x') \geq \overline{u}$ and $p \cdot x' < p \cdot x$.

Then, $x \notin h(p, \overline{u})$, a contradiction.

Hence, $u(x)$ must be equal to $\overline{u}$ for every $x \in h(p, \overline{u})$.

The Hicksian demand correspondence

Proof of Theorem 5 - iii.)

iii-a) Take $\succsim$ convex and let x, x' be two solutions to EMP, i.e. $x \in h(p, \overline{u})$ and $x' \in h(p, \overline{u})$.

We have to show that $\alpha x + (1 - \alpha)x' \equiv x'' \in h(p, \overline{u})$ for every $\alpha \in [0, 1]$.

Since x and x' solve EMP, it must be that $p \cdot x = p \cdot x' = e^*$.

Hence, $p(\alpha x + (1 - \alpha)x') = \alpha p \cdot x + (1 - \alpha) p \cdot x' = e^*$, that is any convex combination of solutions of EMP is itself expenditure minimizing.

In addition, we also know that $u(x) = u(x') = \overline{u}$ by *ii.*). Since $u(.)$ is quasi-concave, $u(x'') = u(\alpha x + (1 - \alpha)x') \geq \overline{u}$.

Hence, x'' is an expenditure minimizing bundle which yields utility $u(x'')$ not lower than $\overline{u}$, therefore is must be $x'' \in h(p, \overline{u})$, too. $h(p, \overline{u})$ is a convex set.

iii-b) Take $\succsim$ strictly convex, i.e. $u(.)$ strictly quasi-concave, then $h(p, \overline{u})$ contains a single element.

Prove it!!

Properties of the Expenditure Function $e(p, \overline{u})$

Theorem 6

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$. Then, the expenditure function has the following properties:

i) it is homogeneous of degree one in prices, i.e. $e(\alpha p, \overline{u}) = \alpha e(p, \overline{u}) \; \forall \; \alpha > 0$;

ii) it is strictly increasing in $\overline{u}$ and non-decreasing in p_l for every $l = 1, \ldots, L$;

iii) it is concave in p;

iv) continuous in p and $\overline{u}$.

Let us prove properties $i) - iii)$ and discuss property $iv)$.

i) Follows immediately from the fact that $h(p, \overline{u})$ is homogeneous of degree zero in $(p, \overline{u})$. Indeed since $h(\alpha p, \overline{u}) = h(p, \overline{u})$ for all $\alpha \in [0, 1]$, then also $\alpha p \cdot h(\alpha p, \overline{u}) = \alpha p \cdot h(p, \overline{u})$.

ii-a) We prove that $e(p, \overline{u})$ is strictly increasing in $\overline{u}$.

Suppose, by contradiction, that $e(p, \overline{u})$ is not strictly increasing in $\overline{u}$, and let x' and x'' denote optimal consumption bundles for utility levels u' and u'', respectively, with $u'' > u'$ and $p \cdot x' \geq p \cdot x'' > 0$.

Consider a bundle $\tilde{x} = \beta x''$, where $\beta \in (0, 1)$.

By continuity of $u(\cdot)$, there exists a β close enough to 1 such that $u(\tilde{x}) > u'$ and $p \cdot x' > p \cdot \tilde{x}$, a contradiction.

ii-b) To show that $e(p, \overline{u})$ is non-decreasing in p_l, consider the price vectors p'' and p' such that $p''_l \geq p'_l$ for commodity l, and $p''_k = p'_k$ for all commodities $k \neq l$.

Let x'' be the solution to the *EMP* for prices p''.

Then, $e(p'', \overline{u}) = p'' \cdot x'' \geq p' \cdot x'' \geq e(p', \overline{u})$, where the latter inequality follows from the definition of $e(p', \overline{u})$.

iii) To show that $e(p, \overline{u})$ is concave in p, we need to prove that

$$e(\alpha p + (1-\alpha)p', \overline{u}) \geq \alpha e(p, \overline{u}) + (1-\alpha)e(p', \overline{u})$$

for every p, p' and for every $\alpha \in [0, 1]$.

Denote $p'' \equiv \alpha p + (1-\alpha)p'$ and let $x'' \in h(\alpha p + (1-\alpha)p', \overline{u})$ be a solution to EMP at price p''.

Then,

$$e(p'', \overline{u}) = p'' \cdot x'' = (\alpha p + (1-\alpha)p') \cdot x'' =$$

$$\alpha p \cdot x'' + (1-\alpha)p' \cdot x'' \geq \alpha p \cdot h(p, \overline{u}) + (1-\alpha)p' \cdot h(p', \overline{u})$$

indeed, x'' is a sub-optimal choice when the prices are either p or p'.

Since $\alpha p \cdot h(p, \overline{u}) + (1-\alpha)p' \cdot h(p', \overline{u}) = \alpha e(p, \overline{u}) + (1-\alpha)e(p', \overline{u})$, the chain of inequalities above expresses the concavity of the expenditure function.

iv) Continuity: skip the proof.$\blacksquare$

The Walrasian Demand Correspondence. The Hicksian Demand Correspondence. The law of demand.

WARP and the law of demand in UMP

Proposition (WARP)

The Walrasian demand function $x(p,w)$ satisfies WARP if and only if, for any compensated price change from (p,w) to $(p',w') = (p', p' \cdot x(p,w))$, we have

$$(p'-p) \cdot [x(p',w') - x(p,w)] \leq 0$$

with strict inequality whenever $x(p',w') \neq x(p,w)$.

- The Weak Axiom imposes a consistency requirement on the Walrasian demand, and implies a form of the law of demand, in that the change in prices and in Walrasian demands move in opposite directions for every compensated price change!!

Compensated change in prices and Walrasian demand

- Consider commodity l and the effect on $x_l(p, w)$ of a compensated change in the price of p_l, only.

- $\Delta p = p' - p = (0, \ldots, \Delta p_l, \ldots, 0)$. We want to measure Δx_l, Proposition WARP implies that if $\Delta p_l > 0$ then $\Delta x_l < 0$.

- We cannot say much about the effect of a price change that is not compensated!

Slutsky matrix and substitution effects

- To summarize, $dp \cdot dx \leq 0$ is equivalent to

$$dp \cdot [D_p x(p,w) + D_w x(p,w) x(p,w)^T] dp \leq 0$$

- Since the Walrasian demands satisfy the weak axiom, the Slutsky matrix is negative semi-definite for every (p,w).

- Negative semi-definiteness of $S(p,w)$ implies that $s_{ll}(p,w) \leq 0$ for every $l = 1,2,...,L$, own substitution effects are non-positive.

The Hicksian demand and the compensated law of demand

Theorem 7

Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $h(p, \overline{u})$ uniquely identifies the optimal bundle for all $p \gg 0$. Then, the Hicksian demand function satisfies the compensated law of demand: for all p, p'

$$(p' - p) \cdot \big(h(p', \overline{u}) - h(p, \overline{u})\big) \leq 0$$

The compensated law of demand

Proof of Theorem 7

- By definition, $h(p, \overline{u})$ solves the expenditure minimization problem at price $p \gg 0$. Hence,

$$p' \cdot h(p', \overline{u}) \leq p' \cdot h(p, \overline{u})$$
$$p \cdot h(p, \overline{u}) \leq p \cdot h(p', \overline{u})$$

- Rearranging the two inequalities, we get

$$p' \cdot \left[h(p', \overline{u}) - h(p, \overline{u})\right] \leq 0$$
$$-p \cdot \left[h(p', \overline{u}) - h(p, \overline{u})\right] \leq 0$$

summing we get the result.$\blacksquare$

- The result we just proved implies that, differently from the Walrasian demand, the change of the Hicksian demand is *always* inverse with respect to any change in prices.

- The inverse relationship holds for each commodity, i.e.

$$(p' - p) \cdot \big(h(p', \overline{u}) - h(p, \overline{u})\big) \leq 0,$$

is a compact way to express that

$$(p'_k - p_k) \cdot \big(h_k(p', \overline{u}) - h_k(p, \overline{u})\big) \leq 0 \qquad \text{for } k = 1, 2, \ldots, L.$$

Duality: implications for the value functions

- We have formally shown that EMP is the dual problem of UMP and viceversa.

- More precisely, recall that, by Theorem 8, if $u(.)$ is a continuous utility function representing LNS $\succsim$ on $X = \mathbb{R}^L_+$ and if $p \gg 0$,

a) if x^* is optimal in *UMP* at $w > 0$, then x^* is optimal in *EMP* at $u(x^*)$. Moreover, the expenditure function of such *EMP* is exactly equal to w, i.e. $p \cdot x^* = w$;

b) if x^* is optimal in *EMP* at $\bar{u} > u(0)$, then x^* is optimal in *UMP* at wealth equal to $p \cdot x^*$. Moreover, the indirect utility of such *UMP* is exactly equal to $\bar{u}$, i.e. $u(x^*) = \bar{u}$.

Duality: summary

Theorem 8 supports the following reasoning.

Let $x(p, w)$ be a solution to *UMP* given $p \gg 0$ and $w > 0$, so that

- $p \cdot x(p, w) = w$ (by Walras' law),
- $u(x(p, w)) = v(p, w) \geq \bar{u}$.

Then,

$$e(p, v(p, w)) = p \cdot x(p, w) = w \tag{5}$$

for all $p \gg 0$ and $w > 0$.

Duality: summary

If $h(p,\overline{u})$ is a solution to *EMP* given $p \gg 0$ and $\overline{u} > u(0)$, so that

- $u(h(p,\overline{u})) = \overline{u}$ (no-excess utility),
- $p \cdot h(p,\overline{u}) = e(p,\overline{u}) = w$.

Then,

$$v(p, e(p,\overline{u})) = u(h(p,\overline{u})) = \overline{u} \tag{6}$$

for all $p \gg 0$ and $\overline{u} > u(0)$.

Fix the price vector $p \gg 0$, equations (6) and (5) imply that the indirect utility function and expenditure function are the inverse of one another.

Duality: implications on demand correspondences

Theorem 8 has also implications on the Walrasian and Hicksian demand correspondences. For all $p \gg 0$ and $\overline{u} > u(0)$,

1) $x_l(p, w) = h_l(p, v(p, w))$ for each commodity $l = 1, \ldots, L$

2) $h_l(p, \overline{u}) = x_l(p, e(p, \overline{u}))$ for each commodity $l = 1, \ldots, L$.

Duality: implications on demand correspondences

Take 2) and let prices vary. For each commodity $l = 1, \ldots, L$

$$h_l(p, \overline{u}) = x_l(p, e(p, \overline{u}))$$

The Hicksian demand $h(p, \overline{u})$ measures the demand that would emerge if we adjust wealth so maintain the consumer at the same level of utility.

This type of compensation is the Hicksian wealth compensation, and explains why $h(p, \overline{u})$ is the compensated demand correspondence.

The relationship between Hicksian and Walrasian demands, indirect utility and expenditure functions.

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Implications of duality

- Suppose that $u(.)$ is a continuous utility function representing a LNS preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $p \gg 0$.

- Assume also that $\succsim$ is strictly-convex, which implies strictly quasi-concave utility function, hence $x(p, w)$ and $h(p, \overline{u})$ identify a unique optimal bundle for UMP and EMP, respectively.

- We start by examining the relationship between the expenditure function and the Hicksian demand.

Hicksian Demand and Expenditure Function

Shepard's Lemma: the relationship between $e(p, \overline{u})$ and $h(p, \overline{u})$

Shepard's Lemma

Suppose that $u(.)$ is a continuous utility function representing a LNS and strictly convex preference relation $\succsim$ on $X = \mathbb{R}^L_+$, and that $p \gg 0$.
If $e(p, \overline{u})$ is differentiable in p then, for all p and $\overline{u}$, the Hicksian demand is the derivative of the expenditure function with respect to prices, i.e.

$$\frac{\partial e(p, \overline{u})}{\partial p_k} = h_k(p, \overline{u}) \qquad \text{for } \; k = 1, \cdots, L.$$

- The proof is an implication of the Envelope Theorem. Indeed,

$$e(p,\overline{u}) = p \cdot h(p,\overline{u}) = p_1 \cdot h_1(p,\overline{u}) + \cdots + p_L \cdot h_L(p,\overline{u})$$

Consider commodity k and differentiate $e(p,\overline{u})$ w.r.t. p_k. By the chain rule

$$\frac{\partial e(p,\overline{u})}{\partial p_k} = h_k(p,\overline{u}) + \sum_{j=1}^{J} p_j \frac{\partial h_j(p,\overline{u})}{\partial p_k}$$

Since $h(p,\overline{u})$ is optimal, a change in prices has no first-order effect on demand, i.e. $\frac{\partial h_j(p,\overline{u})}{\partial p_k} = 0$, hence on expenditure, and

$$\frac{\partial e(p,\overline{u})}{\partial p_k} = h_k(p,\overline{u}).\blacksquare$$

- The proof is an implication of the Envelope Theorem.
- In EMP, prices are parameters of the min problem.
- The Envelope Theorem tells us that, in an optimization problem, when measuring the first-order effects of a change in the parameters of the problem on the value function, we can disregard any change in the maximizer (minimizer), and only consider the direct effects.

  When applied to the EMP, the direct effect of a change in p_k on the minimal expenditure, measures the variation of the expenditure $e(p,\overline{u})$ at fixed demand $h(p,\overline{u})$.

Roy's Identity: the relationship between $v(p, w)$ and $x(p, w)$

- Let $u^* = v(p^*, w^*)$.

- By duality, $v(p, e(p, u^*)) = u^*$ for all p. Differentiate with respect to p_j and evaluate at $p = p^*$, we get

$$\frac{\partial v(p^*, e(p^*, u^*))}{\partial p_j} + \frac{\partial v(p^*, e(p^*, u^*))}{\partial w} \frac{\partial e(p^*, u^*)}{\partial p_j} = 0.$$

- Shepard's lemma implies that $\frac{\partial e(p^*, u^*)}{\partial p_j} = h_j(p^*, u^*)$, substituting we get

$$\frac{\partial v(p^*, e(p^*, u^*))}{\partial p_j} + \frac{\partial v(p^*, e(p^*, u^*))}{\partial w} h_j(p^*, u^*) = 0.$$

$$\frac{\partial v(p^*, e(p^*, u^*))}{\partial p_j} + \frac{\partial v(p^*, e(p^*, u^*))}{\partial w} h_j(p^*, u^*) = 0.$$

- Since $w^* = e(p^*, u^*)$ and $h_j(p^*, u^*) = x_j(p^*, e(p^*, u^*))$, we can write

$$\frac{\partial v(p^*, w^*)}{\partial p_j} + \frac{\partial v(p^*, w^*)}{\partial w} x_j(p^*, w^*) = 0.$$

which gives

$$x_j(p^*, w^*) = -\frac{\dfrac{\partial v(p^*, w^*)}{\partial p_j}}{\dfrac{\partial v(p^*, w^*)}{\partial w}}.$$

Roy's Identity

Suppose that $u(.)$ is a continuous utility function representing a LNS and strictly convex preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $p \gg 0$.
Suppose that $v(p, w)$ is differentiable at $(p^*, w^*) \gg 0$. Then, for every $j = 1, \ldots, L$

$$x_j(p^*, w^*) = -\frac{\dfrac{\partial v(p^*, w^*)}{\partial p_j}}{\dfrac{\partial v(p^*, w^*)}{\partial w}}$$

Walrasian Demand and Indirect Utility functions

- Roy's identity is the analog of Shepard's lemma for the Walrasian demand function.
- When deriving the Walrasian demand from the indirect utility, we have to normalize the price derivative of the indirect utility by the derivative of $v(p, w)$ w.r.t. wealth;

$$x_j(p^*, w^*) = -\frac{\frac{\partial v(p^*, w^*)}{\partial p_j}}{\frac{\partial v(p^*, w^*)}{\partial w}}.$$

- Indeed, utility is an ordinal concept, so is the Walrasian demand, which is then sensitive to the underlying $u(.)$.
- For the compensated demand to be equivalent to the Walrasian demand, which needs to consider the Hicksian wealth compensation (duality).

Slutsky Equation: the relationship between Hicksian and Walrasian demands

Hicksian and Walrasian demand functions

- Fix $\bar{u} = v(\bar{p}, w)$ for some $\bar{p} \gg 0$ and $w > 0$.
- By duality $w = e(\bar{p}, \bar{u})$.
- Duality also implies that for all p and u and for each commodity $l = 1, \dots, L$

$$h_l(p, u) = x_l(p, e(p, u)) \tag{7}$$

- Differentiate both sides of (7) with respect to p_k and evaluate it at $(\bar{p}, \bar{u})$ to get

$$\frac{\partial h_l(\bar{p}, \bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial w} \frac{\partial e(\bar{p}, \bar{u})}{\partial p_k}. \tag{8}$$

Hicksian and Walrasian demand functions

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial w}\frac{\partial e(\bar{p},\bar{u})}{\partial p_k} \qquad (8)$$

- By Shepards' Lemma, $\frac{\partial e(\bar{p},\bar{u})}{\partial p_k} = h_k(\bar{p},\bar{u})$, hence

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial w}h_k(\bar{p},\bar{u}) \qquad (8)$$

- Duality implies that $e(\bar{p},\bar{u}) = w$ and that $h_k(\bar{p},\bar{u}) = x_k(\bar{p},e(\bar{p},\bar{u})) = x_k(\bar{p},w)$, thus (8) can be rewritten as:

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},w)}{\partial p_k} + \frac{\partial x_l(\bar{p},w)}{\partial w}x_k(\bar{p},w).$$

The Slutsky Equation

Suppose that $u(.)$ is a continuous utility function representing a LNS and strictly convex preference relation $\succsim$ on $X = \mathbb{R}^L_+$ and that $p \gg 0$. Then, for all (p, w) and $u = v(p, w)$ we have

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} x_k(p, w) \qquad \text{for all } l, k.$$

The Slutsky Equation

- Let $l = k$, the Slutsky equation tells us that

$$\frac{\partial h_l(p,u)}{\partial p_l} = \frac{\partial x_l(p,w)}{\partial p_l} + \frac{\partial x_l(p,w)}{\partial w} x_l(p,w).$$

- For a given commodity, the Slutsky equation relates the slope of the Hicksian and of the Walrasian demands.
- If commodity l is normal, the Hicksian demand is steeper (more rigid) than the Walrasian demand.
- Indeed, if the price of commodity l increases and its demand falls, we have to increase consumer's wealth to guarantee the same level of utility. If such wealth compensation is absent, as in the Walrasian demand, the fall of the demand for commodity l is more pronounced.

The Slutsky Substitution Matrix

- The matrix that collects all the cross-price derivatives of the Hicksian demands for each commodity l, i.e. $\frac{\partial h_l(p,u)}{\partial p_k}$ for each k, l, is called the *Slutsky substitution matrix*, and labelled $S(p, w)$.

- Since $S(p, w)$ is obtained by taking the price derivative of the Hicksian demand for each commodity, when demand is generated from utility maximization, the matrix $S(p, w)$ inherits some properties of the Hicksian demand and of the expenditure function.

The Slutsky Substitution Matrix

Specifically,

- $S(p, w)$ is negative semidefinite [because of Shepard's Lemma and the concavity of $e(p, \overline{u})$];
- $S(p, w)$ is symmetric [i.e. the compensated cross-price derivatives of any two commodities, l and k, are equal, i.e. $\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial h_k(p,u)}{\partial p_l}$];
- $S(p, w)$ is such that $S(p, w) \cdot p = 0$ [by homogeneity of degree zero of $h(p, \overline{u})$].

The property that $S(p, w) \cdot p = 0$, together with the compensated law of demand imply that every commodity has at least one substitute, i.e. since for commodity k, $\frac{\partial h_k(p,u)}{\partial p_k} \leq 0$ there must exist a commodity, say j, such that $\frac{\partial h_j(p,u)}{\partial p_k} \geq 0$.

The Slutsky Equation

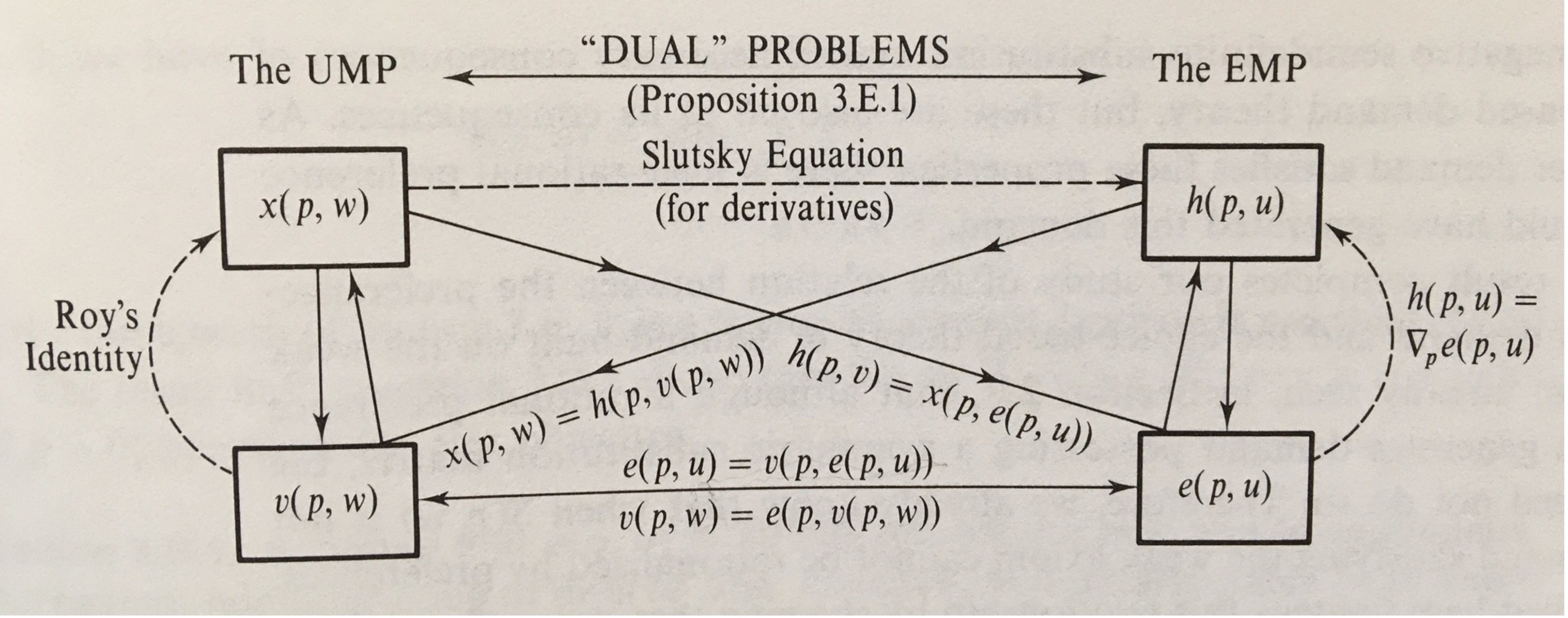
The Slutsky equation can be rewritten as follows:

$$\underbrace{\frac{\partial x_l(p,w)}{\partial p_k}}_{TE} = \underbrace{\frac{\partial h_l(p,u)}{\partial p_k}}_{SE} \underbrace{- \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)}_{IE}$$

The total effect (TE) of a change in price on consumer's demand can be decomposed into two effects: the substitution effect (SE) and the income effect (IE).

SE gives a measure of the effect that a change in price induces in the consumers' demand when wealth is adjusted so to keep the consumer at the same utility level.

IE measures the effect of the same change on the purchasing power of the consumer hence on its Walrasian demand.



"DUAL" PROBLEMS
The UMP
The EMP
(Proposition 3.E.1)
Slutsky Equation
(for derivatives)
$x(p, w)$
$h(p, u)$
Roy's Identity
$h(p, u) = \nabla_p e(p, u)$
$x(p, w) = h(p, v(p, w))$
$h(p, v) = x(p, e(p, u))$
$e(p, u) = v(p, e(p, u))$
$v(p, w) = e(p, v(p, w))$
$v(p, w)$
$e(p, u)$

Correspondences, Upper-hemicontinuity

Definition 1.

Given a set $A \subset \mathbb{R}^n$, a correspondence $f : A \rightarrow \mathbb{R}^n$ is a rule that assigns a set $f(x) \subseteq \mathbb{R}^k$ to every $x \in A$.

Definition 2.

Given a set $A \subset \mathbb{R}^n$ and the closed set $Y \subset \mathbb{R}^k$, the correspondence $f : A \rightarrow Y$ is upper hemicontinuous if it has a closed graph and the images of compact sets are bounded, that is, for every compact set $B \subset A$ the set $f(B) = \{y \in Y : y \in f(x) \text{ for some } x \in B\}$ is bounded.

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