Microeconomics I - MSc Economics

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Introduction

- Microeconomics deals with the analysis of economic choices. Economic agents are consumers, firms, financial intermediaries, who operate in different contexts.
- Consumers typically take consumption and saving decisions, firms deal with investment and production decisions, financial intermediaries handle financial decisions of consumers and firms and manage their own portfolios... all of them interact in (complex) economic systems.
- As a result of their choices, market demand and supply emerge, to which economists combine the notion of equilibrium, typically in terms of price and quantity.

- This course offers a formal (logico-mathematical) approach to the analysis of the choices of consumers and firms operating in market economies and deals with the notion of equilibrium in competitive markets.
- When examining choices, we will pay attention to individual decision making in setting without and with risky alternatives.
- Eventually, we will discuss the two fundamental Theorems of Welfare Economics for competitive market systems which require a general equilibrium perspective.

Introduction

- References:
  - Mas Colell A., Whinston M. and J. Green, Microeconomic Theory, Oxford University Press
  - Varian H. R., Intermediate Microeconomics: A Modern Approach, Norton & Co
- Lecturer and TA
  - **Eloisa Campioni**, office hours on Tuesdays after class, to be arranged via e-mail: eloisa.campioni@uniroma2.it
  - **Lorenzo Bozzoli**, office hours, to be arranged via e-mail: lor.bozzoli@gmail.com

Structure of the course

- Lectures and practices. Practical classes will be held on Fridays, 9-11.
- Problem sets will be assigned on a regular basis and corrected in class during the practices.

Evaluation

- Each student's final evaluation consists of the combination of the following three elements.

1. *Problem sets.* Each student will hand in her/his solutions to the assigned problem sets, cooperative work is encouraged. The student's solutions will be evaluated/graded and will contribute to 30% of the final grade. The exercises/questions will be then corrected during the practices.

2. *Final written exam.* Written closed-book exam (questions and exercises), yields the remaining 70% of the final grade.

3. *In class participation.* Active participation during lectures will also be part of the evaluation. During the practical classes, students will be randomly asked to present their solutions to the assigned problem sets.

Primitives

- In most of this course, we focus on individual economic agents, and make two assumptions about these agents:

  1. *Atomistic:* the agents are small enough compared to the size of the market that their choices do not affect the market price.

  2. *Non–strategic:* agents do not interact when making their choices.

- We start by examining the choice problem of a consumer.

Primitives

- There are four building blocks in modeling consumer choice:
  - *Consumption (Choice) Set:* The set X of all alternatives (complete consumption plans) that the consumer can conceive;
  - *Feasible Set:* The subset B of X that is achievable given the constraints the consumer faces;
  - *Consumer's Preferences:* A rule specifying how the consumer ranks different alternatives;
  - *Behavioral Assumption:* The consumer seeks to identify a feasible alternative that is preferred to all other feasible alternatives.

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Consumption set

- Consider an economy with L commodities. Assume that the consumption set X satisfies:

  i) X is non-empty, specifically $X = \mathbb{R}^L_+$;

  ii) X is closed;

  iii) X is convex;

  iv) $\mathbf{0} \in X$;

  v) consumption of larger quantities is always feasible, i.e. if $x \in X$ and $y \geq x$, then $y \in X$.

- A typical element of X is denoted by $x = (x_1, \ldots, x_L)$, where $x_i \geq 0$ is the amount consumed of good $i = 1, 2, ..., L$.

Walrasian/competitive budget set

- Economic constraints on alternatives: consumer cannot achieve what she cannot afford. Some alternatives may not be (economically) feasible. The *Budget set* identifies the set of economically feasible alternatives.

- For economic decisions, feasibility concerns prices and wealth.

Walrasian/competitive budget set

- Let $p = (p_1, \ldots, p_L) \gg 0$ be the vector of prices of L commoditites, and $w > 0$ the consumer's wealth.

- The *budget set* is given by

$$\mathcal{B}(p,w) = \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}.$$

  with $p \cdot x \; = p_1x_1 + ... + p_Lx_L$.

- The consumer's problem, given price vector p and wealth w, is then to choose $x \in \mathcal{B}(p,w)$ according to some choice criterion, to be specified.

Consumers' choice problem

- Preference approach: the tastes of the decision maker are primitives (given) and embodied in her preferences over alternatives. Axioms of rationality imposed on preferences, then examine behavior.
- Revealed Preference approach: individual's choice behavior first, impose assumptions on choices then reconstruct underlying consistent preferences.
- The two approaches can be reconciled. The first one prevails in courses.

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- Consumer preferences are represented by a binary relation $\succsim$ on elements of the consumption set X,

$$x \succsim y \qquad \textit{x is at least as good as y.}$$

- Define:
  - Strict preference: $x \succ y$ if, and only if, $x \succsim y$ but $y \not\succsim x$;
  - Indifference: $x \sim y$ if, and only if, $x \succsim y$ and $y \succsim x$.

- The binary relation $\succsim$ compares *two* consumption plans at a time.
    - The same is true for strict preference $\succ$ and indifference $\sim$.
- The following axioms determine basic criteria these binary comparisons must adhere to.

- **Axiom 1 (Completeness):** The binary relation $\succsim$ is *complete* if for all $x, y \in X$, we have that

$$x \succsim y \qquad \text{or} \qquad y \succsim x \text{ (or both).}$$

Remark: if $\succsim$ is complete, then $\succsim$ is *reflexive*, i.e., $x \succsim x$ for all $x \in X$.

- **Axiom 2 (Transitivity):** The binary relation $\succsim$ is *transitive* if for all $x, y, z \in X$,

$$\text{if } x \succsim y \text{ and } y \succsim z, \qquad \text{then } x \succsim z.$$

Definition

A *preference relation* is a complete and transitive binary relation.

- Preference relations that satisfy Axiom 1 (completeness) and Axiom 2 (transitivity) are *rational*.
- In this course we will focus on rational preference relations.

Preferences

Sets in X

- Given the preference relation $\succsim$ and a consumption bundle x, we define the following subsets of X:

1. the set of bundles that are *at least as good as x* :
$\succsim(x) = \{x' \in X : x' \succsim x\}$, i.e. the upper contour set of x;

2. the set of bundles that are *no better than x* : $\precsim(x) = \{x' \in X : x \succsim x'\}$, i.e. the lower contour set of x;

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- **Axiom 3 (Monotonicity):** The preference relation $\succsim$ is *monotone* if for each $x \in X$ and $y \gg x$, then $y \succ x$.

- **Axiom** $3'$ **(Strong Monotonicity):** The preference relation $\succsim$ is *strongly monotone* if for each $x \in X$ and $y \geq x$ and $y \neq x$, then $y \succ x$.

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- **Axiom** $3'$ **(Strong Monotonicity):** The preference relation $\succsim$ is *strongly monotone* if for each $x \in X$ and $y \geq x$ and $y \neq x$, then $y \succ x$.

- **Axiom** $3''$ **(Local Non–Satiation):** The preference relation $\succsim$ is *locally non–satiated* if for each $x \in X$ and for each $\varepsilon > 0$, there exists $y \in X$ such that $||y - x|| \leq \epsilon$ and $y \succ x$.

$|| \cdot ||$ is the Euclidean distance, defined as $\left[\sum\limits_{l=1}^{L}(y_l - x_l)^2\right]^{1/2}$

Preferences

- Monotonicity has implications on how upper contour sets and lower contour sets of $x \in X$...
- Local Non-Satiation implies that the Indifference set of $x \in X$ is not thick!!

- **Axiom 4 (Convexity):** The preference relation is *convex* if for every $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ is convex, that is take two bundles $y \succsim x$ and $z \succsim x$, then their convex combination $\alpha y + (1 - \alpha)z \succsim x$ for any $\alpha \in [0, 1]$.

- **Axiom $4'$ (Strict Convexity):** The preference relation is *strictly convex* if for every $x \in X$, we have that $y \succsim x$ and $z \succsim x$ with $y \neq z$ imply $\alpha y + (1 - \alpha)z \succ x$ for any $\alpha \in [0, 1]$.

- **Axiom 4 (Convexity):** The preference relation is *convex* if for every $x \in X$, the upper contour set $\{y \in X : y \succsim x\}$ is convex, that is take two bundles $y \succsim x$ and $z \succsim x$, then their convex combination $\alpha y + (1 - \alpha)z \succsim x$ for any $\alpha \in [0, 1]$.

- **Axiom** $4'$ **(Strict Convexity):** The preference relation is *strictly convex* if for every $x \in X$, we have that $y \succsim x$ and $z \succsim x$ with $y \neq z$ imply $\alpha y + (1 - \alpha)z \succ x$ for any $\alpha \in [0, 1]$.

- **Axiom 5 (Continuity):** The preference relation $\succsim$ is *continuous* if for each sequence of pairs of bundles $\{x^n, y^n\}$,

  that verify $x^n \succsim y^n$ for each n, and

  that are converging $\lim_{n\to\infty} x^n = x$ and $\lim_{n\to\infty} y^n = y$,

  we have that $x \succsim y$.

- The preference relation is preserved under the limit.

- An equivalent statement of *continuity* of $\succsim$ is that for each $x \in X$ the upper contour set $\succsim(x)$ and the lower contour set $\precsim(x)$ are closed subsets of X.

- Since $\sim(x) = \succsim(x) \bigcap \precsim(x)$, $\sim(x)$ is also closed if continuity holds. Indeed, the intersection of closed sets is closed.

Definition

The consumption bundle $x^* \in X$ is a *satiation point* of $\succsim$ if $x^* \succsim x$ for all $x \in X$.

If $\succsim$ is locally non–satiated, then $\succsim$ has no satiation point.

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Utility Functions

- Preference relations satisfying Axioms 1- 5 (or their variants) discipline consumer behavior, but are difficult to work with.
- Microeconomic theory has developed a more suitable approach to represent consumer's preferences.
- First, we establish the existence of a (ordinal) function that represents well–behaved preference relations, i.e. the *utility function*.
- Then we move on to study the properties of such function when the consumer must choose her most preferred alternative.

Utility Functions

Definition

An utility function $u : X \to \mathbb{R}$ represents the binary relation $\succsim$ on X if for all $x, x' \in X$,

$$u(x') \geq u(x) \text{ if and only if } x' \succsim x.$$

Lemma 2

Suppose that $u : X \to \mathbb{R}$ represents the binary relation $\succsim$ on X. Then $\succsim$ is a rational preference relation.

- *Completeness:* For any $x, y \in X$, either $u(x) \geq u(y)$ or $u(x) \leq u(y)$.

  Since u represents $\succsim$, we then have that either $x \succsim y$ or $y \succsim x$, that is $\succsim$ is complete.

- *Transitivity:* Take any $x, y, z \in X$ such that $u(x) \geq u(y)$ and $u(y) \geq u(z)$, then $u(x) \geq u(z)$.

  Since u represents $\succsim$, we have that $x \succsim y$ and $y \succsim z$ imply $x \succsim z$, that is $\succsim$ is transitive. $\blacksquare$

Utility Functions

Proof of Lemma 2

- *Completeness:* For any $x, y \in X$, either $u(x) \geq u(y)$ or $u(x) \leq u(y)$.

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Utility Functions

- Lemma 2 says that if a binary relation is represented by a utility function, then it is complete and transitive, which qualifies a (rational) preference relation.
- In general, is the converse also true? That is, can every preference relation be represented by a utility function?
- The answer is negative, let me show you why by means of an example.

Consumer Theory

November 7, 2024

Utility Functions

Lexicographic order

- Consider a particular preference order: the lexicographic order in $\mathbb{R}^2_+$, that is the binary relation $\succsim_\ell$ such that $x \succsim_\ell y$ is defined by looking at the ordered components of the bundles. That is,

$$(x_1, x_2) \succsim_\ell (y_1, y_2) \text{ if, and only if, } x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 \geq y_2.$$

- Rank elements by comparing the quantities of each good in turn.

- **We show that $\succsim_\ell$ is a preference order, but it cannot be represented by an utility function.**

- The lexicographic order is a preference order, i.e. $\succsim_\ell$ is **complete** and transitive.

- **Completeness.** Take two bundles $x, y \in \mathbb{R}^2_+$. Focus on the first commodity, it must be that either $x_1 > y_1$ or that $x_1 \leq y_1$.

  - If $x_1 > y_1$, then $x \succsim_\ell y$.

  - If $x_1 = y_1$, it could be one of two possibilities: either $x_2 \geq y_2$ or $y_2 \geq x_2$, which respectively lead to either $x \succsim_\ell y$ or $y \succsim_\ell x$.

  - If $x_1 < y_1$, then $y \succsim_\ell x$.

Utility Functions

Lexicographic order

- The lexicographic order is a preference order, i.e. $\succsim_\ell$ is complete and **transitive**.

- **Transitivity.** Take three bundles $x, y, z \in \mathbb{R}^2_+$ such that $x \succsim_\ell y$ and $y \succsim_\ell z$. By the lexicographic order, it could be that
    - $x_1 > y_1 > z_1$, then $x \succsim_\ell z$ is an immediate implication;
    - $x_1 = y_1$ and $x_2 \geq y_2$ and $y_1 > z_1$, in which case again $x \succsim_\ell z$ is implied;
    - $x_1 = y_1$ and $x_2 \geq y_2$, and $y_1 = z_1$ and $y_2 \geq z_2$, in which case again $x \succsim_\ell z$ is implied.

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- The proof goes by contradiction. Suppose that $u : \mathbb{R}^2_+ \to \mathbb{R}$ represents $\succsim_\ell$.
- Since $(x, 1) \succ_\ell (x, 0)$ for all $x \in \mathbb{R}_+$, then $u(x, 1) > u(x, 0)$ for all $x \in \mathbb{R}_+$
- For each $x \in \mathbb{R}_+$, pick a $q_x \in \mathbb{Q}$, such that $u(x, 1) > q_x > u(x, 0)$.
- If $x' > x$, then $(x', 0) \succ_\ell (x, 1)$, it is also true that $q_{x'} > u(x', 0) > u(x, 1) > q_x$.
- Thus, $q(.)$ is a one-to-one function that associates to each real number a rational number.
- This is impossible!! since the set of rational number is a countable set, while the set of reals is uncountable. ■

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Utility Functions
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- The big issue with the lexicographic order is that it is not continuous. This discontinuity allows for sudden reversals of preferences.

- Continuity is a crucial property for the existence of an utility representation of a preference relation.

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Theorem 1

Suppose $\succsim$ is a continuous and rational preference relation on X. Then, there exists a continuous function $u : X \to \mathbb{R}$ that represents $\succsim$.

- We establish the result when $X = \mathbb{R}^L_+$ and $\succsim$ is monotone. For ease of exposition let $L = 2$
- Let $e = (1, 1)$ denote the two-dimensional vector with all elements equal to 1. For each $x \in \mathbb{R}^2_+$, let $A^-(x) = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$ and $A^+(x) = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$.
- Since $\succsim$ is assumed to be monotone, $x \succsim \mathbf{0}$, and therefore $\mathbf{0} \in A^-(x)$ for all $x \in \mathbb{R}^2_+$, i.e. $A^-(x)$ is non–empty.
- Monotonicity of $\succsim$ implies that $A^+(x)$ is also non–empty: indeed, given x we can find $\overline{\alpha}$ such that $\overline{\alpha}e \gg x$ which belongs to $A^+(x)$.

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- Let $e = (1, 1)$ denote the two-dimensional vector with all elements equal to 1. For each $x \in \mathbb{R}_+^2$, let $A^-(x) = \{\alpha \in \mathbb{R}_+ : x \succsim \alpha e\}$ and $A^+(x) = \{\alpha \in \mathbb{R}_+ : \alpha e \succsim x\}$.
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- Monotonicity of $\succsim$ implies that $A^+(x)$ is also non–empty: indeed, given x we can find $\overline{\alpha}$ such that $\overline{\alpha}e \gg x$ which belongs to $A^+(x)$.

- Continuity of $\succsim$ implies the upper contour set and the lower contour set of x are closed. Hence, also $A^-(x)$ and $A^+(x)$ are non-empty and closed for every $x \in \mathbb{R}^2_+$.

- Fix $x \in \mathbb{R}^2_+$. Since $\succsim$ is complete, $\mathbb{R}_+ = A^+(x) \bigcup A^-(x)$.

- Thus, $A^+(x) \bigcap A^-(x) \neq \emptyset$, otherwise $\mathbb{R}_+$ would be the union of two disjoint sets, which is not possible since $\mathbb{R}_+$ is connected.

- Hence, there exists a scalar $\hat{\alpha} \in A^+(x) \bigcap A^-(x)$ such that $\hat{\alpha} e \sim x$.

- Since $\succsim$ is monotone, such scalar is unique. Indeed, by monotonicity $\alpha_1 e \succ \alpha_2 e$ whenever $\alpha_1 > \alpha_2$. Let $\hat{\alpha}(x)$ denote the unique scalar satisfying $\hat{\alpha}(x) e \sim x$.

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- Fix $x \in \mathbb{R}^2_+$. Since $\succsim$ is complete, $\mathbb{R}_+ = A^+(x) \bigcup A^-(x)$.

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Utility Functions

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Utility Functions

Proof of Theorem 1 - constructing u

- Let our utility function, $u : X \to \mathbb{R}$, be such that $u(x) = \hat{\alpha}(x)$ for every $x \in X$, with $\hat{\alpha}(x)$ being the unique real number in $A^+(x) \bigcap A^-(x)$.

- We need to check that:
  a.) u represents $\succsim$, that is for all $x, x' \in X$,

$$u(x') \geq u(x) \text{ if and only if } x' \succsim x.$$

  b.) u is continuous.

Utility Functions

Proof of Theorem 1 - u represents $\succsim$

a.) We want to prove that for all $x, x' \in X$,

$$u(x) \geq u(x') \text{ if and only if } x \succsim x'.$$

- *[If]* Consider, $\hat{\alpha}(x)$ and $\hat{\alpha}(x')$, such that $\hat{\alpha}(x) \geq \hat{\alpha}(x')$. By construction, since $\succsim$ are monotone, $\hat{\alpha}(x)e \succsim \hat{\alpha}(x')e$, that implies $x \succsim x'$.

- *[Only if]* Suppose alternatively that $x \succsim x'$, then $\hat{\alpha}(x)e \succsim \hat{\alpha}(x')e$, which implies that $\hat{\alpha}(x) \geq \hat{\alpha}(x')$.

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Everything goes through with an arbitrary finite L.

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- The idea behind the proof of Theorem 1 is to construct an utility function that selects the value of α that makes the individual indifferent between the bundles x and αe.

- Continuity and monotonicity of preferences are indispensable here to guarantee the uniqueness of such value, in particular when the consumption set is infinite, as $X = \mathbb{R}^L_+$.

- From now on, we focus on continuous preferences $\succsim$, hence representable by a continuous utility function.

Definition. Let $u : X \to \mathbb{R}$ and denote the image of u by $\mathcal{U}$. Consider a strictly increasing function, $\tau : \mathcal{U} \to \mathbb{R}$, we call the function v which is the composition of τ and u, so that $v(x) = \tau(u(x))$, a *monotone transformation* of u.

Notice that $v : X \to \mathbb{R}$ is itself a function from X to $\mathbb{R}$.

Theorem 2

Let $\succsim$ be a preference relation on X and let $u : X \to \mathbb{R}$ be a utility function that represents $\succsim$. Then $v : X \to \mathbb{R}$ also represents $\succsim$ if, and only if, v is a monotone transformation of u.

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- If $u : X \to \mathbb{R}$ represents $\succsim$, then monotonicity of preferences implies that the utility function u is increasing.

- If $u : X \to \mathbb{R}$ represents $\succsim$, then monotonicity of the preference relation implies that the utility function u is increasing.

- Suppose $u : X \to \mathbb{R}$ represents $\succsim$, convexity of the preference relation implies that u is quasi–concave.

- If $u : X \to \mathbb{R}$ represents $\succsim$, then convexity of $\succsim$ implies that the utility function u is quasi–concave.

The function $u : X \to \mathbb{R}_+$ is quasi-concave if :

i.) for every $x \in X$ the set $\{y \in X : u(y) \geq u(x)\}$ is convex,

ii.) or, equivalently, for every $x, y \in X$, $u(\alpha x + (1 - \alpha)y) \geq Min\{u(x), u(y)\}$ for every $\alpha \in [0, 1]$.

- A preference relation $\succsim$ that is *strictly convex* implies that u is *strictly quasi–concave*.

Utility Functions
Other Properties

- If $u : X \to \mathbb{R}$ represents $\succsim$, then convexity of $\succsim$ implies that the utility function u is quasi–concave.

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